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Hardy Banach Spaces, Cauchy Formula and Riesz Theorem

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Abstract. In this paper, Banach function space and the Hardy classes of analytic functions generated by this space are considered. An analogue of the classical Riesz theorem in these classes and the validity of the Cauchy formula for analytic functions from these classes are established. The basicity of the parts of exponential system in the corresponding Hardy classes is proved.

Key Words and Phrases: Hardy Banach spaces, Riesz theorem, Cauchy formula, basicity.

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1. Introduction

Lately, in connection with the new non-standard function spaces, there had been great interest in conjugation problems of the theory of analytic functions in different settings. Every Banach functional space generates a corresponding Banach Hardy (or Smirnov) class of analytic functions. Non-standard spaces include the Lebesgue spaces with a variable summability exponent, Morrey spaces, grand-Lebesgue spaces, etc. Numerous articles, review papers, and monographs have been dedicated to the study of various problems of analysis in these spaces (see [1-9] and references therein). Each of the above-mentioned spaces presents specific difficulties to treat your problem depending on the geometry of the space. The solutions of considered problems depend on the parameters of the space (including the norm it is supplied with) and the problem data, and you have to find the relationships between them to solve your problem. Despite these circumstances, it should be noted that these spaces are basically (unlike, for example,

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Lebesgue spaces with variable summability index) so called rearrangement invariant Banach function spaces (r.i.s. for short). For the theory of these spaces we refer the readers to the monographs [10-12]. Along with this, the Hardy and Smirnov classes associated with these spaces have also begun to be considered and the Riemann-Hilbert problems in these spaces have been studied (see, for example, [13-22]).

In this paper, Banach function spaces and the Hardy classes of analytic functions generated by this space are considered. An analogue of the classical Riesz theorem in these classes and the validity of the Cauchy formula for analytic functions from these classes are established. The basicity of the parts of exponential system in the corresponding Hardy classes is proved.

2. Needful information and auxiliary facts

We will use the following standard notations and concepts. $R_+ = (0, +\infty)$; $\chi_M(\cdot)$ is the characteristic function of the set M; R is the set of real numbers; C is the complex plane; $\omega = \{z \in C : |z| < 1\}$ is a unit disk in C; $\gamma = \partial\omega$ is a unit circle; $\bar{M}$ is the closure of the set M with respect to the corresponding norm; $(\bar{\cdot})$ is the complex conjugate. By $[X]$ we denote the algebra of linear bounded operators acting in a Banach space X.

We will need some concepts and facts from the theory of Banach function spaces (see, e.g., [10, 11]).

Let $(R_0; \mu)$ be a measure space. Let $\mathscr{M}^+$ be the cone of μ-measurable functions on R_0 whose values lie in $[0, +\infty]$.

Definition 1. *A mapping $\rho : \mathscr{M}^+ \to [0, +\infty]$ is called a Banach function norm (or simply a function norm) if, for all $f, g, f_n, n \in N$, in $\mathscr{M}^+$, for all constants $a \geq 0$ and for all μ-measurable subsets $E \subset R_0$, the following properties hold:*

(P1) $\rho(f) = 0 \Leftrightarrow f = 0$ *μ-a.e.;* $\rho(af) = a\rho(f)$; $\rho(f + g) \leq \rho(f) + \rho(g)$;

(P2) $0 \leq g \leq f$ *μ-a.e.* $\Rightarrow \rho(g) \leq \rho(f)$;

(P3) $0 \leq f_n \uparrow f$ *μ-a.e.* $\Rightarrow \rho(f_n) \uparrow \rho(f)$;

(P4) $\mu(E) < +\infty \Rightarrow \rho(\chi_E) < +\infty$;

(P5) $\mu(E) < +\infty \Rightarrow \int_E f d\mu \leq C_E \rho(f)$, *for some constant* C_E :

$$0 < C_E < +\infty.$$

Let $\mathscr{M}$ denote the collection of all extended scalar-valued (real or complex) μ-measurable functions and $\mathscr{M}_0 \subset \mathscr{M}$ denote the subclass of functions that are finite μ-a.e. .

Definition 2. *Let ρ be a function norm. The collection $X = X(\rho)$ of all functions f in $\mathscr{M}$ for which $\rho(|f|) < +\infty$ is called a Banach function space. For each $f \in X$, define $\|f\|_X = \rho(|f|)$.*

The following theorem is valid.

Theorem 1. *Let ρ be a function norm, $X = X(\rho)$ and $\|\cdot\|_X$ be as above. Then under the natural vector space operations, $(X; \|\cdot\|_X)$ is a normed linear space for which the inclusions*

$$\mathscr{M}_s \subset X \subset \mathscr{M}_0$$

hold, where $\mathscr{M}_s$ is the set of μ-simple functions. In particular, if $f_n \to f$ in X, then $f_n \to f$ in measure on sets of finite measure, and hence some subsequence converges pointwise μ-a.e. to f.

To obtain our main results, we will significantly use the following result of [23] (see also [11]). Let α_X and β_X be upper and lower Boyd indices for the space X.

Theorem 2. *For every p and q such that*

$$1 \leq q < \frac{1}{\beta_X} \leq \frac{1}{\alpha_X} < p \leq \infty,$$

we have

$$L_p \subset X \subset L_q,$$

with the inclusion maps being continuous.

We will use some results related to Fourier series in an r.i.s.. Let us state some relevant concepts and notations.

Definition 3. *Let X be an Banach function space. The closure in X of the set of simple functions $\mathscr{M}_s$ is denoted by X_b.*

Definition 4. *Let X be a r.i.s. over a resonant space $(R; \mu)$. For each finite value of t belonging to the range of μ, let E be a subset of R with $\mu(E) = t$ and let*

$$\varphi_X(t) = \|\chi_E\|_X \,.$$

If f belongs to $L_1(\gamma)$, then for each integer n the n-th Fourier coefficient of f is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f\left(e^{i\theta}\right) e^{-in\theta} d\theta\,,\ n \in Z.$$

Let S_n's be partial sums of the Fourier series of the function f :

$$S_n(f) = \sum_{|k|\leq n} \hat{f}(k) e^{ikt}.$$

In the sequel, we will also need the following

Theorem 3. *Suppose X is an r.i.s. on γ whose fundamental function satisfies $\varphi_X(+0) = 0$. Then the following conditions are equivalent:*

1. *Fourier series converge in norm in X_b;*
2. *the partial-sum operators S_n are uniformly bounded on X_b .*

The following theorem is valid.

Theorem 4. *Let X be a separable r.i.s. on $(-\pi, \pi]$. Fourier series converge in norm in X if and only if the Boyd indices of X satisfy $0 < \alpha_X; \beta_X < 1$.*

We will need also the following lemma from [10].

Lemma 1. *[10] Let $X = X(\rho)$ be a Banach function space and suppose $f_n \in X$, $n \in N$.*

i) If $0 \leq f_n \uparrow f$ μ-a.e., then either $f \notin X$ and $\|f_n\|_X \uparrow +\infty$, or $f \in X$ and $\|f_n\|_X \uparrow \|f\|_X$.
ii) (Fatou's lemma) If $f_n \to f$ μ -a.e., and if $\lim_{n\to\infty} \inf \|f_n\|_X < +\infty$, then $f \in X$ and $\|f\|_X \leq \lim_{n\to\infty} \inf \|f_n\|_X$.

In the sequel, we will need the following easily provable lemma.

Lemma 2. *Let the Banach space $\left(Y_1; \|\cdot\|_{Y_1}\right)$ be continuously embedded in the Banach space $\left(Y_2; \|\cdot\|_{Y_2}\right)$. Let $T \in [Y_2; Y_1]$ and $\overline{ImT} = Y_1$ (closure of the image of T). If the set $M \subset Y_2$ is everywhere dense in Y_2, i.e. $\bar{M} = Y_2$, then $\overline{TM} = Y_1$.*

In fact, let $y_1 \in Y_1$ be an arbitrary element and $\varepsilon > 0$ be an arbitrary number. It is clear that $\exists z_1 \in I_m T$:

$$\|z_1 - y_1\|_{Y_1} < \varepsilon.$$

Consequently, $\exists x_2 \in Y_2 : Tx_2 = z_1$. From $\overline{M} = Y_2$ it follows that $\exists m_2 \in M$: $\|m_2 - x_2\|_{Y_2} < \varepsilon$. We have

$$\|Tm_2 - y_1\|_{Y_1} \leq \|Tm_2 - z_1\|_{Y_1} + \varepsilon = \|Tm_2 - Tx_2\|_{Y_1} + \varepsilon \leq$$

$$\leq \|T\| \|m_2 - x_2\|_{Y_2} + \varepsilon = (1 + \|T\|)\varepsilon.$$

From the arbitrariness of ε it directly follows that $\overline{TM} = Y_1$.

In what follows, we will also use the concept of Nevanlinna class of analytic functions. By $\mathscr{N}$ we denote the set of analytic functions $F(\cdot)$ in ω such that

$$\sup_{0<r<1} \int_{-\pi}^{\pi} \log^+ \left|F\left(re^{it}\right)\right| dt < +\infty,$$

where $\log^+ u = \log \max\{1; u\}$, $u \geq 0$. It is known (see, e.g., [7, 30]) that the non-zero function $F(\cdot)$ belongs to the class $\mathscr{N}$ if and only if it can be represented as

$$F(z) = B(z) \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{e^{it} + z}{e^{it} - z} dh(t)\right), \tag{1}$$

where $B(\cdot)$ is a Blaschke function, and $h(\cdot)$ is a function of bounded variation on $[-\pi, \pi]$. By $\mathscr{N}'$ (Nevanlinna class) we denote a class of functions $F \in \mathscr{N}$ such that the function $h(\cdot)$ in (1) is absolutely continuous on $[-\pi, \pi]$.

More details concerning the above results can be found in [10-12; 23-29].

3. Hardy classes $H_X^{\pm}$ and bases in them

Let X be a Banach function space over $[-\pi, \pi]$. By H_X^+ we denote a Hardy class of functions $F(\cdot)$ analytic inside ω equipped with the norm

$$\|F\|_{H_X^{\pm}} = \overline{\lim_{r \to 1-0}} \|F_r(\cdot)\|_X ,$$

where $F_r(t) = F\left(re^{it}\right)$. We also define its subclass

$$H_{X_b}^+ \equiv \left\{F \in H_X^+ : F^+ \in X_b\right\} ,$$

where $F^+(\cdot)$ are the non-tangential boundary values of F on γ.

Similar to classical case, we define the Banach Hardy class ${}_mH_X^-$ of analytic functions outside the unit circle which have a finite order at infinity. Let the function $f(\cdot)$, analytic outside ω, have a Laurent decomposition of the form

$$f(z) = \sum_{n=-\infty}^{m} a_n z^n , \ z \to \infty, \ a_m \neq 0 ,$$

in the vicinity of the infinitely remote point. So, for $m > 0$ the point $z = \infty$ is a pole of order m, and for $m \leq 0$ the point $z = \infty$ is a zero of order $(-m)$. Let $f(\cdot) = f_0(\cdot) + f_1(\cdot)$, where $f_0(\cdot)$ is the principal part, and $f_1(\cdot)$ is the regular part of Laurent decomposition in the vicinity of $z = \infty$. If the function

$g(z) = \overline{f_0\left(\frac{1}{\bar{z}}\right)}$, $|z| < 1$, belongs to the class H_X^+ , then we will say that the function $f(\cdot)$ belongs to the class ${}_mH_X^-$.

Consider the following singular integral with the Cauchy kernel:

$$(S f)(\tau) = \frac{1}{2\pi i} \int_\gamma \frac{f(\xi)\, d\xi}{\xi - \tau}\,,\ \tau \in \gamma,$$

where $f \in X$ is some function

We need the following result of D.Boyd [23].

Theorem 5. *Suppose that $T \in [L_p]$ and $T \in [L_q]$, with $1 < p < q < +\infty$. Let X be an r.i.s. with the Boyd indices α_X and β_X which satisfy $\frac{1}{q} < \alpha_X \leq \beta_X < \frac{1}{p}$. Then $T \in [X]$.*

This theorem has the following immediate corollary.

Corollary 1. *Let X be an r.i.s. over $(-\pi, \pi]$ with the Boyd indices $\alpha_X; \beta_X \in (0,1)$. Then singular operator S acts boundedly in X, i.e. $S \in [X]$.*

This fact helps establish the validity of some classical facts concerning Cauchy type integrals in r.i.s. .

3.1. Cauchy integral formula

Let X be some Banach function space over γ. Then it is clear that the following continuous embedding $X \subset L_1$ is true:

$$\|f\|_{L_1} \leq C\, \|f\|_X\,,\ \forall f \in X, \tag{2}$$

where $C > 0$ is some constant. This estimate directly implies that $H_X^+ \subset H_1^+$. Let $f \in H_X^+$. Denote by f^+ the non-tangential boundary values of $f \in H_X^+$ on $\gamma : f^+ = f/_\gamma$. It follows from $f \in H_1^+$ that the Cauchy formula

$$f(z) = \frac{1}{2\pi i} \int_\gamma \frac{f^+(\tau)}{\tau - z} d\tau\ , z \in \omega,$$

holds. Consequently, if $f \in H_X^+$, then the Cauchy formula is valid for it. Let us expand the function $f(\cdot)$ in a Taylor series in a neighborhood of the point $z = 0$:

$$f(z) = \sum_{n=0}^{\infty} f_n^+ z^n\,, z \in \omega.$$

By the Riesz theorem, we have

$$\int_{-\pi}^{\pi} \left| f\left(re^{it}\right) - f^{+}\left(e^{it}\right) \right| dt \to 0\,,\; r \to 1-0.$$

Hence we immediately obtain

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f^{+}\left(e^{it}\right) e^{-int} dt = \begin{cases} f_n^{+}\,,\; n \geq 0\,, \\ 0\,,\; n < 0\,. \end{cases}$$

Now, consider the following Cauchy type integral

$$F(z) = \frac{1}{2\pi i}\int_{\gamma} \frac{f(\tau)}{\tau - z} d\tau\;,\, z \in \omega, \tag{3}$$

where $f \in X$ is some function. By the Sokhotski-Plemelj formulas, we have

$$F^{\pm}(\tau) = \pm\frac{1}{2} f(\tau) + (Sf)(\tau)\,, \quad \text{a.e. } \tau \in \gamma, \tag{4}$$

where $F^{+}(\cdot)$ $(F^{-}(\cdot))$ are non-tangential boundary values of $F(\cdot)$ inside (outside) ω on γ. It immediately follows from formulas (4) that $F^{\pm}(\cdot)$ belong to X if and only if $(Sf)(\cdot)$ belong to X, i.e. $S \in [X]$. Therefore, as follows from Corollary 1, if X is an r.i.s. with the Boyd indices $\alpha_X; \beta_X \in (0,1)$, then $F^{\pm}(\cdot) \in X$. Thus, the boundary values of the Cauchy-type integral (3) belong to X, if only $S \in [X]$.

Next, let $F(\cdot) \in H_X^{+}$. Therefore, the relation

$$\overline{\lim_{r_n \to 1-0}} \left\| F_{r_n}(\cdot) \right\|_X < +\infty$$

holds for any sequence $r_n \uparrow 1$. It is obvious that $\exists \{r_n\}$:$r_n \uparrow 1$ and $F_{r_n}(\tau) \to F^{+}(\tau)$, $n \to \infty$, a.e. $\tau \in \gamma$. Then it follows from Lemma 1 ii) (Fatou's lemma) that

$$\left\| F^{+} \right\|_X \leq \liminf_{n \to \infty} \left\| F_{r_n} \right\|_X \leq \left\| F \right\|_{H_X^{+}}\,, \tag{5}$$

i.e. $F^{+} \in X$. Thus, if $F \in H_X^{+}$, then $F^{+} \in X$ and the inequality (5) is valid.

Let the non-tangential boundary values $F^{+}(\cdot)$ of the function $F \in \mathscr{N}'$ belong to X_b. Then it is clear that (see e.g. [7]) $F \in H_1^{+}$ (since $F^{+} \in X_b \subset L_1$) and the Cauchy formula (5) holds.

Assume that X is an r.i.s. with the Boyd indices $\alpha_X;\; \beta_X \in (0,1)$. It is clear $\exists\; p;\; q \in (1,+\infty)$:

$$1 < q < \frac{1}{\beta_X} \leq \frac{1}{\alpha_X} < p < +\infty.$$

In this case, we have the following continuous embeddings

$$L_p \subset X \subset L_q,$$

i.e. $\exists\, C_1;\ C_2 > 0$:

$$\|f\|_{L_q} \leq C_1 \|f\|_X \ , \ \forall f \in X, \tag{6}$$

$$\|f\|_X \leq C_2 \|f\|_{L_p} \ , \ \forall f \in L_p. \tag{7}$$

It is quite obvious that L_p is dense in X_b. Thus, the embeddings

$$L_p \subset X_b \subset X \subset L_q$$

hold. It is well known that the direct decomposition

$$L_q = L_q^+ \dot{+}_{-1} L_q^-, 1 < q < +\infty, \tag{8}$$

holds, where $L_q^+ \ \left({}_{-1}L_q^-\right)$ are boundary values of functions from Hardy classes $H_q^+ \left({}_{-1}H_q^-\right)$ on γ. Denote the projector onto the subspace $L_q^+ \ ({}_{-1}L_q^-)$, generated by the decomposition (8), by $P^+ \ (P^-)$. Assume $P^+X_b = X_b^+;\ P^-X_b = {}_{-1}X_b^-$. Trigonometric polynomials are dense in X_b (in norm $\|\cdot\|_X$) and in L_q (in norm $\|\cdot\|_q$). The following expressions for projectors are known

$$P^{\pm} = \frac{1}{2} I \pm S,$$

where I is an identity operator. If $\alpha_X; \beta_X \in (0,1)$, then, by Corollary 1, $P^{\pm} \in [X]$. The following relations hold

$$P^+\left(e^{int}\right) = \begin{cases} e^{int}, \ n \geq 0, \\ 0, \ n < 0, \end{cases} ; P^-\left(e^{int}\right) = \begin{cases} 0, \ n \geq 0, \\ e^{int}, \ n < 0. \end{cases} \tag{9}$$

Let us show that the system of exponents $\left\{e^{int}\right\}_{n\in Z}$ forms a basis for X_b, if $\alpha_X;\ \beta_X \in (0,1)$. Let $f \in X$ be an arbitrary function. Consider the partial sums

$$S_m f = \sum_{n=-m}^{m} f_n e^{int},$$

where f_n are the Fourier coefficients of the function f.

In the sequel, we assume that the function f is continued periodically (with period 2π) to the whole real axis and $f(\tau) = f(\arg \tau)\ ,\ \tau \in \gamma$. We have

$$(S_m f)(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{\sin\left[\left(m + \frac{1}{2}\right)(x - t)\right]}{2 \sin \frac{x-t}{2}} dt =$$

$$= \frac{\sin\left(m + \frac{1}{2}\right) x}{2\pi} \int_{-\pi}^{\pi} f(t) \frac{\cos\left(m + \frac{1}{2}\right) t}{2 \sin \frac{x-t}{2}} dt -$$

$$-\frac{\cos\left(m+\frac{1}{2}\right)x}{2\pi}\int_{-\pi}^{\pi} f(t)\,\frac{\sin\left(m+\frac{1}{2}\right)t}{2\sin\frac{x-t}{2}}dt =$$

$$= \frac{\sin\left(m+\frac{1}{2}\right)x}{2\pi}\left(S_m^{(1)}f\right)(x) - \frac{\cos\left(m+\frac{1}{2}\right)x}{2\pi}\left(S_m^{(2)}f\right)(x),$$

where $S_m^{(k)}f$ denotes the corresponding integrals. Hence it follows directly

$$\|S_m f\|_X \leq \frac{1}{2\pi}\left(\left\|S_m^{(1)}f\right\|_X + \left\|S_m^{(2)}f\right\|_X\right).$$

Taking into account the obvious identity

$$\frac{1}{2\sin\frac{x-t}{2}} = i\frac{e^{i\frac{1}{2}(x+t)}}{e^{ix}-e^{it}},$$

we obtain

$$S_m^{(1)}f = \int_{-\pi}^{\pi}\frac{f_1(t)}{2\sin\frac{x-t}{2}}dt = ie^{i\frac{x}{2}}\int_{-\pi}^{\pi}\frac{f_1(t)\,e^{i\frac{t}{2}}}{e^{ix}-e^{it}}dt = e^{i\frac{x}{2}}\int_{\gamma}\frac{\tilde{f}_1(\tau)\,d\tau}{e^{ix}-\tau},$$

where

$$\tilde{f}_1\left(e^{it}\right) = e^{-i\frac{t}{2}}f(t)\cos\left(m+\frac{1}{2}\right)t.$$

Since, for $\alpha_X;\ \beta_X \in (0,1)$, the singular integral is bounded in X, we obtain

$$\left\|S_m^{(1)}f\right\|_X \leq C_1\left\|\tilde{f}_1\right\|_X \leq C_1\|f\|_X,$$

where $C_1 > 0$ is a constant independent of m. Similarly we have

$$\left\|S_m^{(2)}f\right\|_X \leq C_2\|f\|_X,$$

and finally

$$\|S_m f\|_X \leq C\|f\|_X,$$

where $C > 0$ is a constant independent of m. Consequently, $\sup\limits_m \|S_m\|_{[X]} < +\infty$. Then from Theorem 3 it follows that the Fourier series of every function $f \in X_b$ converges to itself in X. The estimate (6) implies the minimality of the system in X. As a result, the following theorem is true.

Theorem 6. *Let X be an r.i.s. on γ with $\varphi_X(+0) = 0$. Then the system of exponents $\left\{e^{int}\right\}_{n\in Z}$ forms a basis for X_b if and only if the Boyd indices of X satisfy $0 < \alpha_X;\beta_X < 1$.*

Let us show that the system $\{e^{int}\}_{n\in Z_+}$ forms a basis for X_b^+. Take $\forall f^+ \in X_b^+$. Then it is clear that $\exists f \in X_b$:$P^+f = f^+$. Let us expand the function f with the basis $\{e^{int}\}_{n\in Z_+}$:

$$f = \sum_{n=-\infty}^{+\infty} f_n e^{int}.$$

Since $P^+ \in [X]$, taking into account the relations (9), we obtain

$$f^+ = \sum_{n=0}^{+\infty} f_n e^{int}. \qquad (10)$$

From Lemma 2 it immediately follows that X_b^+ is a subspace of X. Then it is clear that the decomposition (10) is unique. This proves the basicity of the system of exponents $\{e^{int}\}_{n\in Z_+}$ in X_b^+. The basicity of the system $\{e^{-int}\}_{n\in N}$ is proved similarly in ${}_{-1}X_b^-$. So, the following theorem is true.

Theorem 7. *Let X be an r.i.s. on γ with the Boyd indices α_X; $\beta_X \in (0,1)$. Let $X_b^+ = \left(\frac{1}{2}I + S\right)X_b$ and ${}_{-1}X_b^- = \left(\frac{1}{2}I - S\right)X_b$, where S is the singular Cauchy integral. Then the system $\{e^{int}\}_{n\in Z_+}$ $\left(\{e^{-int}\}_{n\in N}\right)$ forms a basis for X_b^+ $\left({}_{-1}X_b^-\right)$.*

Let X be an r.i.s. with the Boyd indices α_X; β_X : $1 < q < \frac{1}{\beta_X} \le \frac{1}{\alpha_X} < p < +\infty$. Then from the relations (6) and (7) we immediately get the continuous embeddings

$$H_p^+ \subset H_X^+ \subset H_q^+.$$

The direct decomposition

$$X_b = X_b^+ \dot{+} {}_{-1}X_b^-$$

holds. It follows directly from Theorem 6 and 7. Denote the set of all polynomials with regard to z^n by $\mathscr{A}$. It is absolutely clear that $\mathscr{A} \subset H_X^+$. The closure of $\mathscr{A}$ in H_X^+ (i.e. according to the norm $\|\cdot\|_{H_X^+}$) is denoted by $H_{X_b}^+$. Let us show that the spaces X_b^+ and $H_{X_b}^+$ are isometric. Let $F \in H_{X_b}^+$ be an arbitrary function. As already established, $F^+ \in X$ and the inequality

$$\left\|F^+\right\|_X \le \|F\|_{H_X^+},$$

is true. We now need the following class of harmonic functions in ω. Denote by h_X the class of harmonic functions u in ω with the norm

$$\|u\|_{h_X} = \overline{\lim_{r\to 1-0}} \|u_r(\cdot)\|_X,$$

where $u_r(t) = u(re^{it})$, $0 \le r < 1$,$-\pi < t \le \pi$. It is absolutely clear that $F \in H_X^+$ if and only if ReF ; $ImF \in h_X$. So, let $f \in X_b^+$. It is obvious that $\exists F \in H_1^+$, $F^+ = f$ a.e. on γ. Let $f_1 = Ref$. Consider the following Poisson integral

$$u_r(x) = u(re^{ix}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f_1(s) P_r(s-x)\, ds = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(s) f_1(s-x)\, ds,$$

where

$$P_r(s) = \frac{1-r^2}{1-2r\cos s + r^2}$$

is a Poisson kernel for a unit circle. The following relation is well known

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(s)\, ds = 1\,,\ \forall r \in [0,1)\,.$$

Let us show that $u \in h_X$. We have

$$\|u_r\|_X = \frac{1}{2\pi}\left\|\int_{-\pi}^{\pi} P_r(s) f_1(s-x)\, ds\right\|_X \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(s) \|f_1(s-\cdot)\|_X\, ds.$$

As X is an r.i.s., it is also translation-invariant (see e.g. [10, p.158]), i.e.

$$\|f_1(s-\cdot)\|_X = \|f_1\|_X\ ,\forall s \in [-\pi,\pi]\,.$$

Then from the previous relation we get

$$\|u_r\|_X \le \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(s) \|f_1\|_X\, ds = \|f_1\|_X \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(s)\, ds =$$

$$= \|f_1\|_X \Rightarrow \|u\|_{h_X} \le \|f_1\|_X\,.$$

It is known that $u/_\gamma = f_1$ a.e. on γ. It is clear that $u = ReF$. It can be proved similarly that

$$\|ImF\|_{h_X} \le \|Imf\|_X\,,$$

and, as a result, $ImF \in h_X$.

It is clear that the Poisson integral

$$F(re^{ix}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} P_r(s-x) f(s)\, ds$$

is an analytic function and $F/_\gamma = ReF/_\gamma + iImF/_\gamma = Ref + iImf = f$ a.e. on γ. Moreover, the following estimate is proved in exactly the same way as in the case h_X:

$$\|F\|_{H_X^+} \le \|f\|_X\,.$$

Recalling the previously obtained estimate (5), we obtain $\|F\|_{H_X^+} = \|f\|_X$. From Theorem 7 it follows that the system $\{z^n\}_{n\in z_+}$forms a basis for $H^+_{X_b}$. Hence it directly follows from Theorem 7 again that the subspace X_b^+ and Hardy class $H^+_{X_b}$ are isometrically isomorphic.

Similarly we can prove that if $f \in {}_{-1}X_b^-$, then $\exists F \in {}_{-1}H^-_{X_b} : F^-/_\gamma = f$; $\|F\|_{{}_{-1}H^-_{X_b}} = \|f\|_X$, and moreover, the system $\{z^{-n}\}_{n\in N}$ forms a basis for ${}_{-1}H^-_{X_b}$.

Assume $X^+ = P^+X$; ${}_{-1}X^- = P^-X$. If X is an r.i.s. and α_X; $\beta_X \in (0, 1)$, then it is clear that $P^\pm \in [X]$, and then from the relation $P^+ + P^- = I$ we immediately get the direct decomposition $X = X^+\dot{+}{}_{-1}X^-$ (since $P^\pm$ are mutually disjoint projectors). Proceeding from this relation, absolutely similar to the previous case it is proved that the spaces X^+ $({}_{-1}X^-)$ and H_X^+ $\left({}_{-1}H_X^-\right)$ are isometric. So we have established the following result.

Theorem 8. *Let X be an r.i.s. with the Boyd indices α_X; $\beta_X \in (0, 1)$. Then:*

i) An analytic in ω function $F \in \mathscr{N}$ belongs to a class $H^+_{X_b}$ $\left(H_X^+\right)$ if and only if its boundary values F^+ belong to X_b (X) and the Cauchy formula

$$F(z) = \frac{1}{2\pi i}\int_\gamma \frac{F^+(\tau)}{\tau - z}d\tau\,,\ z \in \omega,$$

holds for it;

ii) Spaces $H^+_{X_b}$ and X_b^+; ${}_{-1}H^-_{X_b}$ and ${}_{-1}X_b^-$ (H_X^+ and X^+; ${}_{-1}H_X^-$ and ${}_{-1}X^-$) can be identified from the point of view of isometry

$$\|F\|_{H_X^+} = \left\|F^+\right\|_X ; \|F\|_{{}_{-1}H_X^-} = \left\|F^-\right\|_X$$

and the direct decompositions

$$X_b = H^+_{X_b}\dot{+}{}_{-1}H^-_{X_b};\ X = H_X^+\dot{+}{}_{-1}H_X^-,$$

hold;

iii) System $\{z^n\}_{n\in z_+}$ $\left(\{z^{-n}\}_{n\in N}\right)$ forms a basis for $H^+_{X_b}$ (in ${}_{-1}H^-_{X_b}$).

3.2. An analogue of the Riesz theorem

Let X be an r.i.s. with the Boyd indices α_X; $\beta_X \in (0, 1)$. As already established, for an arbitrary sequence $r_n \uparrow 1$, $n \in N$, we have $\|F^+\|_X \le \lim_{n\to\infty} \inf \|F_{r_n}\|_X$, and, as a result,

$$\left\|F^+\right\|_X \le \underline{\lim}_{r\to 1-0} \|F_r\|_X \le \overline{\lim}_{r\to 1-0} \|F_r\|_X \le \left\|F^+\right\|_X .$$

Then

$$\lim_{r\to 1-0}\|F_r\|_X = \left\|F^+\right\|_X . \tag{11}$$

We will prove that

$$\lim_{r\to 1-0}\left\|F_r(\cdot) - F^+(\cdot)\right\|_X = 0$$

also holds if $F \in H_X^+$ and F^+ has an absolutely continuous norm $\|\cdot\|_X$. First, assume the following notation

$$\|f\|_{X(M)} = \|f\,\chi_M\|_X ,$$

where $M \subset \gamma$ is a measurable set. Recall that $\|\cdot\|_X$ is absolutely continuous if for $f \in X_b$ and for $\forall\varepsilon > 0$, $\exists\,\delta > 0\,:\, \|f\|_{X(e)} < \varepsilon$, for $\forall\,|e| < \delta$, $e < \gamma$ is a measurable subset, $|\cdot|$ is a Lebesgue measure.

So, let $F \in H_X^+$ be such that F^+ has an absolutely continuous norm $\|\cdot\|_X$. Let $r_n \uparrow 1,\ n \in N$, be an arbitrary sequence and $\varepsilon > 0$ be an arbitrary number. It is known that $F_{r_n} \to F^+$ a.e. on γ (since $F \in H_1^+$). It is obvious that $\exists\delta > 0$: (everywhere we consider only measurable sets)

$$\left\|F^+\right\|_{X(e)} < \varepsilon\,,\ \forall e:\ |e| < \delta. \tag{12}$$

Then by Egorov's theorem, $\exists\,e_0 : |e_0| < \delta$, on $M = \gamma\backslash e_0$, the sequence F_{r_n} converges uniformly to F^+:

$$\left|F_{r_n}(t) - F^+(t)\right| < \varepsilon,\ \forall t \in M,$$

$\forall n > n_0$, where $n_0 \in N$ depends only on ε. Denote by $c_0 = \|1\|_X$ an absolute constant. We have

$$\left\|F_{r_n} - F^+\right\|_{X(M)} \le \|\varepsilon\|_{X(M)} \le c_0\varepsilon.$$

Then from the relation

$$\left|\|F_{r_n}\|_{X(M)} - \left\|F^+\right\|_{X(M)}\right| \le \left\|F_{r_n} - F^+\right\|_{X(M)}$$

it follows immediately that

$$\lim_{n\to\infty}\|F_{r_n}\|_{X(M)} = \left\|F^+\right\|_{X(M)} . \tag{13}$$

We have

$$\left|\|F_{r_n}\|_{X(e_0)} - \left\|F^+\right\|_{X(e_0)}\right| \le \left\|F_{r_n} - F^+\right\|_{X(e_0)} = \left\|F_{r_n}\chi_{e_0} - F^+\chi_{e_0}\right\|_X =$$

$$= \left\| F_{r_n}\chi_{e_0} + F_{r_n}\chi_M - F_{r_n}\chi_M + F^+\chi_M - F^+\chi_M - F^+\chi_{e_0} \right\|_X =$$
$$= \left\| F_{r_n} - F^+ - F_{r_n}\chi_M + F^+\chi_M \right\|_X \leq \left\| F_{r_n} - F^+ \right\|_X +$$
$$+ \left\| F_{r_n}\chi_M - F^+\chi_M \right\|_X = \left\| F_{r_n} - F^+ \right\|_X + \left\| F_{r_n} - F^+ \right\|_{X(M)} .$$

Hence, taking into account relations (11) and (13), we directly obtain

$$\lim_{n\to\infty} \|F_{r_n}\|_{X(e_0)} = \left\| F^+ \right\|_{X(e_0)} .$$

Then from (12) it follows that $\exists n_1 \in N$:

$$\|F_{r_n}\|_{X(e_0)} < \varepsilon \, , \, \forall n \geq n_1.$$

We have

$$\left\| F_{r_n} - F^+ \right\|_X = \left\| \left(F_{r_n} - F^+ \right) \chi_M + \left(F_{r_n} - F^+ \right) \chi_{e_0} \right\|_X \leq$$
$$\leq \left\| F_{r_n} - F^+ \right\|_{X(M)} + \left\| F_{r_n} - F^+ \right\|_{X(e_0)} \leq$$
$$\leq \left\| F_{r_n} - F^+ \right\|_{X(M)} + \|F_{r_n}\|_{X(e_0)} + \left\| F^+ \right\|_{X(e_0)} \leq 2\varepsilon + \left\| F_{r_n} - F^+ \right\|_{X(M)} .$$

Hence, paying attention to relation (13), from the arbitrariness of ε we obtain

$$\lim_{n\to\infty} \left\| F_{r_n} - F^+ \right\|_X = 0,$$

and this, in turn, by the arbitrariness of $r_n \uparrow 1, \ n \in N$, means

$$\lim_{r\to 1-0} \left\| F_r - F^+ \right\|_X = 0. \tag{14}$$

Again from the inequality

$$\left| \|F_r\|_X - \left\| F^+ \right\|_X \right| \leq \left\| F_r - F^+ \right\|_X$$

it follows that

$$\lim_{r\to 1-0} \|F_r\|_X = \left\| F^+ \right\|_X .$$

On the contrary, let the relation (14) hold. Let us show that F^+ has an absolutely continuous norm $\|\cdot\|_X$. In fact, let $\varepsilon > 0$ be an arbitrary number. Then from the inequality

$$\left\| F_r - F^+ \right\|_{X(e)} \leq \left\| F_r - F^+ \right\|_X$$

and from relation (14) it immediately follows that

$$\lim_{r\to 1-0} \left\| F_r - F^+ \right\|_{X(e)} = 0.$$

Then $\exists\, r_0 \in (0,1)$:

$$\left\| F_r - F^+ \right\|_{X(e)} < \varepsilon\,,\ \forall r \in (r_0\,,\,1)\,.$$

Take and fix $r_1 \in (r_0,\,1)$. It is obvious that $F_{r_1}(\cdot)$ is continuous on γ and therefore is bounded, i.e. $\exists c > 0$:

$$|F_{r_1}(t)| \le c\,,\ \forall t \in \gamma.$$

We have

$$\left\| F^+ \right\|_{X(e)} \le \left\| F^+ - F_{r_1} \right\|_{X(e)} + \left\| F_{r_1} \right\|_{X(e)} < \varepsilon + c \left\| \chi_e \right\|_X\,. \tag{15}$$

It follows from the embedding (7) that the characteristic function χ_e has an absolutely continuous norm $\|\cdot\|_X$. Then it follows from (15) that F^+ also has an absolutely continuous norm $\|\cdot\|_X$, and, as a result, $F^+ \in X_b$. Therefore $F \in H^+_{X_b}$. Thus, the following analog of classical Riesz theorem is true.

Theorem 9. *Let X be an r.i.s. with the Boyd indices $\alpha_X;\ \beta_X \in (0,1)$. Then:*

i) If $F \in H^+_X$, then $\lim\limits_{r\to 1-0} \|F_r\|_X = \|F^+\|_X$;

ii) The relation $\lim\limits_{r\to 1-0} \|F_r - F^+\| = 0$ is true if and only if $F \in H^+_{X_b}$.

Similarly we can prove the following

Theorem 10. *Let X be an r.i.s. with the Boyd indices $\alpha_X;\ \beta_X \in (0,1)$. Then:*

i) If $F \in_m H^-_X$, then $\lim\limits_{r\to 1+0} \|F_r\|_X = \|F^-\|_X$;

ii) The relation $\lim\limits_{r\to 1+0} \|F_r - F^-\| = 0$ is true if and only if $F \in_m H^-_{X_b}$.

iii) Analytic function $F(\cdot)$ outside ω belongs to the class ${}_{-1}H^-_X$ $({}_{-1}H^-_{X_b})$ if and only if its boundary values $F^-(\cdot)$ belong to $X\,(X_b)$ and the Cauchy formula

$$F(z) = -\frac{1}{2\pi i}\int_\gamma \frac{F^-(\tau)}{\tau - z}\,d\tau\,,\ |z| > 1\ ,$$

holds.

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