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# Weighted Riesz Bounded Variation Spaces and the Nemytskii operator

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**Abstract.** We define a weighted version of the Riesz bounded variation space. We show that a generalization of the Riesz theorem relating these spaces to the Sobolev space $W^{1,p}(I)$ holds for weighted Riesz bounded variation spaces when the weight belongs to the Muckenhoupt class. As an application, for weights belonging to the Muckenhoupt class, we characterize the globally Lipschitz Nemytskii operators acting in the weighted Riesz bounded variation spaces.

**Key Words and Phrases**: Banach function spaces, variable Lebesgue spaces, $A_p$ weights, Riesz bounded variation spaces, Nemytskii operators.

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## 1. Introduction

In [9], F. Riesz proved that given a closed interval $I \subset \mathbb{R}$ and $1 < p < \infty$, an absolutely continuous function $f$ belongs to $W^{1,p}(I)$ if and only if

$$\sup \sum_j \frac{\left|f(x_j) - f(x_{j-1})\right|^p}{(x_j - x_{j-1})^{p-1}} < \infty, \tag{1}$$

where the supremum is taken over all finite partitions of $I$. Further he proved that this quantity is comparable to $\|f'\|_p^p$. A function for which (1) holds is said to be of Riesz bounded variation, and the collection of all such functions is the Banach space $\mathbf{RBV}^p(I)$. Note that when $p = 1$, this space reduces to the space $\mathbf{BV}(I)$ of functions of bounded variation. This notion of variation and the result by F. Riesz have been generalized to other function spaces, see e.g. [1, 4, 6, 8].

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The goal of this paper is to generalize (1) to define a weighted Riesz bounded variation space. Given a weight $w$, we consider the weighted $p$ variation

$$\sup \sum_j \left( \frac{|f(x_j) - f(x_{j-1})|}{|[x_{j-1}, x_j]|} \right)^p w([x_{j-1}, x_j]) < \infty.$$

We will use this quantity to define a weighted space $\mathbf{RBV}_w^p(I)$ and prove that a weighted version of the Riesz theorem holds whenever the weights belong to the Muckenhoupt class.

The paper is organized as follows: In Section 2 we give our notation and a few preliminary definitions. In Section 3 we define $\mathbf{RBV}_w^p(I)$, show that it is a Banach function space, and give some elementary embedding theorems. In Section 4 we prove the weighted version of the Riesz theorem: functions in $\mathbf{RBV}_w^p(I)$ are in $\mathbf{AC}(I)$ and if $w$ belongs to the Muckenhoupt class $A_p$, then their derivatives are in the weighted space $L^p(w)$. Finally, in Section 5 we give an application of the Riesz theorem: we characterize the globally Lipchitz Nemytskii operators defined on the weighted Riesz variation space $\mathbf{RBV}_w^p(I)$. Our result is related to the characterization of Nemytskii operators in other function spaces: see, for example, [3, 5].

## 2. Preliminaries

Given $1 < p < \infty$, the conjugate exponent of $p$ is the number $q$ satisfying the relation $1/p + 1/q = 1$. When $A \leqslant cB$ for some constant $c > 0$, we write $A \lesssim B$. If, in addition $B \leqslant CA$, for $C > 0$, we use the notation $A \approx B$.

By a weight we mean a non-negative, locally integrable real-valued function. We shall denote the weights by $w$ and $\sigma$. Given a weight $w$ and any measurable set $E \subset \mathbb{R}^n$, we define

$$w(E) = \int_E w\, dx.$$

If $0 < |E| < \infty$, where $|E|$ is the Lebesgue measure of $E$, we define the average of $w$ on $E$ by

$$\fint_E w \,\mathrm{d}x = \frac{1}{|E|} \int_E w \,\mathrm{d}x.$$

Given a weight $w$, let $I \subset \mathbb{R}$ be an interval. The space of all measurable real-valued functions $f$ that satisfy

$$\|f\|_{L^p_w} = \left( \int_I |f(x)|^p w \,\mathrm{d}x \right)^{1/p} < \infty,$$

is called the weighted Lebesgue space $L^p_w(I)$. We recall the definition of the Muckenhoupt class $A_p$, introduced by Muckenhoupt in [7]. The Muckenhoupt $A_p$ class plays a central role in the study of weighted spaces and weighted norm inequalities. See [2] for further information and references.

**Definition 1.** *Given $1 < p, q < \infty$, assume that $1/p + 1/q = 1$. A non-negative locally integrable function $w$ is say to be in the Muckenhoupt class $A_p$ if $0 < w(x) < \infty$ a.e. and*

$$[w]_{A_p} = \sup_Q \left( \fint_Q w \, \mathrm{d}x \right) \left( \fint_Q w^{1-q} \, \mathrm{d}x \right)^{p-1} < \infty, \tag{2}$$

*where the supremum is taken over all the cubes $Q \subset \mathbb{R}^n$.*

Given $w \in A_p$, by $\sigma = w^{1-q}$ we denote the dual weight of $w$. Observe that $w \in A_p$ implies that $\sigma \in A_q$. Moreover, it follows from the Hölder inequality that

$$1 \leqslant \left( \fint_Q w \, \mathrm{d}x \right) \left( \fint_Q w^{1-q} \, \mathrm{d}x \right)^{p-1} \tag{3}$$

for every cube $Q \subset \mathbb{R}^n$.

Let $I = [a, b]$ be a closed interval, and let $\{x_j\}_{j=0}^n$ satisfy

$$a = x_0 < x_1 < x_2 < \cdots < x_n = b.$$

We define the finite partition $\Pi = \{\pi_j\}_{j=1}^n$ to be the collection of closed intervals $\pi_j = [x_{j-1}, x_j]$. For simplicity, from now on we write $\Delta f(\pi_j) = f(x_j) - f(x_{j-1})$.

A function $f$ is in the Lipschitz class $\mathbf{Lip}(I)$ if for any points $x$, $y \in I$, $|f(x) - f(y)| \leqslant C|x-y|$. A function $f$ is absolutely continuous, denoted $f \in \mathbf{AC}(I)$, if for every $\epsilon > 0$, there exists $\delta > 0$ such that given any finite collection $\{(a_j, b_j)\}_{j=1}^n$ in $I$,

$$\sum_{j=1}^n (b_j - a_j) < \delta \quad \text{implies} \quad \sum_{j=1}^n |f(b_j) - f(a_j)| < \epsilon.$$

For $1 < p < \infty$ we have the well-known inclusions

$$\mathbf{Lip}(I) \subset \mathbf{RBV}^p(I) \subset \mathbf{AC}(I) \subset \mathbf{BV}(I).$$

## 3. Weighted Riesz bounded variation

In this section we define the weighted Riesz variation, prove that the collection of functions of weighted Riesz bounded variation form a Banach space, and we prove some embedding results relating it to Lipschitz functions and functions of bounded variation.

**Definition 2.** *Given $1 < p < \infty$, a weight $w$ and $I = [a, b]$, a function $f$ is said to be in the weighted Riesz bounded variation space $\mathbf{RBV}_w^p(I)$ if*

$$\mathbf{V}_w^p(f, I) = \sup_{\Pi} \sum_{\pi_j \in \Pi} \left( \frac{|\Delta f(\pi_j)|}{|\pi_j|} \right)^p w(\pi_j) < \infty,$$

*where the supremum is taken over all finite partitions $\Pi$ of $[a, b]$.*

Observe that if $w \in A_p$, the definition above can be modified up to a constant depending on $p$ as follows:

$$\begin{aligned} \mathbf{V}_w^p(f, I) &= \sup_{\Pi} \sum_{\pi_j \in \Pi} \left( \frac{|\Delta f(\pi_j)|}{|\pi_j|} \right)^p w(\pi_j) \\ &\approx \sup_{\Pi} \sum_{\pi_j \in \Pi} \frac{|\Delta f(\pi_j)|^p}{\sigma(\pi_j)^{p-1}}, \end{aligned}$$

where $\sigma = w^{1-q}$. As in the classical case, functions in $\mathbf{RBV}_w^p(I)$ are (uniformly) continuous.

**Lemma 1.** *Given $w \in A_p$ and $1 < p < \infty$, then*

$$\mathbf{RBV}_w^p(I) \subset \mathbf{C}(I).$$

*Proof.* Given a sequence $\{x_k\}_{k \in \mathbb{N}}$ in $I$ converging to $x \in I$, by an abuse of notation, let $[x, x_k]$ denote the closed interval with endpoints $x$ and $x_k$ and $\sigma$ the dual weight of $w$. Then for each $x_k$ we have

$$|f(x) - f(x_k)|^p = \frac{|f(x) - f(x_k)|^p}{\sigma([x, x_k])^{p-1}} \sigma([x, x_k])^{p-1} \lesssim \mathbf{V}_w^p(f, I) \sigma([x, x_k])^{p-1} \longrightarrow 0;$$

since this is true for every such sequence, $f$ is a continuous function. ◀

Though the functional $\mathbf{V}_w^p$ is not a norm, if we define

$$\|f\|_{\mathbf{RBV}_w^p} = |f(a)| + \mathbf{V}_w^p(f, I)^{1/p}, \tag{4}$$

then this is a norm on $\mathbf{RBV}_w^p(I)$, and with respect to this norm it is a Banach space.

**Theorem 1.** *Given $w \in A_p$, let $1 < p < \infty$. Then, the space $\mathbf{RBV}_w^p(I)$ with norm $\|\cdot\|_{\mathbf{RBV}_w^p}$ is a Banach space.*

*Proof.* We will first prove that $\mathbf{RBV}_w^p(I)$ is a normed linear space. Suppose $f = 0$ on $I$; then it is immediate that $\|f\|_{\mathbf{RBV}_w^p} = 0$. On the other hand, if $\|f\|_{\mathbf{RBV}_w^p} = 0$, then we have $f(a) = 0$ and for any $x \in I$, $x > a$,

$$0 = \mathbf{V}_w^p(f, I) \geqslant \left(\frac{|f(a) - f(x)|}{|[a, x]|}\right)^p w([x, a]).$$

Therefore, $f(x) = f(a) = 0$ for all $x \in I$.

Next, fix $f, g \in \mathbf{RBV}_w^p(I)$. Let $\Pi$ be a finite partition of $I$ and let $\tilde{f}_j = \frac{\Delta f(\pi_j)}{|\pi_j|}$, $\tilde{g}_j = \frac{\Delta g(\pi_j)}{|\pi_j|}$. By the Minkowski inequality in $L^p(w, I)$ and the Minkowski inequality in the sequence space $\ell_p$ we estimate as follows:

$$\begin{aligned}
&\sum_{\pi_j \in \Pi} \left(\frac{\left|\Delta(f+g)(\pi_j)\right|}{|\pi_j|}\right)^p w(\pi_j) \\
&= \sum_{\pi_j \in \Pi} \left(\frac{\left|\Delta f(\pi_j) + \Delta g(\pi_j)\right|}{|\pi_j|}\right)^p w(\pi_j) = \sum_{\pi_j \in \Pi} \left|\tilde{f}_j + \tilde{g}_j\right|^p w(\pi_j) \\
&= \sum_{\pi_j \in \Pi} \int_{\pi_j} \left|\tilde{f}_j + \tilde{g}_j\right|^p w(t)\, \mathrm{d}t = \sum_{\pi_j \in \Pi} \left\|\tilde{f}_j + \tilde{g}_j\right\|^p_{L_w^p(\pi_j)} \\
&\leqslant \sum_{\pi_j \in \Pi} \left(\left\|\tilde{f}_j\right\|_{L_w^p(\pi_j)} + \left\|\tilde{g}_j\right\|_{L_w^p(\pi_j)}\right)^p \\
&\leqslant \left[\left(\sum_{\pi_j \in \Pi} \left\|\tilde{f}_j\right\|^p_{L_w^p(\pi_j)}\right)^{1/p} + \left(\sum_{\pi_j \in \Pi} \left\|\tilde{g}_j\right\|^p_{L_w^p(\pi_j)}\right)^{1/p}\right]^p \\
&= \left[\left(\sum_{\pi_j \in \Pi} \int_{\pi_j} \left|\tilde{f}_j\right|^p w(t)\, \mathrm{d}t\right)^{1/p} + \left(\sum_{\pi_j \in \Pi} \int_{\pi_j} \left|\tilde{g}_j\right|^p w(t)\, \mathrm{d}t\right)^{1/p}\right]^p.
\end{aligned}$$

Thus we have

$$\left[\sum_{\pi_j \in \Pi} \left(\frac{\left|\Delta(f+g)(\pi_j)\right|}{|\pi_j|}\right)^p w(\pi_j)\right]^{1/p} \leqslant \left(\mathbf{V}_w^p(f, I)\right)^{1/p} + \left(\mathbf{V}_w^p(g, I)\right)^{1/p}, \tag{5}$$

from which it follows that

$$\|f + g\|_{\mathbf{RBV}_w^p} \leqslant \|f\|_{\mathbf{RBV}_w^p} + \|g\|_{\mathbf{RBV}_w^p}.$$

Similarly, if we fix $f \in \mathbf{RBV}_w^p(I)$ and $a \in \mathbb{R}$, then

$$\sum_{\pi_j \in \Pi} \left( \frac{|\Delta(af)(\pi_j)|}{|\pi_j|} \right)^p w(\pi_j) = |a| \sum_{\pi_j \in \Pi} \left( \frac{|\Delta(f)(\pi_j)|}{|\pi_j|} \right)^p w(\pi_j),$$

and so $\|af\|_{\mathbf{RBV}_w^p} = |a|\|f\|_{\mathbf{RBV}_w^p}$. Thus $\mathbf{RBV}_w^p(I)$ is a normed linear space.

We now prove that it is a Banach space. Fix a Cauchy sequence $\{f_j\}_{j\in\mathbb{N}}$ in $\mathbf{RBV}_w^p(I)$. Then, given a partition $\Pi$ of $[a,b]$, for any $\epsilon > 0$ there exists $N \geqslant 0$ such that for $n$, $m \geqslant N$,

$$\sum_{\pi_j \in \Pi} \left( \frac{|\Delta(f_n - f_m)(\pi_j)|}{|\pi_j|} \right)^p w(\pi_j) \leqslant \mathbf{V}_w^p\,(f_n - f_m, I) < \epsilon,$$

and $|f_n(a) - f_m(a)| < \epsilon$. Therefore, the sequence $\{f_n(a)\}$ converges. Similarly, for $x > a$,

$$\begin{aligned} |f_n(x) - f_m(x)|^p &\leqslant 2^p |f_n(a) - f_m(a)|^p + 2^p \frac{|\Delta(f_n - f_m)([a,x])|^p}{\sigma([a,x])^{p-1}} \sigma([a,x])^{p-1} \\ &\lesssim 2^p |f_n(a) - f_m(a)|^p + 2^p \mathbf{V}_w^p(f_n - f_m, I)\sigma(I)^{p-1}. \end{aligned}$$

Therefore, the sequence $\{f_n\}$ is uniformly Cauchy, and so by Lemma 1 it converges uniformly to a continuous function $f$. Consequently, we have for $n \geqslant N$,

$$\sum_{\pi_j \in \Pi} \left( \frac{|\Delta(f - f_m)(\pi_j)|}{|\pi_j|} \right)^p w(\pi_j) < \epsilon;$$

since this is true for all such partitions $\Pi$, we have

$$\mathbf{V}_w^p\,(f - f_m, I) \leqslant \epsilon.$$

Therefore, we must have

$$\|f - f_m\|_{\mathbf{RBV}_w^p} \longrightarrow 0, \text{ as } m \to 0.$$

By the triangle inequality,

$$\|f\|_{\mathbf{RBV}_w^p} \leqslant \|f - f_m\|_{\mathbf{RBV}_w^p} + \|f_m\|_{\mathbf{RBV}_w^p} < \infty,$$

so $f \in \mathbf{RBV}_w^p(I)$. Thus, $\mathbf{RBV}_w^p(I)$ is a Banach space. ◀

We now consider the relationship between $\mathbf{RBV}_w^p(I)$ and the spaces $\mathbf{Lip}(I)$ and $\mathbf{BV}(I)$.

**Proposition 1.** *Given a weight $w \in A_p$ and $1 < p < \infty$, then*

$$\mathbf{RBV}_w^p(I) \subseteq \mathbf{BV}(I). \tag{6}$$

*Proof.* Let $\Pi = \{\pi_j\}$ be a finite partition of $I = [a,b]$ and $\sigma$ the dual weight of $w$. Then by the Hölder inequality we have

$$\begin{aligned}
\sum_{\pi_j} |\Delta f(\pi_j)| &= \sum_{\pi_j} |\Delta f(\pi_j)| \frac{\sigma(\pi_j)^{(p-1)/p}}{\sigma(\pi_j)^{(p-1)/p}} \\
&\lesssim \left[ \sum_{\pi_j \in \Pi} \left( \frac{|\Delta f(\pi_j)|}{|\pi_j|} \right)^p w(\pi_j) \right]^{1/p} \left( \sum_{\pi_j \in \Pi} \sigma(\pi_j)^{(p-1)q/p} \right)^{1/q} \\
&\lesssim \mathbf{V}_w^p(f,I)^{1/p} \left( \sum_{\pi_j \in \Pi} \sigma(\pi_j) \right)^{1/q} \leqslant \mathbf{V}_w^p(f,I)^{1/p} \sigma(I)^{1/q} < \infty.
\end{aligned}$$

If we take the supremum over all the finite partition $\Pi$ of $I$, we get the desired result. ◀

**Proposition 2.** *Fix $1 < p < \infty$ and let $1/p + 1/q = 1$. Given a weight $w \in A_p$, then*

$$\mathbf{Lip}(I) \subseteq \mathbf{RBV}_w^p(I).$$

*Proof.* Let $\Pi = \{\pi_j\}$ be a finite partition of $[a,b]$. Then, by the Lipschitz condition we have

$$\sum_{\pi_j \in \Pi} \left( \frac{|\Delta f(\pi_j)|}{|\pi_j|} \right)^p w(\pi_j) \lesssim \sum_{\pi_j \in \Pi} w(\pi_j) \lesssim w(I) < \infty.$$

If we take the supremum over all the finite partition $\Pi$ of $I$, we get the desired result. ◀

Finally, we show that the spaces $\mathbf{RBV}_w^p$ are naturally embedded in one another.

**Proposition 3.** *Given a weight $w \in A_p$, let $1 < p \leqslant q < \infty$. Then we have*

$$\mathbf{RBV}_w^q(I) \subseteq \mathbf{RBV}_w^p(I). \tag{7}$$

*Proof.* The case $p = 1$ follows from Proposition 1, so we may assume that $1 < p < q < \infty$. Let

$$r = \frac{q}{p}, \quad s = \frac{q}{q-p},$$

and observe that $1/r + 1/s = 1$. Therefore, by the Hölder inequality we have

$$\begin{aligned}
\sum_{\pi_j \in \Pi} \frac{\left|\Delta f(\pi_j)\right|^p}{\sigma(\pi_j)^{p-1}} &= \sum_{\pi_j \in \Pi} \frac{\left|\Delta f(\pi_j)\right|^p}{\sigma(\pi_j)^{p-1+1/s}} \sigma(\pi_j)^{1/s} \\
&\leqslant \left( \sum_{\pi_j \in \Pi} \frac{\left|\Delta f(\pi_j)\right|^{pr}}{\sigma(\pi_j)^{(p-1+1/s)r}} \right)^{1/r} \left( \sum_{\pi_j \in \Pi} \sigma(\pi_j) \right)^{1/s} \\
&\lesssim \left[ \sum_{\pi_j \in \Pi} \left( \frac{\left|\Delta f(\pi_j)\right|}{|\pi_j|} \right)^q w(\pi_j) \right]^{1/r} \sigma(I)^{1/s} < \infty.
\end{aligned}$$

Hence, $\mathbf{V}_w^p(f, I) < \infty$ . ◀

## 4. The weighted Riesz theorem

In this section we state and prove a weighted generalization of the Riesz theorem. As a first step, we establish that functions in $\mathbf{RBV}_w^p(I)$ are absolutely continuous.

**Proposition 4.** *Given a weight $w \in A_p$ and $1 < p < \infty$, then*

$$\mathbf{RBV}_w^p(I) \subset \mathbf{AC}(I).$$

*Proof.* Since $\sigma \in L^1(I)$, by the continuity of the integral, for any $\epsilon > 0$, there exists $\delta > 0$ such that if $|E| < \delta$, then $\sigma(E) < \epsilon$. Fix any collection $\{(a_j, b_j)\}_{j=1}^n$ of disjoint open intervals in $I$ such that

$$\sum_{j=1}^n (b_j - a_j) < \delta.$$

Let

$$E = \bigcup_{j=1}^n [a_j, b_j].$$

Then $\sigma(E) < \epsilon$. If we let $1/p + 1/q = 1$, then by Hölder's inequality in sequence spaces,

$$\sum_{j=1}^{n} |f(b_j) - f(a_j)| = \sum_{j=1}^{n} \frac{|f(b_j) - f(a_j)|}{\sigma([a_j, b_j])^{1/q}} \sigma([a_j, b_j])^{1/q}$$

$$\lesssim \left[ \sum_{j=1}^{n} \left( \frac{|f(b_j) - f(a_j)|}{|[a_j, b_j]|} \right)^p w([a_j, b_j]) \right]^{1/p} \left( \sum_{j=1}^{n} \sigma([a_j, b_j]) \right)^{1/q} \leqslant \mathbf{V}_w^p(f, I)^{1/p} \epsilon^{1/q}.$$

Since $\epsilon > 0$ is arbitrary, it follows that $f \in \mathbf{AC}(I)$. ◀

We can now state and prove our weighted version of the Riesz theorem.

**Theorem 2.** *Let $1 < p < \infty$ and let $1/p + 1/q = 1$. Given a weight $w \in A_p$, we have $f \in \mathbf{RBV}_w^p(I)$ if and only if $f \in \mathbf{AC}(I)$ and $f' \in L^p(w)$. Moreover,*

$$\mathbf{V}_w^p(f, I)^{1/p} \approx \|f'\|_{L^p(w)}. \tag{8}$$

*Proof.* Suppose first that $f \in \mathbf{AC}(I)$ and $f' \in L^p(w)$. Fix a finite partition $\Pi = \{\pi_j\}$ of $I = [a, b]$. Using the fact that $f \in \mathbf{AC}(I)$, we can estimate as follows:

$$\begin{aligned}
\left( \frac{|\Delta f(\pi_j)|}{|\pi_j|} \right)^p w(\pi_j) &\lesssim \left| \int_{\pi_j} f'(t)\,\mathrm{d}t \right|^p \sigma\left(\pi_j\right)^{1-p} \\
&\leqslant \left( \int_{\pi_j} |f'(t)| w^{1/p} w^{-1/p}\,\mathrm{d}t \right)^p \sigma\left(\pi_j\right)^{1-p} \\
&\leqslant \left[ \left( \int_{\pi_j} |f'(t)|^p w\,\mathrm{d}t \right)^{1/p} \left( \int_{\pi_j} \sigma\,\mathrm{d}t \right)^{1/q} \right]^p \sigma\left(\pi_j\right)^{1-p} \\
&= \left( \int_{\pi_j} |f'(t)|^p w\,\mathrm{d}t \right) \sigma\left(\pi_j\right)^{p/q} \sigma\left(\pi_j\right)^{1-p} = \int_{\pi_j} |f'(t)|^p w\,\mathrm{d}t.
\end{aligned}$$

If we sum over all the sub-intervals, we obtain

$$\sum_{\pi_j \in \Pi} \left( \frac{|\Delta f(\pi_j)|}{|\pi_j|} \right)^p w(\pi_j) \leqslant \int_I |f'(t)|^p w\,\mathrm{d}t,$$

which implies that

$$\mathbf{V}_w^p(f, I) \leqslant \|f'\|_{L^p(w)}^p.$$

To prove the reverse inequality, now let $f \in \mathbf{RBV}_w^p(I)$. Define the regular partition $\Pi_N = \{[t_{k,N}, t_{k+1,N}]\}_{k=0}^{N-1}$, where $t_{k,N} = a + \frac{k}{N}(b-a)$. For brevity, let $\pi_{j,N} = [t_{j-1,N}, t_{j,N}]$. By Proposition 4, $f \in \mathbf{AC}(I)$, so $f'$ exists in $L^1(I)$. Therefore, we may define

$$g_N(t) = \sum_{\pi_{j,N} \in \Pi_N} \fint_{\pi_{j,N}} f'(x)\,\mathrm{d}x \chi_{\pi_{j,N}}(t).$$

Then, by the Lebesgue differentiation theorem we obtain

$$\lim_{N\to\infty} g_N(t) = f'(t)$$

for almost every $t \in I$. Consequently, by Fatou's Lemma applied with respect to the measure $w\,\mathrm{d}x$ and using the fact that $w \in A_p$, we estimate as follows:

$$\begin{aligned}
\int_I |f'(t)|^p w\,\mathrm{d}t &\leqslant \liminf_{N\to\infty} \int_I |g_N(t)|^p w\,\mathrm{d}t \\
&= \liminf_{N\to\infty} \sum_{\pi_{j,N} \in \Pi_N} \int_{\pi_{j,N}} |g_N(t)|^p w\,\mathrm{d}t \\
&\leqslant \liminf_{N\to\infty} \sum_{\pi_{j,N} \in \Pi_N} \left( \frac{|\Delta f(\pi_{j,N})|}{|\pi_{j,N}|} \right)^p \int_{\pi_{j,N}} w(t)\,\mathrm{d}t \\
&= \liminf_{N\to\infty} \sum_{\pi_{j,N} \in \Pi_N} \left( \frac{|\Delta f(\pi_{j,N})|}{|\pi_{j,N}|} \right)^p w\left(\pi_{j,N}\right) \\
&\lesssim \mathbf{V}_w^p\,(f, I) < \infty.
\end{aligned}$$

This completes the proof of inequality (8). ◀

## 5. Nemytskii operator in weighted Riesz bounded variation spaces

In this section, as an application of Theorem 2 we characterize the globally Lipchitz Nemytskii operators defined on $\mathbf{RBV}_w^p(I)$. Observe that given $w \in A_p$,

$$|f(x)|^p = |f(x) - f(a)|^p + |f(a)| \lesssim V_w^p(f, I)\sigma(I)^{p-1} + |f(a)|$$

yields

$$\|f\|_\infty \lesssim V_w^p(f, I) + |f(a)|. \tag{9}$$

Using this fact we will be able to prove that $\mathbf{RBV}_w^p(I)$ is a Banach algebra. To do so we need the following lemma.

**Lemma 2** (Orlicz-Maligranda Criterion [4]). *Let* $(X,\|\cdot\|)$ *be a Banach space whose elements are bounded functions, which is closed under pointwise multiplication of functions and satisfies* $\|fg\| \leqslant \|f\|_\infty \|g\| + \|f\| \|g\|_\infty$. *Then* $(X, \|\!|\cdot|\!\|)$, *where* $\|\!|f|\!\| = \|f\|_\infty + \|f\|$, *is a Banach algebra.*

To apply this condition we need to show that $\mathbf{RBV}^p_w(I)$ is closed under pointwise multiplication.

**Proposition 5.** *Given* $w \in A_p$, *let* $f, g \in \mathbf{RBV}^p_w(I)$ *and* $1 < p < \infty$. *Then*

$$\|f \cdot g\|_{\mathbf{RBV}^p_w} \leqslant \|f\|_\infty \|g\|_{\mathbf{RBV}^p_w} + \|f\|_{\mathbf{RBV}^p_w} \|g\|_\infty .$$

*Proof.* Fix a finite partition $\Pi$ of $I$. By the triangle inequality (5) we get

$$\left(\sum_{\pi_j \in \Pi} \frac{\left|\Delta(f \cdot g)(\pi_j)\right|^p}{\sigma(\pi_j)^{p-1}}\right)^{1/p} \leqslant \left[\sum_{\pi_j \in \Pi} \frac{\left(\|g\|_\infty \left|\Delta f(\pi_j)\right| + \|f\|_\infty \left|\Delta g(\pi_j)\right|\right)^p}{\sigma(\pi_j)^{p-1}}\right]^{1/p}$$

$$\leqslant \|g\|_\infty \left(\sum_{\pi_j \in \Pi} \frac{\left|\Delta f(\pi_j)\right|^p}{\sigma(\pi_j)^{p-1}}\right)^{1/p} + \|f\|_\infty \left(\sum_{\pi_j \in \Pi} \frac{\left|\Delta g(\pi_j)\right|^p}{\sigma(\pi_j)^{p-1}}\right)^{1/p}$$

$$\leqslant \|g\|_\infty V^p_w(f, I) + \|f\|_\infty V^p_w(g, I).$$

Since $\Pi$ is arbitrary and $|(fg)(a)| \leqslant \|f\|_\infty |g(a)| + \|g\|_\infty |f(a)|$, we get the desired inequality. ◀

By the Orlicz-Maligranda criterion we immediately get the following corollary.

**Corollary 1.** *The Banach space* $\mathbf{RBV}^p_w(I)$ *endowed with the norm*

$$\|\!|f|\!\|_{\mathbf{RBV}^p_w} = \|f\|_\infty + \|f\|_{\mathbf{RBV}^p_w} ,$$

*is a Banach algebra.*

Note that by Corollary 1 and (9) the norms $\|\cdot\|_{\mathbf{RBV}^p_w}$ and $\|\!|\cdot|\!\|_{\mathbf{RBV}^p_w}$ are equivalent.

We now recall the definition of a *Nemytskii operator*, also known as a *superposition operator.*

**Definition 3.** *Let* $I \subset \mathbb{R}$ *be a closed interval. We say that a function* $f : I \times \mathbb{R} \longrightarrow I$ *is a Carathéodory function if:*

*(1)* $f(\cdot, x)$ *is a Lebesgue measurable function for all* $x \in \mathbb{R}$*;*

*(2)* $f(t, \cdot)$ *is a continuous function a.e. in* $\mathbb{R}$ *for all* $t \in I$*.*

**Definition 4** (Nemytskii operator)**.** *Let* $f : I \times \mathbb{R} \longrightarrow I$ *be a Carathéodory function. The operator* $F_f$*, defined by* $(F_f\varphi)(t) = f(t, \varphi(t))$*, for all* $t \in I$ *and* $\varphi : I \longrightarrow \mathbb{R}$*, is called the Nemytskii operator based on* $f$*.*

To state our main result, we give one more definition.

**Definition 5.** *We say that an operator* $T : X \longrightarrow Y$*, where* $X$ *and* $Y$ *are normed spaces, is globally Lipschitz if there exists a global constant* $C$ *such that*

$$\|Tx_1 - Tx_2\|_Y \leqslant C\|x_1 - x_2\|_X\,,$$

*for all* $x_1, x_2 \in X$*.*

We can now characterize globally Lipschitz Nemytskii operators acting in weighted Riesz bounded variation spaces, where the weight belongs to the Muckenhoupt class.

**Theorem 3.** *Let* $w \in A_p$*. Then the Nemytskii operator* $F_f$ *maps* $\mathbf{RBV}^p_w(I)$ *into* $\mathbf{RBV}^p_w(I)$ *and is globally Lipschitz if and only if there are functions* $g, h \in \mathbf{RBV}^p_w(I)$ *such that*

$$f(t, y) = g(t)y + h(t), \quad t \in I, y \in \mathbb{R}. \tag{10}$$

*Proof.* Let $g, h \in \mathbf{RBV}^p_w(I)$ and define $f(t, y)$ by (10). Hereafter we can use either $|||\cdot|||_{\mathbf{RBV}^p_w}$ or $\|\cdot\|_{\mathbf{RBV}^p_w}$ since they are equivalent norms. By Corollary 1, Proposition 5 we have

$$\begin{aligned}|||F_f(u)|||_{\mathbf{RBV}^p_w} &= |||g \cdot u + h|||_{\mathbf{RBV}^p_w}\\ &\leqslant |||g \cdot u|||_{\mathbf{RBV}^p_w} + |||h|||_{\mathbf{RBV}^p_w}\\ &\leqslant |||g|||_{\mathbf{RBV}^p_w}|||u|||_{\mathbf{RBV}^p_w} + |||h|||_{\mathbf{RBV}^p_w} < \infty,\end{aligned}$$

from which it follows that $F_f : \mathbf{RBV}^p_w(I) \longrightarrow \mathbf{RBV}^p_w(I)$.

Now fix $u_1, u_2 \in \mathbf{RBV}^p_w(I)$. Again, by Corollary 1 we estimate as follows:

$$\begin{aligned}|||F_f(u_1) - F_f(u_2)|||_{\mathbf{RBV}^p_w} &= |||f(\cdot, u_1) - f(\cdot, u_2)|||_{\mathbf{RBV}^p_w}\\ &= |||g\,(u_1 - u_2)\,|||_{\mathbf{RBV}^p_w}\\ &\lesssim |||g|||_{\mathbf{RBV}^p_w}|||u_1 - u_2|||_{\mathbf{RBV}^p_w}\\ &\lesssim |||u_1 - u_2|||_{\mathbf{RBV}^p_w}.\end{aligned}$$

Therefore, $F_f$ is globally Lipschitz.

To prove the converse, suppose $F_f$ maps $\mathbf{RBV}_w^p(I)$ into $\mathbf{RBV}_w^p(I)$ and is globally Lipschitz. Observe that given $y \in \mathbb{R}$, the function $x \mapsto f(x,y) \in \mathbf{RBV}_w^p(I)$. Consequently, $f(\cdot,y) \in \mathbf{AC}(I)$ by Corollary 4.

Let $t, t' \in [a,b]$, $t < t'$, $y_1, y_2, y_1', y_2' \in \mathbb{R}$. Define $u_1, u_2$ by

$$u_i(s) = \begin{cases} y_i & a \leqslant s \leqslant t, \\ \dfrac{y_i' - y_i}{t'-t}(s-t) + y_i & t < s \leqslant t', \\ y_i' & t' < s \leqslant b. \end{cases}$$

Since $u_1 - u_2 \in \mathbf{AC}(I)$, $(u_1-u_2)'$ exists *a.e.* and is given by

$$(u_1-u_2)'(s) = \begin{cases} 0 & a \leqslant s < t, \\ \dfrac{y_1' - y_1 - (y_2' - y_2)}{t'-t} & t \leqslant s \leqslant t', \\ 0 & t' < s \leqslant b, \end{cases}$$

and so

$$\begin{aligned} \left\|(u_1-u_2)'\right\|_{L^p(w)} &= \left(\int_t^{t'} \left|\frac{y_1'-y_1-y_2'+y_2}{t'-t}\right|^p w(s)\,\mathrm{d}s\right)^{1/p} \\ &\lesssim \left|\frac{y_1'-y_1-y_2'+y_2}{t'-t}\right| w([t,t'])^{1/p} \\ &\lesssim \left|\frac{y_1'-y_1-y_2'+y_2}{t'-t}\right| w(I)^{1/p}. \end{aligned}$$

Hence, by Theorem 2 we have

$$V_w^p(u_1-u_2)^{1/p} \lesssim \left|\frac{y_1'-y_1-y_2'+y_2}{t'-t}\right|. \tag{11}$$

Clearly, $F_f(u_1) - F_f(u_2) \in \mathbf{RBV}_w^p$, since $\mathbf{RBV}_w^p(I)$ is a normed linear space. Then by the global Lipschitz condition on $F_f$, we see that

$$\begin{aligned} \frac{\left|\big(F_f(u_1)-F_f(u_2)\big)(t') - \big(F_f(u_1)-F_f(u_2)\big)(t)\right|^p}{\sigma([t,t'])^{p-1}} &\lesssim \left\|F_f(u_1)-F_f(u_2)\right\|_{\mathbf{RBV}_w^p}^p \\ &\lesssim \|u_1-u_2\|_{\mathbf{RBV}_w^p}^p. \end{aligned}$$

Then by the definition of $u_1$ and $u_2$ and (11),

$$\frac{\left|f(t',y_1') - f(t',y_2') - f(t,y_1) + f(t,y_2)\right|}{\sigma([t,t'])^{1/q}} \lesssim \left|\frac{y_1'-y_1-y_2'+y_2}{t'-t}\right| + |y_1-y_2|.$$

Set $y_1' = w + z$, $y_2' = w$, $y_1 = z$, $y_2 = 0$. Then

$$|f(t', w + z) - f(t', w) + f(t, 0) - f(t, z)| \leqslant |z|\sigma([t, t'])^{1/q}.$$

Since $f(\cdot, x)$ is continuous, if we take the limit as $t' \to t$, we get

$$f(t, w + z) - f(t, 0) = \big(f(t, w) - f(t, 0)\big) + \big(f(t, z) - f(t, 0)\big).$$

Now define $P_t(\cdot) := f(t, \cdot) - f(t, 0)$; then $P_t$ is linear: $P_t(w + z) = P_t(w) + P_t(z)$. Since $P_t(\cdot)$ is continuous, the Cauchy functional equation has a unique continuous solution which is given by $P_t(y) = g(t)y$ with $g : [a, b] \to \mathbb{R}$. Now define the function $h$ on $[a, b]$ by $h(t) = f(t, 0)$. Then $h \in \mathbf{RBV}_w^p(I)$ and $f(t, y) = g(t)y + h(t)$. Since for $t \in I$,

$$f(t, 1) - f(t, 0) = \big(P_t(1) + f(t, 0)\big) - f(t, 0) = g(t),$$

we have $g \in \mathbf{RBV}_w^p(I)$. This completes the proof. ◀

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