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On Partial Derivatives of the I-Function of r-Variables

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Abstract. The object of this paper is to establish some formulas involving the partial derivatives of the Prasad's I-function of r-variables. On specializing the parameters, the results can be reduced to derivatives of H-function of r-variables, H-function of two variables. As a result, we have four corollaries concerning the multivariable H-function.

Key Words and Phrases: I-function of two and several complex variables, multivariable H-functions, partial differentiation.

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1. Introduction and preliminaries

Recently Sreenivas et al. [16] have studied some formulas involving the partial derivatives of the multivariable I-function defined by Prathima et al. [10]. In this paper, we establish four formulas concerning the multivariable I-function defined by Prasad [9]. We shall give the formulas involving the multivariable H-function defined by Srivastava and Panda [18, 19].

The multivariable I-function generalizes the multivariable H-function. This function of r-variables is defined in term of multiple Mellin-Barnes type integral:

$$I\left(z_1, z_2, \cdots, z_r\right) = I^{0,n_2;0,n_3;\cdots;0,n_r:m^{(1)},n^{(1)};\cdots;m^{(r)},n^{(r)}}_{p_2,q_2,p_3,q_3;\cdots;p_r,q_r:p^{(1)},q^{(1)};\cdots;p^{(r)},q^{(r)}} \left[\begin{array}{c} z_1 \\ \cdot \\ \cdot \\ z_r \end{array}\right| \begin{array}{l} \left(a_{2j};\alpha'_{2j},\alpha''_{2j}\right)_{1,p_2}; \\ \\ \left(b_{2j};\beta'_{2j},\beta''_{2j}\right)_{1,q_2}; \end{array}$$

$$\left.\begin{array}{l} \cdots;\left(a_{rj};\alpha^{(1)}_{rj},\cdots,\alpha^{(r)}_{rj}\right)_{1,p_r} : \left(a^{(1)}_j,\alpha^{(1)}_j\right)_{1,p^{(1)}};\cdots;\left(a^{(r)}_j,\alpha^{(r)}_j\right)_{1,p^{(r)}} \\ \cdots;\left(b_{rj};\beta^{(1)}_{rj},\cdots,\beta^{(r)}_{rj}\right)_{1,q_r} : \left(b^{(1)}_j,\beta^{(1)}_j\right)_{1,q^{(1)}};\cdots;\left(b^{(r)}_j,\beta^{(r)}_j\right)_{1,q^{(r)}} \end{array}\right]$$

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$$= \frac{1}{(2\pi\omega)^r} \int_{\mathcal{L}_1} \cdots \int_{\mathcal{L}_r} \phi(s_1, \cdots, s_r) \prod_{i=1}^{r} \phi_i(s_i)\; z_i^{s_i}\, \mathrm{d}s_1 \cdots \mathrm{d}s_r, \tag{1}$$

where

$$\phi_i(s_i) = \frac{\prod_{j=1}^{m^{(i)}} \Gamma\left(b_j^{(i)} - \beta_j^{(i)} s_i\right) \prod_{j=1}^{n^{(i)}} \Gamma\left(1 - a_j^{(i)} + \alpha_j^{(i)} s_i\right)}{\prod_{j=m^{(i)}+1}^{q^{(i)}} \Gamma\left(1 - b_j^{(i)} + \beta_j^{(i)} s_i\right) \prod_{j=n^{(i)}+1}^{p^{(i)}} \Gamma\left(a_j^{(i)} - \alpha_j^{(i)} s_i\right)},$$

$$(i = 1, \cdots, r), \tag{2}$$

and

$$\phi(s_1, \cdots, s_r) = \frac{\prod_{j=1}^{n_2} \Gamma\left(1 - a_{2j} + \sum_{i=1}^{2} \alpha_{2j}^{(i)} s_i\right) \prod_{j=1}^{n_3} \Gamma\left(1 - a_{3j} + \sum_{i=1}^{3} \alpha_{3j}^{(i)} s_i\right) \cdots}{\prod_{j=n_2+1}^{p_2} \Gamma\left(a_{2j} - \sum_{i=1}^{2} \alpha_{2j}^{(i)} s_i\right) \prod_{j=n_3+1}^{p_3} \Gamma\left(a_{3j} - \sum_{i=1}^{3} \alpha_{3j}^{(i)} s_i\right) \cdots}$$

$$\frac{\cdots \prod_{j=1}^{n_r} \Gamma\left(1 - a_{rj} + \sum_{i=1}^{r} \alpha_{rj}^{(i)} s_i\right)}{\cdots \prod_{j=n_r+1}^{p_r} \Gamma\left(a_{rj} - \sum_{i=1}^{r} \alpha_{rj}^{(i)} s_i\right) \prod_{j=1}^{q_2} \Gamma\left(1 - b_{2j} - \sum_{i=1}^{2} \beta_{2j}^{(i)} s_i\right)}$$

$$\times \frac{1}{\prod_{j=1}^{q_3} \Gamma\left(1 - b_{3j} + \sum_{i=1}^{3} \beta_{3j}^{(i)} s_i\right) \cdots \prod_{j=1}^{q_r} \Gamma\left(1 - b_{rj} - \sum_{i=1}^{r} \beta_{rj}^{(i)} s_i\right)}. \tag{3}$$

For more details, see Y.N Prasad [9]. Throughout this paper, we assume that the existence and convergence conditions of the multivariable I-function are satisfied. The condition for absolute convergence of multiple Mellin-Barnes type contour (1) can be obtained by extension of the corresponding conditions for multivariable H-function given as

$|\arg z_i| < \frac{1}{2}\Omega_i \pi$, where

$$\Omega_i = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \left(\sum_{k=1}^{n_2} \alpha_{2k}^{(i)} - \sum_{k=n_2+1}^{p_2} \alpha_{2k}^{(i)}\right) +$$

$$\cdots + \left(\sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)}\right) - \left(\sum_{k=1}^{q_2} \beta_{2k}^{(i)} + \sum_{k=1}^{q_3} \beta_{3k}^{(i)} + \cdots + \sum_{k=1}^{q_s} \beta_{sk}^{(i)}\right), \tag{4}$$

and $z_i \in \mathbb{C}, z_i \neq 0$ for $i = 1, \ldots, r$. We may establish the asymptotic expansion in the following convenient form:

$$I(z_1, \ldots, z_r) = 0\left(|z_1|^{\alpha_1}, \ldots, |z_r|^{\alpha_r}\right), \;\; \max\left(|z_1|, \ldots, |z_r|\right) \to 0;$$

$$I\left(z_1, \cdots, z_r\right) = 0\left(|z_1|^{\beta_1}, \ldots, |z_r|^{\beta_r}\right), \quad \min\left(|z_1|, \ldots, |z_r|\right) \to \infty.$$

Here, $k = 1, \ldots, r : \alpha'_k = \min\left[\Re\left(b_j^{(k)}/\beta_j^{(k)}\right)\right],\ j = 1, \ldots, m^{(k)}$ and $\beta'_k = \max\left[\Re\left(\left(a_j^{(k)} - 1\right)/\alpha_j^{(k)}\right)\right],\ j = 1, \ldots, n^{(k)}$.

In this paper, we will use the following notations:

$$U = p_2, q_2; p_3, q_3; \ldots; p_{r-1}, q_{r-1}, \tag{5}$$

$$V = 0, n_2; 0, n_3; \ldots; 0, n_{r-1}, \tag{6}$$

$$X = m^{(1)}, n^{(1)}; \ldots; m^{(r)}, n^{(r)}, \tag{7}$$

$$Y = p^{(1)}, q^{(1)}; \ldots; p^{(r)}, q^{(r)}, \tag{8}$$

$$\mathbb{A} = \left(a_{2k}; \alpha_{2k}^{(1)}, \alpha_{2k}^{(2)}\right)_{1,p_2}; \ldots; \left(a_{(r-1)k}; \alpha_{(r-1)k}^{(1)}, \alpha_{(r-1)k}^{(2)}, \ldots, \alpha_{(r-1)k}^{(r-1)}\right)_{1,p_{r-1}}, \tag{9}$$

$$A = \left(a_{rk}; \alpha_{rk}^{(1)}, \alpha_{rk}^{(2)}, \cdots, \alpha_{rk}^{(r)}\right)_{1,p_r}, \tag{10}$$

$$\mathbf{A} = \left(a_k^{(1)}, \alpha_k^{(1)}\right)_{1,p^{(1)}}; \ldots; \left(a_k^{(r)}, \alpha_k^{(r)}\right)_{1,p^{(r)}}, \tag{11}$$

$$\mathbb{B} = \left(b_{2k}; \beta_{2k}^{(1)}, \beta_{2k}^{(2)}\right)_{1,q_2}; \ldots; \left(b_{(r-1)k}; \beta_{(r-1)k}^{(1)}, \beta_{(r-1)k}^{(2)}, \ldots, \beta_{(r-1)k}^{(r-1)}\right)_{1,q_{r-1}}, \tag{12}$$

$$B = \left(b_{rk}; \beta_{rk}^{(1)}, \beta_{rk}^{(2)}, \ldots, \beta_{rk}^{(r)}\right)_{1,q_r}, \tag{13}$$

$$\mathbf{B} = \left(b_k^{(1)}, \beta_k^{(1)}\right)_{1,q^{(1)}}; \ldots; \left(b_k^{(r)}, \beta_k^{(r)}\right)_{1,q^{(r)}}. \tag{14}$$

2. Partial differentiation formulas for the Prasad's I-function of r-variables

In this section, we give four formulas involving the partial differentiation for the multivariable I-function. Here, we also treat partial differentiation operators $D_x = \frac{\partial}{\partial x}$ and $D_y = \frac{\partial}{\partial y}$.

Theorem 1.

$$\prod_{i=1}^{m} \left[D_x\left(ax + by + c\right) - \lambda_i\right] \left\{(ax + by + c)^{\mu} \right.$$
$$\left. \times\ I\left[z_1\left(ax + by + c\right)^{h_1}, \ldots, z_r\left(ax + by + c\right)^{h_r}\right]\right\} = a^m\left(ax + by + c\right)^{\mu}$$

$$\times\ I^{V;0,n_r+m;X}_{U:p_r+m,q_r+m;Y}\left(\begin{array}{c|c} z_1\,(ax+by+c)^{h_1} & \mathbb{A}:\ (\rho_i-\mu-1;h_1,\cdots,h_r)_{1,m}\,,\ A:\mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\,(ax+by+c)^{h_r} & \mathbb{B}:\ (\rho_i-\mu;h_1,\cdots,h_r)_{1,m}\,,\ B:\mathbf{B} \end{array}\right). \tag{15}$$

Provided that
$\mu>0, h_j>0$ for $j=1,\ldots,r$. a,b,c are complex numbers, $a\neq 0, b\neq 0$, m is a positive integer. Also, $|\arg z_j\,(ax+by+c)^{h_i}|<\frac{1}{2}\Omega_j\pi$, where Ω_j is defined by (4); $\lambda_i=a\rho_i$ for $i=1,\ldots,m$.

Proof. L.H.S. of (15) (say $\mathcal{P}_1$)=

$$\prod_{i=1}^{m}\left[D_x\,(ax+by+c)-\lambda_i\right]\{(ax+by+c)^{\mu}$$

$$I\left[z_1\,(ax+by+c)^{h_1},\ldots,z_r\,(ax+by+c)^{h_r}\right]\right\}.$$

$$\mathcal{P}_1=\prod_{i=1}^{m}\Big\{D_x\,(ax+by+c)^{\mu+1}\ I\left[z_1\,(ax+by+c)^{h_1},\ldots,z_r\,(ax+by+c)^{h_r}\right]-$$
$$\lambda_i\,(ax+by+c)^{\mu}\ I\left[z_1\,(ax+by+c)^{h_1},\ldots,z_r\,(ax+by+c)^{h_r}\right]\Big\}$$

$$=\prod_{i=1}^{m}\Big\{a\,(\mu+1)\,(ax+by+c)^{\mu}\,I\left[z_1\,(ax+by+c)^{h_1},\cdots,z_r\,(ax+by+c)^{h_r}\right]+$$
$$+\,(ax+by+c)^{\mu+1}$$
$$\times\ D_x\,\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi\,(s_1,\ldots,s_r)\left\{\prod_{i=1}^{r}\phi_i\,(s_i)\left[z_i\,(ax+by+c)^{h_i}\right]^{s_i}\mathrm{d}s_1\ldots\mathrm{d}s_r\right\}$$
$$-\ \lambda_i\,(ax+by+c)^{\mu}\ I\left[z_1\,(ax+by+c)^{h_1},\ldots,z_r\,(ax+by+c)^{h_r}\right]\Big\}$$

$$=a^m\,(ax+by+c)^{\mu}\ \frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi\,(s_1,\ldots,s_r)\prod_{i=1}^{r}\phi_i\,(s_i)\left[z_i\,(ax+by+c)^{h_i}\right]^{s_i}$$
$$\times\ \prod_{i=1}^{m}\left(\mu+1+\sum_{j=1}^{r}h_js_j-\rho_i\right)\mathrm{d}s_1\ldots\mathrm{d}s_r.$$

Now, using the relation

$$\mu + 1 + \sum_{j=1}^{r} h_j s_j - \rho_i = \frac{\Gamma\left(\mu + 2 + \sum_{j=1}^{r} h_j s_j - \rho_i\right)}{\Gamma\left(\mu + 1 + \sum_{j=1}^{r} h_j s_j - \rho_i\right)}, \tag{16}$$

and interpreting the multiple integrals with the help of Mellin-Barnes contour integral (1), we arrive at the required result (15). ◄

Theorem 2.

$$\prod_{i=1}^{m} \left[D_y (ax + by + c) - k_i\right] \left\{(ax + by + c)^{\mu}\right.$$

$$\left.\times\ I\left[z_1 (ax + by + c)^{h_1}, \ldots, z_r (ax + by + c)^{h_r}\right]\right\} = b^m (ax + by + c)^{\mu}$$

$$\times\ I^{V;0,n_r+m;X}_{U:p_r+m,q_r+m;Y}\left(\begin{array}{c|c} z_1 (ax + by + c)^{h_1} & \mathbb{A} :\ (\rho_i - \mu - 1; h_1, \ldots, h_r)_{1,m},\ A :\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r (ax + by + c)^{h_r} & \mathbb{B} :\ B, (\rho_i - \mu; h_1, \ldots, h_r)_{1,m} :\ \mathbf{B} \end{array}\right). \tag{17}$$

Provided that
$\mu > 0, h_j > 0$ for $j = 1, \ldots, r$. a, b, c are complex numbers, $a \neq 0, b \neq 0$, m is a positive integer. $|\arg z_j (ax + by + c)^{h_i}| < \frac{1}{2}\Omega_j \pi$, where Ω_j is defined by (4); and $k_i = b\rho_i$ for $i = 1, \ldots, m$.

Proof. L.H.S. of (17) (say $\mathcal{P}_2$)=

$$\prod_{i=1}^{m} \left[D_y (ax + by + c) - k_i\right] \times$$

$$\times \left\{(ax + by + c)^{\mu}\, I\left[z_1 (ax + by + c)^{h_1}, \ldots, z_r (ax + by + c)^{h_r}\right]\right\}.$$

$$\mathcal{P}_2 = \prod_{i=1}^{m} \left\{D_y (ax + by + c)^{\mu+1}\ I\left[z_1 (ax + by + c)^{h_1}, \ldots, z_r (ax + by + c)^{h_r}\right] - \right.$$

$$\left. k_i (ax + by + c)^{\mu}\ I\left[z_1 (ax + by + c)^{h_1}, \ldots, z_r (ax + by + c)^{h_r}\right]\right\}$$

$$= \prod_{i=1}^{m} \{b (\mu + 1) (ax + by + c)^{\mu} \times$$

$$\times I\left[z_1\left(ax+by+c\right)^{h_1},\cdots,z_r\left(ax+by+c\right)^{h_r}\right]+\left(ax+by+c\right)^{\mu+1}$$
$$\times\ D_y\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi\left(s_1,\ldots,s_r\right)\left\{\prod_{i=1}^{r}\phi_i\left(s_i\right)\left[z_i\left(ax+by+c\right)^{h_i}\right]^{s_i}\mathrm{d}s_1\ldots\mathrm{d}s_r\right\}$$
$$-k_i\left(ax+by+c\right)^{\mu}\ I\left[z_1\left(ax+by+c\right)^{h_1},\ldots,z_r\left(ax+by+c\right)^{h_r}\right]\right\}$$

$$=b^m\left(ax+by+c\right)^{\mu}\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi\left(s_1,\ldots,s_r\right)\prod_{i=1}^{r}\phi_i\left(s_i\right)\left[z_i\left(ax+by+c\right)^{h_i}\right]^{s_i}$$
$$\times\prod_{i=1}^{m}\left(\mu+1+\sum_{j=1}^{r}h_js_j-\rho_i\right)\mathrm{d}s_1\ldots\mathrm{d}s_r.$$

Now, using the relation (16) and then interpreting the multiple integrals with the help of (1), we obtain required result (17). ◀

Theorem 3. *Let* $\mu>0,h_j>0\ (j=1,\cdots,r),\ \lambda_i=a\rho_i\ (i=1,\ldots,m),\ a,b,c\in\mathbb{C}$*, also* $a\neq0,b\neq0$*, and* m *be a positive integer. Let* $|\arg z_j\left(ax+by+c\right)^{h_i}|<\frac{1}{2}\Omega_j\pi$*, where* Ω_j *is given by* (4)*. Then*

$$\prod_{i=1}^{m}\left[(ax+by+c)\ D_x-\lambda_i\right]\left\{(ax+by+c)^{\mu}\right.$$
$$\times\ I\left[z_1\left(ax+by+c\right)^{h_1},\ldots,z_r\left(ax+by+c\right)^{h_r}\right]\right\}=a^m\left(ax+by+c\right)^{\mu}$$
$$\times\ I^{V;0,n_r+m;X}_{U:p_r+m,q_r+m;Y}\left(\begin{array}{c|c}z_1\left(ax+by+c\right)^{h_1} & \mathbb{A}:\ (\rho_i-\mu;h_1,\ldots,h_r)_{1,m},A:\ \mathbf{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r\left(ax+by+c\right)^{h_r} & \mathbb{B}:\ B,(\rho_i-\mu+1;h_1,\ldots,h_r)_{1,m}:\ \mathbf{B}\end{array}\right).\qquad(18)$$

Proof. L.H.S. of (18) (say $\mathcal{P}_3$)=

$$\prod_{i=1}^{m}\left[(ax+by+c)\,D_x-\lambda_i\right]\left\{(ax+by+c)^{\mu}\right.$$
$$I\left[z_1\left(ax+by+c\right)^{h_1},\ldots,z_r\left(ax+by+c\right)^{h_r}\right]\right\}$$

$$=\prod_{i=1}^{m}\left\{(ax+by+c)\ D_x\left(ax+by+c\right)^{\mu}\right.$$

$$I\left[z_1\left(ax+by+c\right)^{h_1},\ldots,z_r\left(ax+by+c\right)^{h_r}\right]$$
$$-\lambda_i\left(ax+by+c\right)^{\mu}\ I\left[z_1\left(ax+by+c\right)^{h_1},\ldots,z_r\left(ax+by+c\right)^{h_r}\right]\Big\}$$

$$=\prod_{i=1}^{m}\left\{a\mu\left(ax+by+c\right)^{\mu}\times\right.$$
$$\times\ I\left[z_1\left(ax+by+c\right)^{h_1},\ldots,z_r\left(ax+by+c\right)^{h_r}\right]+\left(ax+by+c\right)^{\mu+1}$$
$$\times\ D_x\frac{1}{\left(2\pi\omega\right)^r}\int_{L_1}\cdots\int_{L_r}\phi\left(s_1,\ldots,s_r\right)\left\{\prod_{i=1}^{r}\phi_i\left(s_i\right)\left[z_i\left(ax+by+c\right)^{h_i}\right]^{s_i}\mathrm{d}s_1\ldots\mathrm{d}s_r\right\}$$
$$\left.-\lambda_i\left(ax+by+c\right)^{\mu}\ I\left[z_1\left(ax+by+c\right)^{h_1},\ldots,z_r\left(ax+by+c\right)^{h_r}\right].\right\}$$

$$\mathcal{P}_3=a^m\left(ax+by+c\right)^{\mu}\frac{1}{\left(2\pi\omega\right)^r}\int_{L_1}\cdots\int_{L_r}\phi\left(s_1,\ldots,s_r\right)$$
$$\times\prod_{i=1}^{r}\phi_i\left(s_i\right)\left[z_i\left(ax+by+c\right)^{h_i}\right]^{s_i}\times\prod_{i=1}^{m}\left(\mu+\sum_{j=1}^{r}h_js_j-\rho_i\right)\mathrm{d}s_1\ldots\mathrm{d}s_r.$$

Now, using the relation

$$\mu+\sum_{j=1}^{r}h_js_j-\rho_i=\frac{\Gamma\left(\mu+1+\sum_{j=1}^{r}h_js_j-\rho_i\right)}{\Gamma\left(\mu+\sum_{j=1}^{r}h_js_j-\rho_i\right)},\tag{19}$$

and then interpreting the multiple integrals with the help of Mellin-Barnes contour integral as given by (1), we finally arrive at the required result (18). ◄

Theorem 4. *Let* $\mu>0,h_j>0\ (j=1,\cdots,r),\ \lambda_i=b\rho_i\ (i=1,\ldots,m),\ a,b,c\in\mathbb{C}$*, also* $a\neq0,b\neq0$*, and* m *be a positive integer. Let* $|\arg z_j\left(ax+by+c\right)^{h_i}|<\frac{1}{2}\Omega_j\pi$*. Then*

$$\prod_{i=1}^{m}\left[\left(ax+by+c\right)\ D_y-k_i\right]\left\{\left(ax+by+c\right)^{\mu}\right.$$
$$\left.\times\ I\left[z_1\left(ax+by+c\right)^{h_1},\ldots,z_r\left(ax+by+c\right)^{h_r}\right]\right\}=b^m\left(ax+by+c\right)^{\mu}$$
$$\times\ I_{U:p_r+m,q_r+m;Y}^{V;0,n_r+m;X}\left(\begin{array}{c|c}z_1\left(ax+by+c\right)^{h_1} & \mathbb{A}:\ \left(\rho_i-\mu;h_1,\ldots,h_r\right)_{1,m},A:\ \mathbf{A}\\ \cdot & \cdot\\ \cdot & \cdot\\ z_r\left(ax+by+c\right)^{h_r} & \mathbb{B}:\ B,\left(\rho_i-\mu+1;h_1,\ldots,h_r\right)_{1,m}:\ \mathbf{B}\end{array}\right).\tag{20}$$

Proof. L.H.S. of (20) (say $\mathcal{P}_4$)=

$$\prod_{i=1}^{m}\left[(ax+by+c)\,D_y-k_i\right]\left\{(ax+by+c)^{\mu}\times\right.$$
$$\left.\times I\left[z_1\,(ax+by+c)^{h_1},\ldots,z_r\,(ax+by+c)^{h_r}\right]\right\}$$

$$=\prod_{i=1}^{m}\left\{(ax+by+c)\,D_y\,(ax+by+c)^{\mu}\right.$$
$$I\left[z_1\,(ax+by+c)^{h_1},\ldots,z_r\,(ax+by+c)^{h_r}\right]$$
$$\left.-k_i\,(ax+by+c)^{\mu}\;I\left[z_1\,(ax+by+c)^{h_1},\ldots,z_r\,(ax+by+c)^{h_r}\right]\right\}$$

$$\prod_{i=1}^{m}\left\{b\mu\,(ax+by+c)^{\mu}\times\right.$$
$$\times I\left[z_1\,(ax+by+c)^{h_1},\ldots,z_r\,(ax+by+c)^{h_r}\right]+(ax+by+c)^{\mu+1}$$
$$\times\;D_y\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi\,(s_1,\cdots,s_r)\left\{\prod_{i=1}^{r}\phi_i\,(s_i)\left[z_i\,(ax+by+c)^{h_i}\right]^{s_i}\mathrm{d}s_1\ldots\mathrm{d}s_r\right\}$$
$$\left.-k_i\,(ax+by+c)^{\mu}\;\;I\left[z_1\,(ax+by+c)^{h_1},\ldots,z_r\,(ax+by+c)^{h_r}\right]\right\}$$

$$=b^m\,(ax+by+c)^{\mu}\,\frac{1}{(2\pi\omega)^r}\int_{L_1}\cdots\int_{L_r}\phi\,(s_1,\ldots,s_r)\prod_{i=1}^{r}\phi_i\,(s_i)\left[z_i\,(ax+by+c)^{h_i}\right]^{s_i}$$
$$\times\;\prod_{i=1}^{m}\left(\mu+\sum_{j=1}^{r}h_js_j-\rho_i\right)\mathrm{d}s_1\ldots\mathrm{d}s_r.$$

Now, using the relation (19) and then interpreting the multiple integrals with the help of (1), we obtain the required result (20). ◄

3. Special cases

In this section, we assume $U=V=\mathbb{A}=\mathbb{B}=0$. Then the multivariable I-function defined in (1) reduces to multivariable H-function [2, 3, 4, 8].

Note that

$$\Omega_i' = \sum_{k=1}^{n^{(i)}} \alpha_k^{(i)} - \sum_{k=n^{(i)}+1}^{p^{(i)}} \alpha_k^{(i)} + \sum_{k=1}^{m^{(i)}} \beta_k^{(i)} - \sum_{k=m^{(i)}+1}^{q^{(i)}} \beta_k^{(i)} + \sum_{k=1}^{n_s} \alpha_{sk}^{(i)} - \sum_{k=n_s+1}^{p_s} \alpha_{sk}^{(i)} - \sum_{k=1}^{q_s} \beta_{sk}^{(i)}. \tag{21}$$

We have the following corollaries:

Corollary 1. *Let* $\mu > 0, h_j > 0 \ (j = 1, \cdots, r)\,, \lambda_i = a\rho_i \ (i = 1, \cdots, m)\,,\ a, b, c \in \mathbb{C}$, *also* $a \neq 0, b \neq 0$, *and* m *be a positive integer. Let* $|\arg z_j \left(ax + by + c\right)^{h_i}| < \frac{1}{2}\Omega_j'\pi$, *where* Ω_j' *is defined as* (21)*. Then*

$$\begin{aligned}
&\prod_{i=1}^{m} \left[D_x \left(ax + by + c\right) - \lambda_i\right] \{(ax + by + c)^{\mu} \\
&\times H\left[z_1 \left(ax + by + c\right)^{h_1}, \ldots, z_r \left(ax + by + c\right)^{h_r}\right]\Big\} = a^m \left(ax + by + c\right)^{\mu} \\
&\times\ H^{0,n_r+m;X}_{p_r+m,q_r+m;Y}\left(\begin{array}{c|c} z_1 \left(ax + by + c\right)^{h_1} & \left(\rho_i - \mu - 1; h_1, \ldots, h_r\right)_{1,m}, A:\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r \left(ax + by + c\right)^{h_r} & B, \left(\rho_i - \mu; h_1, \ldots, h_r\right)_{1,m}:\ \mathbf{B} \end{array}\right).
\end{aligned} \tag{22}$$

Corollary 2. *Let* $k_i = b\rho_i$ *for* $i = 1, \ldots, m$, *and* $a, b, c \in \mathbb{C}$, $\mu > 0, h_j > 0$ *for* $j = 1, \cdots, r$; *also* $a \neq 0, b \neq 0$, m *be a positive integer,* $|\arg z_j \left(ax + by + c\right)^{h_i}| < \frac{1}{2}\Omega_j'\pi$*. Then*

$$\begin{aligned}
&\prod_{i=1}^{m} \left[D_y \left(ax + by + c\right) - k_i\right] \{(ax + by + c)^{\mu} \\
&\times H\left[z_1 \left(ax + by + c\right)^{h_1}, \ldots, z_r \left(ax + by + c\right)^{h_r}\right]\Big\} = b^m \left(ax + by + c\right)^{\mu} \\
&\times\ H^{0,n_r+m;X}_{p_r+m,q_r+m;Y}\left(\begin{array}{c|c} z_1 \left(ax + by + c\right)^{h_1} & \left(\rho_i - \mu - 1; h_1, \ldots, h_r\right)_{1,m}, A:\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r \left(ax + by + c\right)^{h_r} & B, \left(\rho_i - \mu; h_1, \ldots, h_r\right)_{1,m}:\ \mathbf{B} \end{array}\right).
\end{aligned} \tag{23}$$

Corollary 3.

$$\prod_{i=1}^{m} \left[\left(ax + by + c\right)\ D_x - \lambda_i\right] \{(ax + by + c)^{\mu}$$

$$\times H\left[z_1\left(ax+by+c\right)^{h_1},\ldots,z_r\left(ax+by+c\right)^{h_r}\right]\Big\} = a^m\left(ax+by+c\right)^{\mu}$$

$$\times\ H^{0,n_r+m;X}_{p_r+m,q_r+m;Y}\left(\begin{array}{c|c} z_1\left(ax+by+c\right)^{h_1} & \left(\rho_i-\mu;h_1,\cdots,h_r\right)_{1,m},A:\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\left(ax+by+c\right)^{h_r} & B,\left(\rho_i-\mu+1;h_1,\cdots,h_r\right)_{1,m}:\ \mathbf{B} \end{array}\right). \tag{24}$$

Provided that
$\mu > 0, h_j > 0$ *for* $j = 1,\ldots,r$; a,b,c *are complex numbers; and* $a \neq 0, b \neq 0$, m *is a positive integer. Here,* $|\arg z_j\left(ax+by+c\right)^{h_i}| < \frac{1}{2}\Omega'_j\pi$, *where* Ω'_j *is defined by* (21). *Also,* $\lambda_i = a\rho_i$ *for* $i = 1,\cdots,m$.

Corollary 4. *Let* $k_i = b\rho_i\,(i=1,\ldots,m)$, $a,b,c \in \mathbb{C}, m \in \mathbb{Z}^+$, $\mu > 0, h_j > 0\,(j=1,\ldots,r)$; $a \neq 0, b \neq 0$, *and* $|\arg z_j\left(ax+by+c\right)^{h_i}| < \frac{1}{2}\Omega'_j\pi$. *Then*

$$\prod_{i=1}^{m}\left[\left(ax+by+c\right)D_y - k_i\right]\left\{\left(ax+by+c\right)^{\mu}\right.$$

$$\times H\left[z_1\left(ax+by+c\right)^{h_1},\ldots,z_r\left(ax+by+c\right)^{h_r}\right]\Big\} = b^m\left(ax+by+c\right)^{\mu}$$

$$\times\ H^{0,n_r+m;X}_{p_r+m,q_r+m;Y}\left(\begin{array}{c|c} z_1\left(ax+by+c\right)^{h_1} & \left(\rho_i-\mu;h_1,\ldots,h_r\right)_{1,m},A:\ \mathbf{A} \\ \cdot & \cdot \\ \cdot & \cdot \\ z_r\left(ax+by+c\right)^{h_r} & B,\left(\rho_i-\mu+1;h_1,\ldots,h_r\right)_{1,m}:\ \mathbf{B} \end{array}\right). \tag{25}$$

Remark 1. *By using the similar methods, we can obtain the similar relations with the multivariable A-function defined by Gautam and Asgar [6], the multivariable Aleph-function defined by Ayant [1] (also see, Saxena et al. [12]), the Aleph-function of two variables defined by Kumar [7] (see also, Sharma [15]), the I-function of two variables defined by Sharma and Mishra [14], the H-function of two variables defined by Srivastava et al. [17] (see also, Ram and Kumar [11]), the Aleph-function of one variable defined by Südland [20, 21], the I-function of one variable defined by Saxena [13] and the A-function of one variable defined by Gautam and Asgar [5]. We can generalize these formulas considering the argument* $\sum_{i=0}^{n} a_i x_i$ *and the partial derivatives* $\frac{\partial_i}{\partial x_i}$.

4. Concluding remarks

The multivariable I-function can also be suitably specialized to a remarkably wide variety of useful special functions of one or several variables or product of several such special functions which are expressible in terms of E, G and H-functions of one, two and more variables available in the literature. Thus the integral formulas established in this paper would serve as key formulas from which, upon specializing the parameters, as many as desired results involving E, G and H-functions of one, two and more variables and other simpler special functions can be obtained.

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