Azerbaijan Journal of Mathematics
V. 10, No 2, 2020, July
ISSN 2218-6816

Nonsingular Integral Operator and Its Commutators on Vanishing Generalized Orlicz-Morrey Spaces

M.N. Omarova

Abstract. We obtain sufficient conditions for the boundedness of the nonsingular integral operator and its commutators on vanishing generalized Orlicz-Morrey spaces $M^{\Phi,\varphi}(\mathbb{R}^n_+)$ including their weak versions.

Key Words and Phrases: vanishing generalized Orlicz-Morrey spaces, nonsingular integral, commutator, BMO.

2010 Mathematics Subject Classifications: 42B20, 42B35, 46E30

1. Introduction

In connection with elliptic partial differential equations, C. Morrey proposed a weak condition for the solution to be continuous enough in [26]. Later on, his condition became a family of normed spaces and they are called Morrey spaces. Although the notion is originally from the partial differential equations, the space turned out to be important in many branches of mathematics. Despite the fact that such spaces allow to describe local properties of functions better than Lebesgue spaces, they have some unpleasant issues. It is well known that Morrey spaces are non separable and the usual classes of nice functions are not dense in such spaces. Moreover, various Morrey spaces are defined in the process of study. Guliyev, Mizuhara and Nakai [9, 25, 27] introduced and studied the boundedness of some classical integral operators in the generalized Morrey spaces $M^{p,\varphi}(\mathbb{R}^n)$.

In [4], the generalized Orlicz-Morrey space $M^{\Phi,\varphi}(\mathbb{R}^n)$ was introduced to unify Orlicz and generalized Morrey spaces. Other definitions of generalized Orlicz-Morrey spaces can be found in [28] and [34]. In words of [15], the generalized Orlicz-Morrey space is the third kind and the ones in [28] and [34] are the first kind and the second kind, respectively. According to the examples in [8], one can say that the generalized Orlicz-Morrey spaces of the first kind and the second kind are different and that second kind and third kind are different. However, we do not know anything about the relationship between the first and the second kind.

Note that, Orlicz-Morrey spaces unify Orlicz and generalized Morrey spaces. We extend some results on generalized Morrey space in the papers [1, 5, 10, 11, 12, 16, 18] to the case of Orlicz-Morrey space in [4, 13, 14, 15].

Based on the results of [10, 11], the following conditions were introduced in [4] (see also [13]) for the boundedness of the singular integral operators on $M^{\Phi,\varphi}(\mathbb{R}^n)$:

$$\int_r^{\infty} \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \right) \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C\, \varphi_2(x,r), \tag{1}$$

where C does not depend on x and r.

Consider the half-space $\mathbb{R}^n_+ = \mathbb{R}^{n-1} \times (0,\infty)$. For $x = (x', x_n) \in \mathbb{R}^n_+$, let $\tilde{x} = (x', -x_n)$ be the "reflected point" and $\mathcal{B}^+(x,r) = B(x,r) \cap \mathbb{R}^n_+$. Let $x \in \mathbb{R}^n_+$. The nonsingular integral operator $\widetilde{T}$ is defined by

$$\widetilde{T} f(x) = \int_{\mathbb{R}^n_+} \frac{|f(y)|}{|\tilde{x} - y|^n}\, dy, \quad \tilde{x} = (x', -x_n). \tag{2}$$

The commutators generated by $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and the operator $\widetilde{T}$ are defined by

$$[b, \widetilde{T}] f(x) = \int_{\mathbb{R}^n_+} \frac{b(x) - b(y)}{|\tilde{x} - y|^n} f(y) dy.$$

The operator $\widetilde{T}$ and its commutator appear in [3] in connection with boundary estimates for solutions to elliptic equations.

In [6], the boundedness of the nonsingular integral operator $\widetilde{T}$ and its commutators $[b, \widetilde{T}]$ on generalized Orlicz-Morrey spaces of the third kind $M^{\Phi,\varphi}(\mathbb{R}^n_+)$ was studied.

The main purpose of this paper is to find sufficient conditions on general Young function Φ and functions φ_1, φ_2 which ensure the boundedness of the nonsingular integral operator $\widetilde{T}$ from one vanishing generalized Orlicz-Morrey space $VM^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ (see definition in Section 2) to another $VM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$, from $VM^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to vanishing weak generalized Orlicz-Morrey space $VWM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$ and the boundedness of the commutator of the nonsingular integral operator $[b, \widetilde{T}]$ from $VM^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $VM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$.

The following results are the fundamental theorems in this paper:

Theorem 1. *Let Φ be a Young function with $\Phi \in \Delta_2$. Let also $\varphi_1, \varphi_2 \in \Omega_{\Phi,1}$ satisfy*

$$c_\delta := \int_\delta^\infty \sup_{x \in \mathbb{R}^n_+} \varphi_1(x,t) \frac{dt}{t} < \infty \tag{3}$$

for every $\delta > 0$, and

$$\frac{1}{\varphi_2(x,r)} \int_r^\infty \varphi_1(x,t) \, \frac{dt}{t} \le C_0, \tag{4}$$

where C_0 does not depend on $x \in \mathbb{R}^n_+$ and $r > 0$. Then the nonsingular integral operator $\widetilde{T}$ is bounded from $VM^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $VWM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$. If, in addition, $\Phi \in \nabla_2$, then the operator $\widetilde{T}$ is bounded from $VM^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $VM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$.

Theorem 2. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $b \in BMO(\mathbb{R}^n_+)$. $\varphi_1, \varphi_2 \in \Omega_{\Phi,1}$ satisfy*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \varphi_1(x,t) \, \frac{dt}{t} \le C_0 \varphi_2(x,r), \tag{5}$$

where C_0 does not depend on $x \in \mathbb{R}^n_+$ and $r > 0$, and the conditions

$$\lim_{r \to 0} \frac{\ln \frac{1}{r}}{\inf_{x \in \mathbb{R}^n_+} \varphi_2(x,r)} = 0 \tag{6}$$

and

$$c_\delta := \int_\delta^\infty (1 + |\ln t|) \sup_{x \in \mathbb{R}^n_+} \varphi_1(x,t) \frac{dt}{t} < \infty \tag{7}$$

hold for every $\delta > 0$. Then the commutator of the nonsingular integral operator $[b, \widetilde{T}]$ is bounded from $VM^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $VM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$.

By $A \lesssim B$ we mean that $A \le CB$ with some positive constant C independent of appropriate quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. Definitions and Preliminary Results

2.1. On Young Functions and Orlicz Spaces

We recall the definition of Young functions.

Definition 1. *A function* $\Phi : [0,\infty) \to [0,\infty]$ *is called a Young function if* Φ *is convex, left-continuous,* $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ *and* $\lim\limits_{r\to\infty} \Phi(r) = \infty$*.*

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0,\infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0,\infty)$ and bijective from $[0,\infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. We note that

$$\Phi(\Phi^{-1}(r)) \leq r \leq \Phi^{-1}(\Phi(r)) \quad \text{for } 0 \leq r < \infty.$$

It is well known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \qquad \text{for } r \geq 0, \tag{8}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0,\infty)\} & , \quad r \in [0,\infty) \\ \infty & , \quad r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted also as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq k\Phi(r) \text{for } r > 0$$

for some $k > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted also by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2k}\Phi(kr), \qquad r \geq 0,$$

for some $k > 1$.

Definition 2. *(Orlicz Space). For a Young function* Φ*, the set*

$$L^{\Phi}(\mathbb{R}^n_+) = \left\{ f \in L^1_{\mathrm{loc}}(\mathbb{R}^n_+) : \int_{\mathbb{R}^n_+} \Phi(k|f(x)|)dx < \infty \textit{ for some } k > 0 \right\}$$

is called Orlicz space. If $\Phi(r) = r^p$, $1 \leq p < \infty$*, then* $L^{\Phi}(\mathbb{R}^n_+) = L^p(\mathbb{R}^n_+)$*. If* $\Phi(r) = 0$, $(0 \leq r \leq 1)$ *and* $\Phi(r) = \infty$, $(r > 1)$*, then* $L^{\Phi}(\mathbb{R}^n_+) = L^{\infty}(\mathbb{R}^n_+)$*. The space* $L^{\Phi}_{\mathrm{loc}}(\mathbb{R}^n_+)$ *is defined as the set of all functions* f *such that* $f\chi_B \in L^{\Phi}(\mathbb{R}^n_+)$ *for all balls* $B \subset \mathbb{R}^n_+$*.*

$L^{\Phi}(\mathbb{R}^n_+)$ is a Banach space with respect to the norm

$$\|f\|_{L^{\Phi}(\mathbb{R}^n_+)} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n_+} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \le 1\right\}.$$

We note that

$$\int_{\mathbb{R}^n_+} \Phi\Big(\frac{|f(x)|}{\|f\|_{L^{\Phi}(\mathbb{R}^n_+)}}\Big)dx \le 1. \tag{9}$$

The weak Orlicz space

$$WL^{\Phi}(\mathbb{R}^n_+) = \{f \in L^1_{\rm loc}(\mathbb{R}^n_+) : \|f\|_{WL^{\Phi}(\mathbb{R}^n_+)} < +\infty\}$$

is defined by the norm

$$\|f\|_{WL^{\Phi}(\mathbb{R}^n_+)} = \inf\Big\{\lambda > 0 \ : \ \sup_{t>0} \Phi(t) m\Big(\frac{f}{\lambda}, \ t\Big) \ \le 1\Big\}.$$

2.2. Vanishing generalized Orlicz-Morrey Space

Various versions of generalized Orlicz-Morrey spaces were introduced in [28], [34] and [4]. We used the definition of [4] which runs as follows.

We now define generalized Orlicz-Morrey spaces of the third kind. The *generalized Orlicz-Morrey space* $M^{\Phi,\varphi}(\mathbb{R}^n_+)$ *of the third kind* is defined as the set of all measurable functions f for which the norm

$$\|f\|_{M^{\Phi,\varphi}(\mathbb{R}^n_+)} \equiv \sup_{x\in\mathbb{R}^n_+, \, r>0} \frac{1}{\varphi(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right) \ \|f\|_{L^{\Phi}(\mathcal{B}^+(x,r))}$$

is finite, where $\mathcal{B}^+(x,r) = B(x,r) \cap \mathbb{R}^n_+$. Also by $WM^{\Phi,\varphi}(\mathbb{R}^n_+)$ we denote the *weak generalized Orlicz-Morrey space of the third kind* of all functions $f \in WL^{\Phi}_{\rm loc}(\mathbb{R}^n_+)$ for which

$$\|f\|_{WM^{\Phi,\varphi}(\mathbb{R}^n_+)} = \sup_{x\in\mathbb{R}^n_+, r>0} \varphi(x,r)^{-1} \Phi^{-1}(|\mathcal{B}^+(x,r)|^{-1}) \, \|f\|_{WL^{\Phi}(\mathcal{B}^+(x,r))} < \infty,$$

where $WL^{\Phi}(\mathcal{B}^+(x,r))$ denotes the weak L^{Φ}-space of measurable functions f for which

$$\|f\|_{WL^{\Phi}(\mathcal{B}^+(x,r))} \equiv \|f\chi_{\mathcal{B}^+(x,r)}\|_{WL^{\Phi}(\mathbb{R}^n_+)}.$$

Note that $M^{\Phi,\varphi}(\mathbb{R}^n_+)$ covers many classical function spaces.

Example 1. *Let $1 \le q \le p < \infty$ and $\Phi \in \Delta_2 \cap \nabla_2$. From the following special cases, we see that our results will cover the Lebesgue space $L^p(\mathbb{R}^n_+)$, the classical Morrey space $M^p_q(\mathbb{R}^n_+)$, the generalized Morrey space $M^{p,\varphi}(\mathbb{R}^n_+)$ and the Orlicz space $L^\Phi(\mathbb{R}^n_+)$ with norm coincidence:*

1. *If $\Phi(t) = t^p$ and $\varphi(t) = t^{-\frac{n}{p}}$, then $M^{\Phi,\varphi}(\mathbb{R}^n_+) = L^p(\mathbb{R}^n_+)$ with norm equivalence.*
2. *If $\Phi(t) = t^q$ and $\varphi(t) = t^{-\frac{n}{p}}$, then $M^{\Phi,\varphi}(\mathbb{R}^n_+)$, which is denoted by $M^p_q(\mathbb{R}^n_+)$, is the classical Morrey space.*
3. *If $\Phi(t) = t^p$, then $M^{\Phi,\varphi}(\mathbb{R}^n_+) = M^{p,\varphi}(\mathbb{R}^n_+)$ is the generalized Morrey space discussed in [10, 25, 27].*
4. *If $\varphi(t) = \Phi^{-1}(t^{-n})$, then $M^{\Phi,\varphi}(\mathbb{R}^n_+) = L^\Phi(\mathbb{R}^n_+)$, which is beyond the reach of generalized Orlicz-Morrey spaces of the second kind defined in [8] according to an example constructed in [34].*

Other definitions of generalized Orlicz-Morrey spaces can be found in [8, 28, 29, 30]. Therefore, our definition of generalized Orlicz-Morrey spaces here is named “third kind”.

In the case $\varphi(x,r) = \frac{\Phi^{-1}\left(|\mathcal{B}^+(x,r)|^{-1}\right)}{\Phi^{-1}\left(|\mathcal{B}^+(x,r)|^{-\lambda/n}\right)}$, we get the Orlicz-Morrey space $\mathcal{M}^{\Phi,\lambda}(\mathbb{R}^n_+)$ from generalized Orlicz-Morrey space $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n_+)$. We refer to [13, Lemmas 2.8 and 2.9] for more information about Orlicz-Morrey spaces.

Lemma 1. *[13, Lemma 2.12] Let Φ be a Young function and φ be a positive measurable function on $\mathbb{R}^n_+ \times (0,\infty)$.*

(i) If

$$\sup_{t<r<\infty} \frac{\Phi^{-1}(|\mathcal{B}^+(x,r)|^{-1})}{\varphi(x,r)} = \infty \quad \text{for some } t > 0 \text{ and for all } x \in \mathbb{R}^n_+, \tag{10}$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n_+) = \Theta$.

(ii) If

$$\sup_{0<r<\tau} \varphi(x,r)^{-1} = \infty \quad \text{for some } \tau > 0 \text{ and for all } x \in \mathbb{R}^n_+, \tag{11}$$

then $\mathcal{M}^{\Phi,\varphi}(\mathbb{R}^n_+) = \Theta$.

Remark 1. *Let Φ be a Young function. We denote by Ω_Φ the set of all positive measurable functions φ on $\mathbb{R}^n_+ \times (0,\infty)$ such that for all $t>0$,*

$$\sup_{x\in\mathbb{R}^n} \Big\| \frac{\Phi^{-1}(|\mathcal{B}^+(x,r)|^{-1})}{\varphi(x,r)} \Big\|_{L^\infty(t,\infty)} < \infty,$$

and

$$\sup_{x\in\mathbb{R}^n} \Big\| \varphi(x,r)^{-1} \Big\|_{L^\infty(0,t)} < \infty,$$

respectively. In what follows, keeping in mind Lemma 1, we always assume that $\varphi \in \Omega_\Phi$.

Definition 3. *(vanishing generalized Orlicz-Morrey Space) The vanishing generalized Orlicz-Morrey space $VM^{\Phi,\varphi}(\mathbb{R}^n_+)$ is defined as the space of functions $f \in M^{\Phi,\varphi}(\mathbb{R}^n_+)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n_+} \frac{1}{\varphi(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right) \|f\|_{L^\Phi(\mathcal{B}^+(x,r))} = 0.$$

Definition 4. *(vanishing weak generalized Orlicz-Morrey Space) The vanishing weak generalized Orlicz-Morrey space $VWM^{\Phi,\varphi}(\mathbb{R}^n_+)$ is defined as the space of functions $f \in WM^{\Phi,\varphi}(\mathbb{R}^n_+)$ such that*

$$\lim_{r\to 0} \sup_{x\in\mathbb{R}^n_+} \frac{1}{\varphi(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right) \|f\|_{L^\Phi(\mathcal{B}^+(x,r))} = 0.$$

The vanishing Morrey space $VM_{p,\lambda}(\mathbb{R}^n)$ of the classical Morrey spaces $M_{p,\lambda}(\mathbb{R}^n)$ was introduced by Vitanza in [35] and applied there to obtain a regularity result for elliptic partial differential equations. Later in [36] Vitanza proved an existence theorem for a Dirichlet problem, under weaker assumptions than those introduced by Miranda in [24], and obtained a $W^{3,2}$ regularity result assuming that the partial derivatives of the coefficients of the highest and lower order terms belong to the vanishing Morrey spaces depending on the dimension. Also M.A. Ragusa [32] proved a sufficient condition for commutators of fractional integral operators to belong to vanishing Morrey spaces $VM_{p,\lambda}(\mathbb{R}^n_+)$. About commutator operators in vanishing Morrey spaces see the papers [2, 7, 17, 31, 32].

Remark 2. *We denote by $\Omega_{\Phi,1}$ the set of all positive measurable functions φ on $\mathbb{R}^n_+ \times (0,1)$ such that*

$$\lim_{r\to 0} \frac{1}{\Phi^{-1}(r^{-n}) \inf_{x\in\mathbb{R}^n_+} \varphi(x,r)} = 0 \tag{12}$$

and

$$\inf_{x\in\mathbb{R}^n_+}\inf_{r>\delta}\varphi(x,r)>0, \quad \textit{for some} \quad \delta>0. \tag{13}$$

For the non-triviality of the space $VM^{\Phi,\varphi}(\mathbb{R}^n_+)$ we always assume that $\varphi\in\Omega_{\Phi,1}$.

The spaces $VM_{\Phi,\varphi}(\mathbb{R}^n_+)$ and $WVM_{\Phi,\varphi}(\mathbb{R}^n_+)$ are Banach spaces with respect to the norm

$$\|f\|_{VM^{\Phi,\varphi}}\equiv\|f\|_{M^{\Phi,\varphi}}=\sup_{x\in\mathbb{R}^n_+,r>0}\frac{1}{\varphi(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right)\,\|f\|_{L^\Phi(\mathcal{B}^+(x,r))},$$

$$\|f\|_{VWM^{\Phi,\varphi}}\equiv\|f\|_{WM^{\Phi,\varphi}}=\sup_{x\in\mathbb{R}^n_+,r>0}\frac{1}{\varphi(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right)\,\|f\|_{WL^\Phi(\mathcal{B}^+(x,r))},$$

respectively. The spaces $VM^{\Phi,\varphi}(\mathbb{R}^n_+)$ and $VWM^{\Phi,\varphi}(\mathbb{R}^n_+)$ are closed subspaces of the Banach spaces $M^{\Phi,\varphi}(\mathbb{R}^n_+)$ and $WM^{\Phi,\varphi}(\mathbb{R}^n_+)$, respectively, which may be shown by standard means.

3. Nonsingular integral operators in the space $VM^{\Phi,\varphi}(\mathbb{R}^n_+)$

For any $x\in\mathbb{R}^n_+$ define $\tilde{x}=(x',-x_n)$ and recall that $x^0=(x',0)$. Also define $\mathcal{B}^+_r\equiv\mathcal{B}^+(x^0,r)=B(x^0,r)\cap\mathbb{R}^n_+$, $2\mathcal{B}^+_r=\mathcal{B}^+(x^0,2r)$.

For proving our main results, we need the following estimate, which was proved in [6].

Lemma 2. *Let $\widetilde{T}$ be a nonsingular integral operator, defined by (2), Φ be any Young function and $f\in L^\Phi_{\rm loc}(\mathbb{R}^n_+)$ be such that*

$$\int_1^\infty \|f\|_{L^\Phi(\mathcal{B}^+(x^0,t))}\,\Phi^{-1}\big(t^{-n}\big)\,\frac{dt}{t}<\infty \tag{14}$$

i) If $\Phi\in\Delta_2\bigcap\nabla_2$, then

$$\|\widetilde{T}f\|_{L^\Phi(\mathcal{B}^+(x^0,r))}\le\frac{C}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty \|f\|_{L^\Phi(\mathcal{B}^+(x^0,t))}\,\Phi^{-1}\big(t^{-n}\big)\,\frac{dt}{t}. \tag{15}$$

ii) If $\Phi\in\Delta_2$, then

$$\|\widetilde{T}f\|_{WL^\Phi(\mathcal{B}^+(x^0,r))}\le\frac{C}{\Phi^{-1}\big(r^{-n}\big)}\int_{2r}^\infty \|f\|_{L^\Phi(\mathcal{B}^+(x^0,t))}\,\Phi^{-1}\big(t^{-n}\big)\,\frac{dt}{t}, \tag{16}$$

where the constants are independent of x^0, r and f.

By using Lemma 2 the following statement was proved in [6].

Theorem 3. *Let* $\widetilde{T}$ *be a nonsingular integral operator, defined by* (2), $\Phi \in \Delta_2$ *and* $\varphi_1, \varphi_2 \in \Omega_\Phi$ *satisfy* (1).
i) Then the operator $\widetilde{T}$ *is bounded from* $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ *to* $WM^{\Phi,\varphi_2}(\mathbb{R}^n_+)$ *and*

$$\|\widetilde{T}f\|_{M^{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C\|f\|_{WM^{\Phi,\varphi_1}(\mathbb{R}^n_+)}$$

with constants independent of f.
ii) If $\Phi \in \nabla_2$, *then the operator* $\widetilde{T}$ *is bounded from* $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ *to* $M^{\Phi,\varphi_2}(\mathbb{R}^n_+)$ *and*

$$\|\widetilde{T}f\|_{M^{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C\|f\|_{M^{\Phi,\varphi_1}(\mathbb{R}^n_+)} \tag{17}$$

with constants independent of f.

Proof of Theorem 1. The statement is derived from Theorem 3.
So we only have to prove that

$$\begin{aligned}
&\lim_{r\to 0}\sup_{x\in\mathbb{R}^n_+}\frac{1}{\varphi_1(x,r)}\,\Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right)\,\|f\|_{L^\Phi(\mathcal{B}^+(x,r))}=0\\
&\quad\Rightarrow\quad \lim_{r\to 0}\sup_{x\in\mathbb{R}^n_+}\frac{1}{\varphi_2(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right)\,\|\widetilde{T}f\|_{L^\Phi(\mathcal{B}^+(x,r))}=0,
\end{aligned}\tag{18}$$

and

$$\begin{aligned}
&\lim_{r\to 0}\sup_{x\in\mathbb{R}^n_+}\frac{1}{\varphi_1(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right)\,\|f\|_{L^\Phi(\mathcal{B}^+(x,r))}=0\\
&\quad\Rightarrow\quad \lim_{r\to 0}\sup_{x\in\mathbb{R}^n_+}\frac{1}{\varphi_2(x,r)}\Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right)\,\|\widetilde{T}f\|_{WL^\Phi(\mathcal{B}^+(x,r))}=0.
\end{aligned}\tag{19}$$

In this estimation we follow some ideas of [33] in such passage to the limit in the case $\Phi(r) = r^p$, but base ourselves on Lemma 2.

To show that $\sup\limits_{x\in\mathbb{R}^n_+}\frac{1}{\varphi_2(x,r)}\,\Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right)\,\|\widetilde{T}f\|_{L^\Phi(\mathcal{B}^+(x,r))} < \varepsilon$ for small r, we split the right-hand side of (15):

$$\varphi_2(x,r)^{-1}\,\Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right)\,\|\widetilde{T}f\|_{L_\Phi(\mathcal{B}^+(x,r))} \le C[I_\delta(x,r)+J_\delta(x,r)], \tag{20}$$

where $\delta_0 > 0$ (we may take $\delta_0 < 1$), and

$$I_\delta(x,r) := \frac{1}{\varphi_2(x,r)}\left(\int_r^{\delta_0}\frac{\varphi_1(x,t)}{t}(\varphi_1(x,t)^{-1}\,\|f\|_{L_\Phi(\mathcal{B}^+(x,t))})dt\right)$$

and

$$J_\delta(x,r) := \frac{1}{\varphi_2(x,r)} \left(\int_{\delta_0}^{\infty} \frac{\varphi_1(x,t)}{t} (\varphi_1(x,t)^{-1} \, \|f\|_{L_\Phi(\mathcal{B}^+(x,t))}) dt \right).$$

Besides, it is supposed that $r < \delta_0$. Now we choose any fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n_+} \varphi_1(x,t)^{-1} \, \Phi^{-1} \left(\frac{1}{|\mathcal{B}^+(x,r)|} \right) \, \|f\|_{L^\Phi(\mathcal{B}^+(x,r))} < \frac{\varepsilon}{2CC_0},$$

where C and C_0 are constants from (20) and (4). This allows us to estimate the first term uniformly in $r \in (0,\delta_0)$:

$$\sup_{x\in\mathbb{R}^n_+} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0.$$

The estimation of the second term now may be made by choosing r sufficiently small. Indeed, thanks to the condition (12) we have

$$J_\delta(x,r) \le c_{\delta_0} \|f\|_{VM^{\Phi,\varphi}} \frac{1}{\varphi(x,r)},$$

where c_{δ_0} is the constant from (3). Then, by (12) it suffices to choose r small enough such that

$$\sup_{x\in\mathbb{R}^n_+} \frac{1}{\varphi(x,r)} \le \frac{\varepsilon}{2c_{\delta_0}\|f\|_{VM^{\Phi,\varphi}}},$$

which completes the proof of (18).

The proof of (19) is similar to the proof of (18). ◄

4. Commutators of nonsingular integrals in the space $M^{\Phi,\varphi}(\mathbb{R}^n_+)$

For a function $b \in BMO$ define the commutator $[b,\widetilde{T}]f = b\widetilde{T}f - \widetilde{T}(bf)$. Our aim is to show boundedness of $[b,\widetilde{T}]$ in $M^{\Phi,\varphi}(\mathbb{R}^n_+)$. For this goal we recall some well known properties of the BMO functions.

Lemma 3. (John-Nirenberg lemma, [19]) *Let $b \in BMO$ and $p \in (1,\infty)$. Then for any ball $\mathcal{B}$ the inequality*

$$\left(\frac{1}{|\mathcal{B}|} \int_{\mathcal{B}} |b(y) - b_{\mathcal{B}}|^p dy \right)^{\frac{1}{p}} \le C(p)\|b\|_* \tag{21}$$

holds.

Definition 5. *A Young function Φ is said to be of upper type p (resp. lower type p) for some $p \in [0, \infty)$, if there exists a positive constant C such that, for all $t \in [1, \infty)$(resp. $t \in [0, 1]$) and $s \in [0, \infty)$,*

$$\Phi(st) \leq Ct^p \Phi(s).$$

Remark 3. *We know that if Φ is lower type p_0 and upper type p_1 with $1 < p_0 \leq p_1 < \infty$, then $\Phi \in \Delta_2 \cap \nabla_2$. Conversely, if $\Phi \in \Delta_2 \cap \nabla_2$, then Φ is lower type p_0 and upper type p_1 with $1 < p_0 \leq p_1 < \infty$ (see [22]).*

Before proving the main theorems, we need the following lemma.

Lemma 4. *[20] Let $b \in BMO(\mathbb{R}^n_+)$. Then there is a constant $C > 0$ such that*

$$\left| b_{\mathcal{B}^+_r} - b_{\mathcal{B}^+_t} \right| \leq C\|b\|_* \ln \frac{t}{r} \quad \textit{for} \quad 0 < 2r < t,$$

where C is independent of b, x, r, and t.

In the following lemma which was proved in [14] we provide a generalization of the property (21) from L^p-norms to Orlicz norms.

Lemma 5. *Let $b \in BMO(\mathbb{R}^n_+)$ and Φ be a Young function. Let Φ be lower type p_0 and upper type p_1 with $1 \leq p_0 \leq p_1 < \infty$. Then*

$$\|b\|_* \approx \sup_{x \in \mathbb{R}^n_+, r > 0} \Phi^{-1}\big(r^{-n}\big) \left\| b(\cdot) - b_{\mathcal{B}^+(x,r)} \right\|_{L^\Phi(\mathcal{B}^+(x,r))}.$$

Remark 4. *Note that the Lemma 5 for the variable exponent Lebesgue space $L^{p(\cdot)}$ was proved in [21].*

Definition 6. *Let Φ be a Young function. Let*

$$a_\Phi := \inf_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}, \qquad b_\Phi := \sup_{t \in (0,\infty)} \frac{t\Phi'(t)}{\Phi(t)}.$$

Remark 5. *It is known that $\Phi \in \Delta_2 \cap \nabla_2$ if and only if $1 < a_\Phi \leq b_\Phi < \infty$ (see, for example, [23]).*

Remark 6. *Remark 5 and Remark 3 show us that a Young function Φ is lower type p_0 and upper type p_1 with $1 < p_0 \leq p_1 < \infty$ if and only if $1 < a_\Phi \leq b_\Phi < \infty$.*

To estimate the commutator we shall employ the same idea which we used in the proof of Lemma 2.

Lemma 6. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $b \in BMO(\mathbb{R}^n_+)$. Suppose that for all $f \in L^{\Phi}_{\rm loc}(\mathbb{R}^n_+)$ and $r > 0$ the inequality*

$$\int_1^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \, \|f\|_{L^{\Phi}(\mathcal{B}^+(x^0,t))} \, \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} < \infty \tag{22}$$

holds. Then

$$\|[b,\widetilde{T}]f\|_{L^{\Phi}(\mathcal{B}^+_r)} \le \frac{C\|b\|_*}{\Phi^{-1}\big(r^{-n}\big)} \int_{2r}^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \|f\|_{L^{\Phi}(\mathcal{B}^+(x^0,t))} \, \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t}. \tag{23}$$

Theorem 4. *Let $b \in BMO(\mathbb{R}^n_+)$, $\widetilde{T}$ be a nonsingular integral operator, defined by (2), and $\Phi \in \Delta_2 \cap \nabla_2$, $\varphi_1, \varphi_2 \in \Omega_{\Phi}$ satisfy the condition*

$$\int_r^{\infty} \Big(1 + \ln \frac{t}{r}\Big) \left(\operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(x,s)}{\Phi^{-1}\big(s^{-n}\big)} \right) \, \Phi^{-1}\big(t^{-n}\big) \frac{dt}{t} \le C \, \varphi_2(x,r), \tag{24}$$

where C does not depend on x and r. Then the operator $[b,\widetilde{T}]$ is bounded from $M^{\Phi,\varphi_1}(\mathbb{R}^n_+)$ to $M^{\Phi,\varphi_2}(\mathbb{R}^n_+)$ and

$$\|[b,\widetilde{T}]f\|_{M^{\Phi,\varphi_2}(\mathbb{R}^n_+)} \le C\|b\|_* \, \|f\|_{M^{\Phi,\varphi_1}(\mathbb{R}^n_+)} \tag{25}$$

with a constant independent of f.

Proof of Theorem 2. The proof follows more or less the same lines as for Theorem 3, but now the arguments are different due to the necessity to introduce the logarithmic factor into the assumptions.

The norm inequality having already been provided by Theorem 4, we only have to prove the implication

$$\begin{aligned} &\lim_{r\to 0} \sup_{x\in\mathbb{R}^n_+} \frac{1}{\varphi_1(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right) \, \|f\|_{L^{\Phi}(\mathcal{B}^+(x,r))} = 0 \\ &\quad \Longrightarrow \quad \frac{1}{\varphi_2(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right) \, \|[b,\widetilde{T}]f\|_{L^{\Phi}(\mathcal{B}^+(x,r))} = 0. \end{aligned} \tag{26}$$

To check that

$$\sup_{x\in\mathbb{R}^n_+} \frac{1}{\varphi_2(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right) \, \|[b,\widetilde{T}]f\|_{L^{\Phi}(\mathcal{B}^+(x,r))} < \varepsilon \quad \text{for small} \quad r,$$

we use the estimate (23):

$$\frac{1}{\varphi_2(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right) \, \|[b,\widetilde{T}]f\|_{L^{\Phi}(\mathcal{B}^+(x,r))} \lesssim$$

$$\lesssim \frac{\|b\|_*}{\varphi_2(x,r)} \int_r^1 \left(1+\ln\frac{t}{r}\right) \|f\|_{L_\Phi(B(x_0,t))} \frac{dt}{t}.$$

We take $r < \delta_0$, where δ_0 is chosen small enough, and split the integration:

$$\frac{1}{\varphi_2(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right) \|[b,\widetilde{T}]f\|_{L^\Phi(\mathcal{B}^+(x,r))} \leq C[I_{\delta_0}(x,r) + J_{\delta_0}(x,r)], \quad (27)$$

where

$$I_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int_r^{\delta_0} \left(1+\ln\frac{t}{r}\right) \|f\|_{L^\Phi(\mathcal{B}^+(x,r))} \frac{dt}{t}$$

and

$$J_{\delta_0}(x,r) := \frac{1}{\varphi_2(x,r)} \int_{\delta_0}^{\infty} \left(1+\ln\frac{t}{r}\right) \|f\|_{L^\Phi(\mathcal{B}^+(x,r))} \frac{dt}{t}.$$

We choose a fixed $\delta_0 > 0$ such that

$$\sup_{x\in\mathbb{R}^n_+} \frac{1}{\varphi_1(x,r)} \Phi^{-1}\left(\frac{1}{|\mathcal{B}^+(x,r)|}\right) \|f\|_{L^\Phi(\mathcal{B}^+(x,r))} < \frac{\varepsilon}{2CC_0}, \quad t \leq \delta_0,$$

where C and C_0 are constants from (27) and (5), which yields the estimate of the first term uniform in $r \in (0,\delta_0) : \sup\limits_{x\in\mathbb{R}^n_+} CI_{\delta_0}(x,r) < \frac{\varepsilon}{2}, \quad 0 < r < \delta_0$.

For the second term, writing $1+\ln\frac{t}{r} \leq 1+|\ln t| + \ln\frac{1}{r}$, we obtain

$$J_{\delta_0}(x,r) \leq \frac{c_{\delta_0} + \widetilde{c_{\delta_0}}\ \ln\frac{1}{r}}{\varphi_2(x,r)} \|f\|_{M^{\Phi,\varphi}},$$

where c_{δ_0} is the constant from (7) with $\delta = \delta_0$ and $\widetilde{c_{\delta_0}}$ is a similar constant with omitted logarithmic factor in the integrand. Then, by (6) we can choose small r such that $\sup_{x\in\mathbb{R}^n_+} J_{\delta_0}(x,r) < \frac{\varepsilon}{2}$, which completes the proof. ◄

Acknowledgements

The author would like to express her gratitude to the referees for their valuable comments and suggestions.

The research of M. Omarova was partially supported by the Grant of the 1st Azerbaijan-Russia Joint Grant Competition (Agreement Number No. EIF-BGM-4-RFTF-1/2017-21/01/1).

References

[1] A. Akbulut, V.H. Hamzayev, Z.V. Safarov, *Rough fractional multilinear integral operators on generalized weighted Morrey space*, Azerb. J. Math., **6(2)**, 2016, 128-142.

[2] X.A. Badalov, *Maximal and singular integral operators and their commutators in the vanishing generalized weighted Morrey spaces with variable exponent*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **38(1)**, 2018, Mathematics, 30-42.

[3] F. Chiarenza, M. Frasca, P. Longo, *Interior $W^{2,p}$-estimates for nondivergence ellipic equations with discontinuous coefficients*, Ricerche Mat., **40**, 1991, 149-168.

[4] F. Deringoz, V.S. Guliyev, S. Samko, *Boundedness of maximal and singular operators on generalized Orlicz-Morrey spaces*, Operator Theory, Operator Algebras and Applications, Series: Operator Theory: Advances and Applications, **242**, 2014, 139-158.

[5] A. Eroglu, *Boundedness of fractional oscillatory integral operators and their commutators on generalized Morrey spaces*, Bound. Value Probl., **2013:70**, 2013, 12 pp.

[6] A. Eroglu, V.S. Guliyev, M.N. Omarova, *Characterizations for the nonsingular integral operator and its commutators on generalized Orlicz-Morrey spaces*, Azerb. J. Math, **7(2)**, 2017, 66-88.

[7] A. Eroglu, F.A. Isayev, F.M. Namazov, *Fractional integral associated to Schrödinger operator on the Heisenberg groups in vanishing generalized Morrey spaces*, Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci., **38(4)**, 2018, Mathematics, 54-62.

[8] S. Gala, Y. Sawano, H. Tanaka, *A remark on two generalized Orlicz-Morrey spaces*, J. Approx. Theory, **98**, 2015, 1-9.

[9] V.S. Guliyev, Integral operators on function spaces on the homogeneous groups and on domains in $\mathbb{R}^n$. Doctor's degree dissertation, Mat. Inst. Steklov, Moscow, 1994, 329 pp. (in Russian)

[10] V.S. Guliyev, *Boundedness of the maximal, potential and singular operators in the generalized Morrey spaces*, J. Inequal. Appl., 2009, Art. ID 503948, 20 pp.

[11] V.S. Guliyev, S.S. Aliyev, T. Karaman, P.S. Shukurov, *Boundedness of sublinear operators and commutators on generalized Morrey space*, Integral Equations Operator Theory, **71(3)**, 2011, 327-355.

[12] V.S. Guliyev, *Local generalized Morrey spaces and singular integrals with rough kernel*, Azerb. J. Math., **3(2)**, 2013, 79-94.

[13] V.S. Guliyev, F. Deringoz, J.J. Hasanov, Φ*-admissible singular operators and their commutators on vanishing generalized Orlicz-Morrey spaces*, J. Inequal. Appl., **2014:143**, 2014:143, 18 pp

[14] V.S. Guliyev, F. Deringoz, *On the Riesz potential and its commutators on generalized Orlicz-Morrey spaces*, J. Funct. Spaces, 2014, Article ID 617414, 11 pp.

[15] V.S. Guliyev, S.G. Hasanov, Y. Sawano, T. Noi, *Non-smooth atomic decompositions for generalized Orlicz-Morrey spaces of the third kind*, Acta Appl. Math., **145(1)**, 2016, 133-174.

[16] V.S. Guliyev, M.N. Omarova, *Multilinear singular and fractional integral operators on generalized weighted Morrey spaces*, Azerb. J. Math., **5(1)**, 2015, 104-132.

[17] V.S. Guliyev, R.V. Guliyev, M.N. Omarova, *Riesz transforms associated with Schrödinger operator on vanishing generalized Morrey spaces*, Appl. Comput. Math., **17(1)**, 2018, 56-71.

[18] D.I. Hakim, Y. Sawano, T. Shimomura, *Boundedness of generalized fractional integral operators from the Morrey space $L_{1,\phi}(X;\mu)$ to the Campanato space $\mathcal{L}_{1,\phi}(X;\mu)$ over non-doubling measure spaces*, Azerb. J. Math., **6(2)**, 2016, 117-122.

[19] F. John, L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math., **14**, 1961, 415-426.

[20] S. Janson, *Mean oscillation and commutators of singular integral operators*, Ark. Mat., **16**, 1978, 263-270.

[21] M. Izuki, Y. Sawano, *Variable Lebesgue norm estimates for BMO functions*, Czechoslovak Math. J., **62(3)**, 2012, 717-727.

[22] V. Kokilashvilli, M. Krbec, *Weighted Inequalities in Lorentz and Orlicz spaces*, World Scientific, 1991.

[23] M.A. Krasnoselskii, Ya.B. Rutickii, *Convex Functions and Orlicz Spaces*, English translation P. Noordhoff Ltd., Groningen, 1961.

[24] C. Miranda, *Sulle equazioni ellittiche del secondo ordine di tipo non variazionale, a coefficienti discontinui*, Ann. Math. Pura E Appl., **63(4)**, 1963, 353-386.

[25] T. Mizuhara, *Boundedness of some classical operators on generalized Morrey spaces*, Harmonic Analysis, ICM 90 Statellite Proceedings, Springer, Tokyo, 1991, 183-189.

[26] C.B. Morrey, *On the solutions of quasi-linear elliptic partial differential equations*, Trans. Amer. Math. Soc., **43**, 1938, 126-166.

[27] E. Nakai, *Hardy-Littlewood maximal operator, singular integral operators, and the Riesz potential on generalized Morrey spaces*, Math. Nachr., **166**, 1994, 95-103.

[28] E. Nakai, *Generalized fractional integrals on Orlicz-Morrey spaces*, In: Banach and Function Spaces, (Kitakyushu, 2003), Yokohama Publishers, Yokohama, (2004), 323-333.

[29] E. Nakai, *Orlicz-Morrey spaces and Hardy–Littlewood maximal function*, Studia Math., **188**, 2008, 193-221.

[30] E. Nakai, *Orlicz-Morrey spaces and their preduals*, in: Banach and Function spaces (Kitakyushu, 2009); Yokohama Publ. Yokohama (2011), 187-205.

[31] L.E. Persson, M.A. Ragusa, N. Samko, P. Wall, *Commutators of Hardy operators in vanishing Morrey spaces*, AIP Conf. Proc. 1493, 859 (2012); http://dx.doi.org/10.1063/1.4765588.

[32] M.A. Ragusa, *Commutators of fractional integral operators on vanishing-Morrey spaces*, J. Global Optim., **40(1-3)**, 2008, 361-368.

[33] N. Samko, *Maximal, Potential and Singular Operators in Vanishing Generalized Morrey Spaces*, J. Global Optim., **57**, 2013, 1385–1399.

[34] Y. Sawano, S. Sugano, H. Tanaka, *Orlicz-Morrey spaces and fractional operators*, Potential Anal., **36(4)**, 2012, 517-556.

[35] C. Vitanza, Functions with vanishing Morrey norm and elliptic partial differential equations. In: Proceedings of Methods of Real Analysis and Partial Differential Equations, Capri, Springer, 1990, 147-150.

[36] C. Vitanza, *Regularity results for a class of elliptic equations with coefficients in Morrey spaces*, Ricerche di Matematica, **42(2)**, 1993, 265-281.

Mehriban N. Omarova
Baku State University, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Baku, Azerbaijan
E-mail: mehriban_omarova@yahoo.com

Received 25 January 2020
Accepted 28 March 2020