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One Distribution Function on the Moran Sets

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Abstract. In the present article, topological, metric, and fractal properties of certain sets are investigated. These sets are images of sets whose elements have restrictions on using digits or combinations of digits in own s-adic representations, under the map f, that is a certain distribution function.

Key Words and Phrases: s-adic representation, Moran set, Hausdorff dimension, monotonic function, distribution function.

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1. Introduction

Let us consider the space $\mathbb{R}^n$. In [7], P. A. P. Moran introduced the following construction of sets and calculated the Hausdorff dimension of the limit set:

$$E = \bigcap_{n=1}^{\infty} \bigcup_{i_1,\dots,i_n \in A_{0,p}} \Delta_{i_1 i_2 \dots i_n}. \tag{1}$$

Here p is a fixed positive integer, $A_{0,p} = \{1, 2, \dots, p\}$, and sets $\Delta_{i_1 i_2 \dots i_n}$ are basic sets having the following properties:

- any set $\Delta_{i_1 i_2 \dots i_n}$ is closed and disjoint;
- for any $i \in A_{0,p}$ the condition $\Delta_{i_1 i_2 \dots i_n i} \subset \Delta_{i_1 i_2 \dots i_n}$ holds;
- $$\lim_{n\to\infty} d\left(\Delta_{i_1 i_2 \dots i_n}\right) = 0, \text{where } d(\cdot) \text{ is the diameter of a set;}$$
- each basic set is the closure of its interior;
- at each level the basic sets do not overlap (their interiors are disjoint);

- any basic set $\Delta_{i_1 i_2 \ldots i_n i}$ is geometrically similar to $\Delta_{i_1 i_2 \ldots i_n}$;
- $$\frac{d\left(\Delta_{i_1 i_2 \ldots i_n i}\right)}{d\left(\Delta_{i_1 i_2 \ldots i_n}\right)} = \sigma_i,$$
  where $\sigma_i \in (0,1)$ for $i = \overline{1,p}$.

The Hausdorff dimension α_0 of the set E is the unique root of the following equation:

$$\sum_{i=1}^{p} \sigma_i^{\alpha_0} = 1.$$

It is easy to see that the set (1) is a Cantor-like set and a self-similar fractal. The set E is called *the Moran set*.

Much research has been dedicated to Moran-like constructions and Cantor-like sets (see, e.g., [3, 4, 6, 8, 1, 2, 5, 23, 17] and references therein).

Fractal sets are widely applicated in computer design, algorithms of information compression, quantum mechanics, solid-state physics, analysis and categorizations of signals of various forms appearing in different areas (e.g. the analysis of exchange rate fluctuations in economics, etc.). In addition, such sets are useful for checking the Hausdorff dimension by certain functions [22, 23]. However, for many classes of fractals the problem of Hausdorff dimension calculation is difficult and the estimation of parameters on which the Hausdorff dimension of certain classes of fractal sets depends is left out of consideration.

Let $s > 1$ be a fixed positive integer. Let us consider the s-adic representation of numbers from $[0,1]$:

$$x = \Delta^{s}_{\alpha_1 \alpha_2 \ldots \alpha_n \ldots} = \sum_{n=1}^{\infty} \frac{\alpha_n}{s^n},$$

where $\alpha_n \in A = \{0, 1, \ldots, s-1\}$.

In addition, we say that the following representation

$$x = \Delta^{-s}_{\alpha_1 \alpha_2 \ldots \alpha_n \ldots} = \sum_{n=1}^{\infty} \frac{\alpha_n}{(-s)^n},$$

is a nega-s-adic representation of numbers from $\left[-\frac{s}{s+1}, \frac{1}{s+1}\right]$. Here $\alpha_n \in A$ as well.

Some articles (see [1, 2, 17, 9, 10, 11, 12, 13, 14, 16]) were dedicated to sets whose elements have certain restrictions on using combinations of digits in own s-adic representation. Let us consider the following results.

Suppose $s > 2$ is a fixed positive integer.

Let us consider a class Υ_s of sets $\mathbb{S}_{(s,u)}$ represented in the form

$$\mathbb{S}_{(s,u)} = \left\{x : x = \frac{u}{s-1} + \sum_{n=1}^{\infty} \frac{\alpha_n - u}{s^{\alpha_1+\dots+\alpha_n}}, (\alpha_n) \in L, \alpha_n \neq u, \alpha_n \neq 0\right\},$$

where $u = \overline{0, s-1}$, the parameters u and s are fixed for the set $\mathbb{S}_{(s,u)}$. In other words, the class Υ_s contains the sets $\mathbb{S}_{(s,0)}, \mathbb{S}_{(s,1)}, \dots, \mathbb{S}_{(s,s-1)}$. We say that Υ is a class of sets which contains the classes $\Upsilon_3, \Upsilon_4, \dots, \Upsilon_n, \dots$.

It is easy to see that the set $\mathbb{S}_{(s,u)}$ can be defined by the s-adic representation in the following form:

$$\mathbb{S}_{(s,u)} = \left\{x : x = \Delta^s_{\underbrace{u\dots u}_{\alpha_1-1}\alpha_1\underbrace{u\dots u}_{\alpha_2-1}\alpha_2\dots\underbrace{u\dots u}_{\alpha_n-1}\alpha_n\dots}, (\alpha_n) \in L, \alpha_n \neq u, \alpha_n \neq 0\right\},$$

Theorem 1 ([13, 16, 17])**.** *For an arbitrary $u \in A$, the set $\mathbb{S}_{(s,u)}$ is an uncountable, perfect, nowhere dense set of zero Lebesgue measure and a self-similar fractal whose Hausdorff dimension $\alpha_0(\mathbb{S}_{(s,u)})$ satisfies the following equation:*

$$\sum_{p\neq u, p\in A_0} \left(\frac{1}{s}\right)^{p\alpha_0} = 1.$$

Remark 1. *Theorem 1 is true for all sets $\mathbb{S}_{(s,0)}, \mathbb{S}_{(s,1)}, \dots, \mathbb{S}_{(s,s-1)}$ (for fixed parameters $u = \overline{0, s-1}$ and any fixed $2 < s \in \mathbb{N}$) except for the sets $S_{(3,1)}$ and $S_{(3,2)}$.*

Theorem 2 ([13, 16, 14, 17])**.** *Let E be a set, whose elements contain (in own s-adic or nega-s-adic representation) only digits or combinations of digits from a certain fixed finite set $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$ of s-adic digits or combinations of digits.*

Then the Hausdorff dimension α_0 of E satisfies the following equation:

$$N(\sigma^1_m)\left(\frac{1}{s}\right)^{\alpha_0} + N(\sigma^2_m)\left(\frac{1}{s}\right)^{2\alpha_0} + \dots + N(\sigma^k_m)\left(\frac{1}{s}\right)^{k\alpha_0} = 1,$$

where $N(\sigma^k_m)$ is a number of k-digit combinations σ^k_m from the set $\{\sigma_1, \sigma_2, \dots, \sigma_m\}$, $k \in \mathbb{N}$, and $N(\sigma^1_m) + N(\sigma^2_m) + \dots + N(\sigma^k_m) = m$.

Now we will describe the main function of our investigation. Let η be a random variable defined by the s-adic representation

$$\eta = \frac{\xi_1}{s} + \frac{\xi_2}{s^2} + \frac{\xi_3}{s^3} + \dots + \frac{\xi_k}{s^k} + \dots = \Delta^s_{\xi_1\xi_2\dots\xi_k\dots},$$

where $\xi_k = \alpha_k$ and digits ξ_k $(k = 1, 2, 3, \dots)$ are random and taking the values $0, 1, \dots, s-1$ with positive probabilities $p_0, p_1, \dots, p_{s-1}$. That is, ξ_k are independent and $P\{\xi_k = i_k\} = p_{i_k}$, $i_k \in A$.

From the definition of distribution function and the expressions

$$\{\eta < x\} = \{\xi_1 < \alpha_1(x)\} \cup \{\xi_1 = \alpha_1(x), \xi_2 < \alpha_2(x)\} \cup \dots$$

$$\cup\{\xi_1 = \alpha_1(x), \xi_2 = \alpha_2(x), \dots, \xi_{k-1} = \alpha_{k-1}(x), \xi_k < \alpha_k(x)\} \cup \dots,$$

$$P\{\xi_1 = \alpha_1(x), \xi_2 = \alpha_2(x), \dots, \xi_{k-1} = \alpha_{k-1}(x), \xi_k < \alpha_k(x)\} = \beta_{\alpha_k(x)} \prod_{j=1}^{k-1} p_{\alpha_j(x)},$$

where

$$\beta_{\alpha_k} = \begin{cases} \sum_{i=0}^{\alpha_k(x)-1} p_i(x) & \text{whenever } \alpha_k(x) > 0 \\ 0 & \text{whenever } \alpha_k(x) = 0, \end{cases}$$

it is easy to see that the following statement is true.

Statement 1. *The distribution function f_η of the random variable η can be represented in the following form:*

$$f_\eta(x) = \begin{cases} 0 & \textit{whenever } x < 0 \\ \beta_{\alpha_1(x)} + \sum_{k=2}^{\infty} \left(\beta_{\alpha_k(x)} \prod_{j=1}^{k-1} p_{\alpha_j(x)}\right) & \textit{whenever } 0 \le x < 1 \\ 1 & \textit{whenever } x \ge 1, \end{cases}$$

where $p_{\alpha_{j(x)}} > 0$.

The function

$$f(x) = \beta_{\alpha_1(x)} + \sum_{n=2}^{\infty} \left(\beta_{\alpha_n(x)} \prod_{j=1}^{n-1} p_{\alpha_j(x)}\right)$$

can be used as a representation of numbers from $[0, 1]$. That is,

$$x = \Delta^P_{\alpha_1(x)\alpha_2(x)\dots\alpha_n(x)\dots} = \beta_{\alpha_1(x)} + \sum_{n=2}^{\infty} \left(\beta_{\alpha_n(x)} \prod_{j=1}^{n-1} p_{\alpha_j(x)}\right),$$

where $P = \{p_0, p_1, \dots, p_{s-1}\}$, $p_0+p_1+\dots+p_{s-1} = 1$, and $p_i > 0$ for all $i = \overline{0, s-1}$. The last-mentioned representation is *the P-representation of numbers from* $[0, 1]$.

Let us remark that the function f is the Salem function for $s = 2$. Some researches are devoted to generalizations of the Salem function (see [20, 21, 24, 15]) including the cases when arguments of such generalizations are defined in terms of certain (see [18, 19]) alternating representations of real numbers.

In this article, we consider properties of the images of the sets considered in Theorem 1 and Theorem 2 under the map f.

We begin with definitions.

Let s be a fixed positive integer, $s > 2$. Let $c_1, c_2, \dots, c_m$ be an ordered tuple of integers such that $c_i \in \{0, 1, \dots, s-1\}$ for $i = \overline{1, m}$.

Definition 1. *A cylinder of rank m with base $c_1c_2\dots c_m$ is a set $\Delta^P_{c_1c_2\dots c_m}$ formed by all numbers of the segment $[0,1]$ with P-representations in which the first m digits coincide with $c_1, c_2, \dots, c_m$, respectively, i.e.,*

$$\Delta^P_{c_1c_2\dots c_m} = \left\{x : x = \Delta^P_{\alpha_1\alpha_2\dots\alpha_n\dots}, \alpha_j = c_j, j = \overline{1, m}\right\}.$$

Cylinders $\Delta^P_{c_1c_2\dots c_m}$ have the following properties:

1. any cylinder $\Delta^P_{c_1c_2\dots c_m}$ is a closed interval;

2. $$\inf \Delta^P_{c_1c_2\dots c_m} = \Delta^P_{c_1c_2\dots c_m000\dots}, \sup \Delta^P_{c_1c_2\dots c_m} = \Delta^P_{c_1c_2\dots c_m[s-1][s-1][s-1]\dots};$$

3. $$|\Delta^P_{c_1c_2\dots c_m}| = p_{c_1}p_{c_2}\cdots p_{c_m};$$

4. $$\Delta^P_{c_1c_2\dots c_mc} \subset \Delta^P_{c_1c_2\dots c_m};$$

5. $$\Delta^P_{c_1c_2\dots c_m} = \bigcup_{c=0}^{s-1} \Delta^P_{c_1c_2\dots c_mc};$$

6. $$\lim_{m\to\infty} |\Delta^P_{c_1c_2\dots c_m}| = 0;$$

7. $$\frac{|\Delta^P_{c_1c_2\dots c_mc_{m+1}}|}{|\Delta^P_{c_1c_2\dots c_m}|} = p_{c_{m+1}};$$

8. $$\sup \Delta^P_{c_1c_2\dots c_mc} = \inf \Delta^P_{c_1c_2\dots c_m[c+1]},$$

   where $c \neq s-1$;

9.

$$\bigcap_{m=1}^{\infty} \Delta^P_{c_1c_2...c_m} = x = \Delta^P_{c_1c_2...c_m...}.$$

Definition 2. *A number $x \in [0,1]$ is called P-rational if*

$$x = \Delta^P_{\alpha_1\alpha_2...\alpha_{n-1}\alpha_n 000...}$$

or

$$x = \Delta^P_{\alpha_1\alpha_2...\alpha_{n-1}[\alpha_n-1][s-1][s-1][s-1]...}.$$

The other numbers in $[0,1]$ are called P-irrational.

2. The objects of research

Let $2 < s$ be a fixed positive integer, $A = \{0,1,\dots,s-1\}$, $A_0 = A \setminus \{0\} = \{1,2,\dots,s-1\}$, and

$$L \equiv (A_0)^\infty = (A_0) \times (A_0) \times (A_0) \times \dots$$

be the space of one-sided sequences of elements of A_0.

Let $P = \{p_0, p_1, \dots, p_{s-1}\}$ be a fixed set of positive numbers such that $p_0 + p_1 + \dots + p_{s-1} = 1$.

Let us consider a class Γ that contains classes Γ_{P_s} of sets $\mathbb{S}_{(P_s,u)}$ represented in the form

$$\mathbb{S}_{(P_s,u)} \equiv \left\{ x : x = \Delta^P_{\underbrace{u...u}_{\alpha_1-1}\alpha_1\underbrace{u...u}_{\alpha_2-1}\alpha_2...\underbrace{u...u}_{\alpha_n-1}\alpha_n...}, (\alpha_n) \in L, \alpha_n \neq u, \alpha_n \neq 0 \right\}, \quad (2)$$

where $u = \overline{0, s-1}$, the parameters u and s are fixed for the set $\mathbb{S}_{(P_s,u)}$. That is, the class Γ_{P_s} contains the sets $\mathbb{S}_{(P_s,0)}, \mathbb{S}_{(P_s,1)}, \dots, \mathbb{S}_{(P_s,s-1)}$.

Lemma 1. *An arbitrary set $\mathbb{S}_{(P_s,u)}$ is an uncountable set.*

Proof. Let us consider the mapping $g : \mathbb{S}_{(P_s,u)} \to S_u$. That is,

$$\forall (\alpha_n) \in L : x = \Delta^P_{\underbrace{u...u}_{\alpha_1-1}\alpha_1\underbrace{u...u}_{\alpha_2-1}\alpha_2...\underbrace{u...u}_{\alpha_n-1}\alpha_n...} \xrightarrow{g} \Delta^s_{\alpha_1\alpha_2...\alpha_n...} = y = g(x).$$

It follows from the definition of an arbitrary set S_u that s-adic-rational numbers of the form

$$\Delta^s_{\alpha_1\alpha_2...\alpha_{n-1}\alpha_n 000...}$$

do not belong to S_u (since the condition $\alpha_n \notin \{0,u\}$ holds). Hence each element of S_u has the unique s-adic representation.

For any $x \in \mathbb{S}_{(P_s,u)}$ there exists $y = g(x) \in S_u$ and for any $y \in S_u$ there exists $x = g^{-1}(y) \in \mathbb{S}_{(P_s,u)}$. Since P-rational numbers do not belong to $\mathbb{S}_{(P_s,u)}$, we have $f(x_1) \neq f(x_2)$ for every $x_1 \neq x_2$.

So from the uncountability of S_u we get the uncountability of the set $\mathbb{S}_{(P_s,u)}$. ◄

To investigate topological and metric properties of $\mathbb{S}_{(P_s,u)}$, we will study properties of cylinders.

Let $c_1, c_2, \dots, c_n$ be an ordered tuple of integers such that $c_i \in \{0, 1, \dots, s-1\}$ for $i = \overline{1,n}$.

Definition 3. *A cylinder of rank n with base $c_1c_2\dots c_n$ is a set $\Delta^{(P,u)}_{c_1c_2\dots c_n}$ of the form*

$$\Delta^{(P,u)}_{c_1c_2\dots c_n} =$$

$$= \left\{ x : x = \Delta^P_{\underbrace{u\dots u}_{c_1-1}c_1\underbrace{u\dots u}_{c_2-1}c_2\dots\underbrace{u\dots u}_{c_n-1}c_n\underbrace{u\dots u}_{\alpha_{n+1}-1}\alpha_{n+1}\underbrace{u\dots u}_{\alpha_{n+2}-1}\alpha_{n+2}\dots}, \alpha_j = c_j, j = \overline{1,n} \right\}.$$

By $(a_1a_2\dots a_k)$ we denote the period $a_1a_2\dots a_k$ in the representation of a periodic number.

Lemma 2. *Cylinders $\Delta^{(P,u)}_{c_1\dots c_n}$ have the following properties:*

1.

$$\inf \Delta^{(P,u)}_{c_1\dots c_n} = \begin{cases} \Delta^P_{\underbrace{0\dots0}_{c_1-1}c_1\underbrace{0\dots0}_{c_2-1}c_2\dots\underbrace{0\dots0}_{c_n-1}c_n(\underbrace{0\dots0}_{s-2}[s-1])} & \text{if } u = 0 \\ \Delta^P_{\underbrace{1\dots1}_{c_1-1}c_1\underbrace{1\dots1}_{c_2-1}c_2\dots\underbrace{1\dots1}_{c_n-1}c_n(\underbrace{1\dots1}_{s-2}[s-1])} & \text{if } u = 1 \\ \Delta^P_{\underbrace{u\dots u}_{c_1-1}c_1\underbrace{u\dots u}_{c_2-1}c_2\dots\underbrace{u\dots u}_{c_n-1}c_n(1)} & \text{if } u \in \{2,3,\dots,s-1\}, \end{cases}$$

$$\sup \Delta^{(P,u)}_{c_1\dots c_n\dots} =$$

$$= \begin{cases} \Delta^P_{\underbrace{[s-1]...[s-1]}_{c_1-1}c_1...\underbrace{[s-1]...[s-1]}_{c_n-1}c_n(\underbrace{[s-1]...[s-1]}_{s-3}[s-2])} & \textit{if } u = s-1 \\ \Delta^P_{\underbrace{u...u}_{c_1-1}c_1\underbrace{u...u}_{c_2-1}c_2...\underbrace{u...u}_{c_n-1}c_n(\underbrace{u...u}_{u}[u+1])} & \textit{if } u \in \{1,\dots,s-2\} \\ \Delta^P_{\underbrace{0...0}_{c_1-1}c_1\underbrace{0...0}_{c_2-1}c_2...\underbrace{0...0}_{c_n-1}c_n(1)} & \textit{if } u = 0. \end{cases}$$

2. *If $d(\cdot)$ is the diameter of a set, then*

$$d(\Delta^{(P,u)}_{c_1...c_n}) = d(\mathbb{S}_{(P_s,u)})p_u^{c_1+c_2+\dots+c_n-n}\prod_{j=1}^{n}p_{c_j}.$$

3.
$$\frac{d(\Delta^{(P,u)}_{c_1...c_nc_{n+1}})}{d(\Delta^{(P,u)}_{c_1...c_n})} = p_{c_{n+1}}p_u^{c_{n+1}-1}.$$

4.
$$\Delta^{(P,u)}_{c_1c_2...c_n} = \bigcup_{i=1}^{s-1}\Delta^{(P,u)}_{c_1c_2...c_ni} \quad \forall c_n \in A_0, \quad n \in \mathbb{N},\ i \neq u.$$

5. *The following relations are true:*

   (a) *if $u \in \{0,1\}$, then*
$$\inf \Delta^{(P,u)}_{c_1...c_np} > \sup \Delta^{(P,u)}_{c_1...c_n[p+1]};$$

   (b) *if $u \in \{2,3,\dots,s-3\}$, then*
$$\begin{cases} \sup \Delta^{(P,u)}_{c_1...c_np} < \inf \Delta^{(P,u)}_{c_1...c_n[p+1]} & \textit{for all } p+1 \le u \\ \inf \Delta^{(P,u)}_{c_1...c_np} > \sup \Delta^{(P,u)}_{c_1...c_n[p+1]}, & \textit{for all } u < p; \end{cases}$$

   (c) *if $u \in \{s-2, s-1\}$, then*
$$\sup \Delta^{(P,u)}_{c_1...c_np} < \inf \Delta^{(P,u)}_{c_1...c_n[p+1]}$$
(*in this case, the condition $p \neq s-1$ holds*).

Proof. The first property follows from the equality

$$x = \Delta^P_{\underbrace{u...u}_{c_1-1}c_1\underbrace{u...u}_{c_2-1}c_2...\underbrace{u...u}_{c_n-1}c_n\underbrace{u...u}_{\alpha_{n+1}-1}\alpha_{n+1}\underbrace{u...u}_{\alpha_{n+2}-1}\alpha_{n+2}...}$$

$$= \Delta^P_{\underbrace{u...u}_{c_1-1}c_1\underbrace{u...u}_{c_2-1}c_2...\underbrace{u...u}_{c_n-1}c_n(0)} + p_u^{c_1+\dots+c_n-n}\left(\prod_{k=1}^{n} p_{c_k}\right)\Delta^P_{\underbrace{u...u}_{\alpha_{n+1}-1}\alpha_{n+1}\underbrace{u...u}_{\alpha_{n+2}-1}\alpha_{n+2}...}$$

and the definition of $\mathbb{S}_{(P_s,u)}$.

It is easy to see that *the second property* follows from the first property, *the third property* is a corollary of the first and second properties, and *Property 4* follows from the definition of the set.

Let us show that *Property 5* is true. Let us prove that the first inequality holds for $u = 1$. In fact,

$$\inf \Delta^{(P,0)}_{c_1...c_n p} - \sup \Delta^{(P,0)}_{c_1...c_n[p+1]} =$$

$$= \Delta^P_{\underbrace{0...0}_{c_1-1}c_1\underbrace{0...0}_{c_2-1}c_2...\underbrace{0...0}_{c_n-1}c_n\underbrace{0...0}_{p-1}p(\underbrace{0...0}_{s-2}[s-1])} - \Delta^P_{\underbrace{0...0}_{c_1-1}c_1\underbrace{0...0}_{c_2-1}c_2...\underbrace{0...0}_{c_n-1}c_n\underbrace{0...0}_{p}[p+1](1)}$$

$$= \beta_p p_0^{c_1+\ldots+c_n-n+p-1}\prod_{j=1}^{n} p_{c_j} + p_p p_0^{c_1+\ldots+c_n-n+p-1}\left(\prod_{j=1}^{n} p_{c_j}\right)\inf \mathbb{S}_{(P_s,0)}$$

$$-\beta_{p+1} p_0^{c_1+\ldots+c_n-n+p}\prod_{j=1}^{n} p_{c_j} - p_{p+1} p_0^{c_1+\ldots+c_n-n+p}\left(\prod_{j=1}^{n} p_{c_j}\right)\sup \mathbb{S}_{(P_s,0)}$$

$$= p_0^{c_1+\ldots+c_n-n+p}\left(\prod_{j=1}^{n} p_{c_j}\right)\left(\beta_p p_0^{-1} + p_p p_0^{-1}\inf \mathbb{S}_{(P_s,0)} - \beta_{p+1} - p_{p+1}\sup \mathbb{S}_{(P_s,0)}\right)$$

$$= p_0^{c_1+\ldots+c_n-n+p-1}\left(\prod_{j=1}^{n} p_{c_j}\right)(p_0(1-p_0-p_p-p_{p+1}\sup \mathbb{S}_{(P_s,0)})+$$

$$+(1-p_0)(p_1+\ldots+p_{p-1}) + p_p \inf \mathbb{S}_{(P_s,0)}) > 0,$$

because

$$1-p_0-p_p-p_{p+1}\sup \mathbb{S}_{(P_s,0)} = 1-p_0-p_p-p_{p+1}\frac{p_0}{1-p_1} =$$

$$= \frac{\sum_{i\notin\{0,1,p,p+1\}} p_i + p_{p+1}(1-p_0) + p_0 p_1 + p_1 p_p}{1-p_1} > 0.$$

Also,

$$\inf \Delta^{(P,1)}_{c_1\ldots c_n p} - \sup \Delta^{(P,1)}_{c_1\ldots c_n[p+1]} =$$

$$= \Delta^P_{\underbrace{1\ldots1}_{c_1-1}c_1\underbrace{0\ldots0}_{c_2-1}c_2\ldots\underbrace{1\ldots1}_{c_n-1}c_n\underbrace{1\ldots1}_{p-1}p(\underbrace{1\ldots1}_{s-2}[s-1])} - \Delta^P_{\underbrace{1\ldots1}_{c_1-1}c_1\underbrace{1\ldots1}_{c_2-1}c_2\ldots\underbrace{1\ldots1}_{c_n-1}c_n\underbrace{1\ldots1}_{p}[p+1](12)}$$

$$= \beta_p p_1^{c_1+\ldots+c_n+p-n-1}\prod_{j=1}^{n} p_{c_j} + p_p p_1^{c_1+\ldots+c_n-n+p-1}\left(\prod_{j=1}^{n} p_{c_j}\right)\inf \mathbb{S}_{(P_s,1)}$$

$$-\beta_{p+1} p_1^{c_1+\ldots+c_n+p-n}\prod_{j=1}^{n} p_{c_j} - p_{p+1} p_1^{c_1+\ldots+c_n-n+p}\left(\prod_{j=1}^{n} p_{c_j}\right)\sup \mathbb{S}_{(P_s,1)}$$

$$= p_1^{c_1+\ldots+c_n-n+p-1}\left(\prod_{j=1}^{n} p_{c_j}\right)\left(\beta_p + p_p \inf \mathbb{S}_{(P_s,1)} - \beta_{p+1}p_1 - p_{p+1}p_1 \sup \mathbb{S}_{(P_s,1)}\right)$$

$$= p_1^{c_1+\ldots+c_n-n+p-1}\left(\prod_{j=1}^{n} p_{c_j}\right)(p_p \inf \mathbb{S}_{(P_s,1)} + p_1(1-p_1-p_p-p_{p+1}\sup \mathbb{S}_{(P_s,1)})+$$

$$+(1-p_1)(p_0+p_2+\ldots+p_{p-1})) > 0,$$

since

$$\sup \mathbb{S}_{(P_s,1)} = \Delta^P_{(12)} = \beta_1 + \sum_{k=1}^{\infty}\beta_1 p_1^k p_2^k + \sum_{k=1}^{\infty}\beta_2 p_1^k p_2^k = \frac{p_0 + p_0 p_1 + p_1^2}{1-p_1p_2} > 0$$

and

$$1 = p_0 + p_1 + \cdots + p_{s-1}.$$

Let us prove the system of inequalities. Consider the first inequality. For the case where $p+1 \le u$ we get

$$\inf \Delta^{(P,u)}_{c_1\ldots c_n[p+1]} - \sup \Delta^{(P,u)}_{c_1\ldots c_n p} = \Delta^P_{\underbrace{u\ldots u}_{c_1-1}c_1\underbrace{u\ldots u}_{c_2-1}c_2\ldots\underbrace{u\ldots u}_{c_n-1}c_n\underbrace{u\ldots u}_{p}[p+1](1)} -$$

$$-\Delta^P_{\underbrace{u\ldots u}_{c_1-1}c_1\underbrace{u\ldots u}_{c_2-1}c_2\ldots\underbrace{u\ldots u}_{c_n-1}c_n\underbrace{u\ldots u}_{p-1}p(\underbrace{u\ldots u}_{u}[u+1])}$$

$$= \beta_u p_u^{c_1+\ldots+c_n-n+p-1}\prod_{j=1}^{n} p_{c_j} + \beta_{p+1} p_u^{c_1+\ldots+c_n-n+p}\prod_{j=1}^{n} p_{c_j} +$$

$$+p_{p+1}p_u^{c_1+\ldots+c_n-n+p}\left(\prod_{j=1}^{n}p_{c_j}\right)\cdot\Delta^P_{(1)}$$

$$-\beta_p p_u^{c_1+\ldots+c_n-n+p-1}\prod_{j=1}^{n}p_{c_j}-p_p p_u^{c_1+\ldots+c_n-n+p-1}\left(\prod_{j=1}^{n}p_{c_j}\right)\cdot\Delta^P_{(\underbrace{u\ldots u}_{u}[u+1])}$$

$$=p_u^{c_1+\ldots+c_n-n+p-1}\left(\prod_{j=1}^{n}p_{c_j}\right)\left(\beta_u+\beta_{p+1}p_u+p_{p+1}p_u\Delta^P_{(1)}-\beta_p-p_p\Delta^P_{(\underbrace{u\ldots u}_{u}[u+1])}\right)$$

$$=p_u^{c_1+\ldots+c_n-n+p-1}\left(\prod_{j=1}^{n}p_{c_j}\right)\times$$

$$\times\left(p_{p+1}p_u\Delta^P_{(1)}+(\beta_u-\beta_p)+p_u p_p+\beta_p p_u-p_p\Delta^P_{(\underbrace{u\ldots u}_{u}[u+1])}\right)>0$$

since the conditions $p<u$, $\beta_u-\beta_p>0$, and $\beta_{p+1}=\beta_p+p_p$ hold.

Let us prove that the second inequality is true. Here $p>u$, i.e., $p-u\geq 1$. Similarly,

$$\inf\Delta^{(P,u)}_{c_1\ldots c_n p}-\sup\Delta^{(P,u)}_{c_1\ldots c_n[p+1]}=\Delta^P_{\underbrace{u\ldots u}_{c_1-1}c_1\underbrace{u\ldots u}_{c_2-1}c_2\ldots\underbrace{u\ldots u}_{c_n-1}c_n\underbrace{u\ldots u}_{p-1}p(1)}-$$

$$-\Delta^P_{\underbrace{u\ldots u}_{c_1-1}c_1\underbrace{u\ldots u}_{c_2-1}c_2\ldots\underbrace{u\ldots u}_{c_n-1}c_n\underbrace{u\ldots u}_{p}[p+1](\underbrace{u\ldots u}_{u}[u+1])}$$

$$=\beta_p p_u^{c_1+\ldots+c_n-n+p-1}\prod_{j=1}^{n}p_{c_j}+p_p p_u^{c_1+\ldots+c_n-n+p-1}\left(\prod_{j=1}^{n}p_{c_j}\right)\cdot\Delta^P_{(1)}$$

$$-\beta_u p_u^{c_1+\ldots+c_n-n+p-1}\prod_{j=1}^{n}p_{c_j}-\beta_{p+1}p_u^{c_1+\ldots+c_n-n+p}\prod_{j=1}^{n}p_{c_j}-p_{p+1}p_u^{c_1+\ldots+c_n-n+p}\times$$

$$\times\left(\prod_{j=1}^{n}p_{c_j}\right)\cdot\Delta^P_{(\underbrace{u\ldots u}_{u}[u+1])}$$

$$=p_u^{c_1+\ldots+c_n-n+p-1}\left(\prod_{j=1}^{n}p_{c_j}\right)\left(\beta_p+p_p\Delta^P_{(1)}-\beta_u-\beta_{p+1}p_u-p_{p+1}p_u\Delta^P_{(\underbrace{u\ldots u}_{u}[u+1])}\right)$$

$$= p_u^{c_1+...+c_n-n+p-1}\left(\prod_{j=1}^{n} p_{c_j}\right)\Big(p_p\Delta^P_{(1)}+$$

$$+(p_{u+1}+...+p_{p+1})+p_u(p_{p+1}+...+p_{s-1}-$$

$$-p_{p+1}\Delta^P_{(\underbrace{u...u}_{u}[u+1])})\Bigg)>0$$

since the conditions $p>u$, $\beta_p-\beta_u=p_u+p_{u+1}+...+p_{p-1}$, and $1-\beta_{p+1}=p_{p+1}+...+p_{s-1}$ hold.

Suppose that $u=s-2$. Then

$$\inf\Delta^{(P,s-2)}_{c_1c_2...c_n[p+1]}-\sup\Delta^{(P,s-2)}_{c_1c_2...c_np}$$

$$=\Delta^P_{\underbrace{[s-2]...[s-2]}_{c_1-1}c_1\underbrace{[s-2]...[s-2]}_{c_2-1}c_2...\underbrace{[s-2]...[s-2]}_{c_n-1}c_n\underbrace{[s-2]...[s-2]}_{p}[p+1](1)}$$

$$-\Delta^P_{\underbrace{[s-2]...[s-2]}_{c_1-1}c_1...\underbrace{[s-2]...[s-2]}_{c_n-1}c_n\underbrace{[s-2]...[s-2]}_{p-1}p(\underbrace{[s-2]...[s-2]}_{s-2}[s-1])}$$

$$=\beta_{s-2}p_{s-2}^{c_1+...+c_n-n+p-1}\prod_{j=1}^{n}p_{c_j}+\beta_{p+1}p_{s-2}^{c_1+...+c_n-n+p}\prod_{j=1}^{n}p_{c_j}$$

$$+p_{p+1}p_{s-2}^{c_1+...+c_n-n+p}\left(\prod_{j=1}^{n}p_{c_j}\right)\cdot\Delta^P_{(1)}-\beta_p p_{s-2}^{c_1+...+c_n-n+p-1}\prod_{j=1}^{n}p_{c_j}$$

$$-p_p p_{s-2}^{c_1+...+c_n-n+p-1}\left(\prod_{j=1}^{n}p_{c_j}\right)\cdot\Delta^P_{(\underbrace{[s-2]...[s-2]}_{s-2}[s-1])}$$

$$=p_{s-2}^{c_1+...+c_n-n+p-1}\left(\prod_{j=1}^{n}p_{c_j}\right)\times$$

$$\times(\beta_{s-2}+\beta_{p+1}p_{s-2}+p_{s-2}p_{p+1}\Delta^P_{(1)}-\beta_p-p_p\Delta^P_{(\underbrace{[s-2]...[s-2]}_{s-2}[s-1])})$$

$$=p_{s-2}^{c_1+...+c_n-n+p-1}\left(\prod_{j=1}^{n}p_{c_j}\right)(p_p(1-\Delta^P_{(\underbrace{[s-2]...[s-2]}_{s-2}[s-1])})+(p_{p+1}+...+p_{s-3})+$$

$$+\beta_{p+1}p_{s-2}+p_{s-2}p_{p+1}\Delta^{P}_{(1)})>0$$

since $\beta_{s-2}-\beta_p=p_p+p_{p+1}+\dots+p_{s-3}$. Here $p\ne s-1$.

Suppose that $u=s-1$. Then

$$\inf\Delta^{(P,s-1)}_{c_1c_2\dots c_n[p+1]}-\sup\Delta^{(P,s-1)}_{c_1c_2\dots c_np}$$

$$=\Delta^{P}_{\underbrace{[s-1]\dots[s-1]}_{c_1-1}c_1\dots\underbrace{[s-1]\dots[s-1]}_{c_n-1}c_n\underbrace{[s-1]\dots[s-1]}_{p}[p+1](1)}$$

$$-\Delta^{P}_{\underbrace{[s-1]\dots[s-1]}_{c_1-1}c_1\dots\underbrace{[s-1]\dots[s-1]}_{c_n-1}c_n\underbrace{[s-1]\dots[s-1]}_{p-1}p(\underbrace{[s-1]\dots[s-1]}_{s-3}[s-2])}$$

$$=\beta_{s-1}p_{s-1}^{c_1+\dots+c_n-n+p-1}\prod_{j=1}^{n}p_{c_j}+\beta_{p+1}p_{s-1}^{c_1+\dots+c_n-n+p}\prod_{j=1}^{n}p_{c_j}$$

$$+p_{p+1}p_{s-1}^{c_1+\dots+c_n-n+p}\left(\prod_{j=1}^{n}p_{c_j}\right)\cdot\Delta^{P}_{(1)}-\beta_pp_{s-1}^{c_1+\dots+c_n-n+p-1}\prod_{j=1}^{n}p_{c_j}$$

$$-p_pp_{s-1}^{c_1+\dots+c_n-n+p-1}\left(\prod_{j=1}^{n}p_{c_j}\right)\cdot\Delta^{P}_{(\underbrace{[s-1]\dots[s-1]}_{s-3}[s-2])}$$

$$=p_{s-1}^{c_1+\dots+c_n-n+p-1}\times$$

$$\times\left(\prod_{j=1}^{n}p_{c_j}\right)(\beta_{s-1}+\beta_{p+1}p_{s-1}+p_{s-1}p_{p+1}\Delta^{P}_{(1)}-\beta_p-p_p\Delta^{P}_{(\underbrace{[s-1]\dots[s-1]}_{s-3}[s-2])})>0.$$

◄

Theorem 3. *The set $\mathbb{S}_{(P_s,u)}$ is a perfect and nowhere dense set of zero Lebesgue measure.*

Proof. Let us prove that *the set $\mathbb{S}_{(P_s,u)}$ is a nowhere dense set.* By definition, there exist cylinders $\Delta^{(P,u)}_{c_1\dots c_n}$ of rank n in an arbitrary subinterval of the segment $I=[\inf\mathbb{S}_{(P_s,u)},\sup\mathbb{S}_{(P_s,u)}]$. Since Property 5 from Lemma 2 is true for these cylinders, for any subinterval of I there exists a subinterval which does not contain points from $\mathbb{S}_{(P_s,u)}$. So $\mathbb{S}_{(P_s,u)}$ is a nowhere dense set.

Let us show that $\mathbb{S}_{(P_s,u)}$ *is a set of zero Lebesgue measure.* Suppose that $I^{(P_s,u)}_{c_1c_2\ldots c_n}$ is a closed interval whose endpoints coincide with the endpoits of the cylinder $\Delta^{(P,u)}_{c_1c_2\ldots c_n}$,

$$|I^{(P_s,u)}_{c_1c_2\ldots c_n}| = d(\Delta^{(P,u)}_{c_1c_2\ldots c_n}) = d(\mathbb{S}_{(P_s,u)})p_u^{c_1+c_2+\cdots+c_n-n}\prod_{j=1}^{n} p_{c_j},$$

and

$$\mathbb{S}_{(P_s,u)} = \bigcap_{k=1}^{\infty} E_k^{(P_s,u)},$$

where

$$E_1^{(P_s,u)} = \bigcup_{c_1\in A_0\setminus\{u\}} I^{(P_s,u)}_{c_1},$$

$$E_2^{(P_s,u)} = \bigcup_{c_1,c_2\in A_0\setminus\{u\}} I^{(P_s,u)}_{c_1c_2},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

$$E_k^{(P_s,u)} = \bigcup_{c_1,c_2,\ldots,c_k\in A_0\setminus\{u\}} I^{(P_s,u)}_{c_1c_2\ldots c_k},$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

In addition, since $E^{(P_s,u)}_{k+1}\subset E^{(P_s,u)}_{k}$, we have

$$E_k^{(P_s,u)} = E^{(P_s,u)}_{k+1}\cup \bar{E}^{(P_s,u)}_{k+1}.$$

Let I be an initial closed interval such that $\lambda(I)=d_0$ and $[\inf\mathbb{S}_{(P_s,u)},\sup\mathbb{S}_{(P_s,u)}] = I$, $\lambda(\cdot)$ be the Lebesgue measure of a set. Then

$$\lambda(E_1^{(P_s,u)}) = \sum_{c_1\in A_0\setminus\{u\}} |I^{(P_s,u)}_{c_1}| = d(\mathbb{S}_{(P_s,u)})\sum_{c_1\in A_0\setminus\{u\}} p_u^{c_1-1} = \gamma_0.$$

We get

$$\lambda(\bar{E}_1^{(P_s,u)}) = d_0 - \lambda(E_1^{(P_s,u)}) = d_0-\gamma_0 d_0 = d_0(1-\gamma_0).$$

Similarly,

$$\lambda(\bar{E}_2^{(P_s,u)}) = \lambda(E_1^{(P_s,u)}) - \lambda(E_2^{(P_s,u)}) = \gamma_0 d_0 - \gamma_0^2 d_0 = d_0\gamma_0(1-\gamma_0),$$

$$\lambda(\bar{E}_3^{(P_s,u)}) = \lambda(E_2^{(P_s,u)}) - \lambda(E_3^{(P_s,u)}) = \gamma_0^2 d_0 - \gamma_0^3 d_0 = (1-\gamma_0)\gamma_0^2 d_0,$$

$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$

So,

$$\lambda(\mathbb{S}_{(P_s,u)}) = d_0 - \sum_{k=1}^{n} \lambda(\bar{E}_k^{(P_s,u)}) = d_0 - \sum_{k=1}^{n} \gamma_0^{k-1} d_0(1-\gamma_0) = d_0 - \frac{d_0(1-\gamma_0)}{1-\gamma_0} = 0.$$

The set $\mathbb{S}_{(P_s,u)}$ is a set of zero Lebesgue measure.

Let us prove that $\mathbb{S}_{(P_s,u)}$ *is a perfect set.* Since

$$E_k^{(P_s,u)} = \bigcup_{c_1,c_2,\dots,c_k \in A_0 \setminus \{u\}} I^{(P_s,u)}_{c_1c_2\dots c_k}$$

is a closed set ($E_k^{(P_s,u)}$ is a union of segments), we see that

$$\mathbb{S}_{(P_s,u)} = \bigcap_{k=1}^{\infty} E_k^{(P_s,u)}$$

is a closed set.

Let $x \in \mathbb{S}_{(P_s,u)}$, P be any interval that contains x, and J_n be a segment of $E_n^{(P_s,u)}$ that contains x. Choose a number n such that $J_n \subset P$. Suppose that x_n is the endpoint of J_n such that the condition $x_n \neq x$ holds. Hence $x_n \in \mathbb{S}_{(P_s,u)}$ and x is a limit point of the set.

Since $\mathbb{S}_{(P_s,u)}$ is a closed set and does not contain isolated points, we conclude that $\mathbb{S}_{(P_s,u)}$ is a perfect set. ◄

Theorem 4. *The set $\mathbb{S}_{(P_s,u)}$ is a self-similar fractal and the Hausdorff dimension $\alpha_0(\mathbb{S}_{(P_s,u)})$ of this set satisfies the following equality:*

$$\sum_{i \in A_0 \setminus \{u\}} \left(p_i p_u^{i-1}\right)^{\alpha_0} = 1.$$

Proof. From $\mathbb{S}_{(P_s,u)} \subset I$ and $\mathbb{S}_{(P_s,u)}$ being a perfect set, it follows that $\mathbb{S}_{(P_s,u)}$ is a compact set. In addition,

$$\mathbb{S}_{(P_s,u)} = \bigcup_{i \in A_0 \setminus \{u\}} \left[I_i^{(P_s,u)} \cap \mathbb{S}_{(P_s,u)}\right]$$

and $\left[I_i^{(P_s,u)} \cap \mathbb{S}_{(P_s,u)}\right] \overset{p_i p_u^{i-1}}{\sim} \mathbb{S}_{(P_s,u)}$ for all $i \in A_0 \setminus \{u\}$.

The set $\mathbb{S}_{(P_s,u)}$ is a compact self-similar set of space $\mathbb{R}^1$. Then the self-similar dimension of this set is equal to the Hausdorff dimension of $\mathbb{S}_{(P_s,u)}$. So the set $\mathbb{S}_{(P_s,u)}$ is a self-similar fractal, and its Hausdorff dimension α_0 satisfies the equality

$$\sum_{i \in A_0 \setminus \{u\}} \left(p_i p_u^{i-1}\right)^{\alpha_0} = 1.$$

◄

Theorem 5. *Let E be a set whose elements are represented in terms of the P-representation by a finite number of fixed combinations $\tau_1, \tau_2, \dots, \tau_m$ of digits from the alphabet A. Then the Hausdorff dimension α_0 of E satisfies the following equality:*

$$\sum_{j=1}^{m}\left(\prod_{i=0}^{s-1} p_i^{N_i(\tau_j)}\right)^{\alpha_0} = 1,$$

where $N_i(\tau_k)$ $(k = \overline{1,m})$ is a number of the digit i in τ_k from the set $\{\tau_1, \tau_2, \dots, \tau_m\}$.

Proof. Let $\{\tau_1, \tau_2, \dots, \tau_m\}$ be a set of fixed combinations of digits from A and the P-representation of any number from E contains only such combinations of digits.

It is easy to see that there exist combinations τ', τ'' from the set $\Xi = \{\tau_1, \tau_2, \dots, \tau_m\}$ such that $\Delta^P_{\tau'\tau'\dots} = \inf E$, $\Delta^P_{\tau''\tau''\dots} = \sup E$, and

$$d(E) = \sup E - \inf E = \Delta^P_{\tau''\tau''\dots} - \Delta^s_{\tau'\tau'\dots}.$$

A cylinder $\Delta^{(P,E)}_{\tau_1'\tau_2'\dots\tau_n'}$ of rank n with base $\tau_1'\tau_2'\dots\tau_n'$ is a set formed by all numbers of E with the P-representations in which the first n combinations of digits are fixed and coincide with $\tau_1', \tau_2', \dots, \tau_n'$, respectively ($\tau_j' \in \Xi$ for all $j = \overline{1,n}$).

It is easy to see that

$$d(\Delta^{(P,E)}_{\tau_1'\tau_2'\dots\tau_n'}) = d(E)\cdot p_0^{N_0(\tau_1'\tau_2'\dots\tau_n')} p_1^{N_1(\tau_1'\tau_2'\dots\tau_n')} \cdots p_{s-1}^{N_{s-1}(\tau_1'\tau_2'\dots\tau_n')},$$

where $N_i(\tau_1'\tau_2'\dots\tau_n')$ is a number of the digit $i \in A$ in $\tau_1'\tau_2'\dots\tau_n'$.

Since E is a closed set, $E \subset [\inf E, \sup E]$, and

$$\frac{d\left(\Delta^{(P,E)}_{\tau_1'\tau_2'\dots\tau_n'\tau_{n+1}'}\right)}{d\left(\Delta^{(P,E)}_{\tau_1'\tau_2'\dots\tau_n'}\right)} = \prod_{i=0}^{s-1} p_i^{N_i(\tau_{n+1}')},$$

$$E = [I_{\tau_1} \cap E] \cup [I_{\tau_2} \cap E] \cup \dots \cup [I_{\tau_m} \cap E],$$

where $I_{\tau_j} = [\inf \Delta^{(P,E)}_{\tau_j}, \sup \Delta^{(P,E)}_{\tau_j}]$ and $j = 1, 2, \dots, m$, we have

$$[I_{\tau_j} \cap E] \overset{\omega_j}{\sim} E \text{ for all } j = \overline{1,m},$$

where

$$\omega_j = \prod_{i=0}^{s-1} p_i^{N_i(\tau_j)}.$$

This completes the proof. ◄

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