Azerbaijan Journal of Mathematics
V. 10, No 2, 2020, July
ISSN 2218-6816

On the equation $(2^k - 1)(3^\ell - 1) = 5^m - 1$

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Abstract. In this paper, we solve completely the title diophantine equation. The main tools are linear forms of logarithm of algebraic numbers, reduction methods, modular arithmetic. We use Maple to carry out some calculations corresponding to the LLL algorithm, Baker-Davenport reduction, and certain approximations.

Key Words and Phrases: exponential diophantine equation, linear forms in logarithms.

2010 Mathematics Subject Classifications: 11A25, 11B39, 11J86

1. Introduction

The purpose of the paper is to solve completely the title diophantine equation. It can be considered as an equation between products of terms of two binary recurrences and the terms of a third binary recurrence. This type of problem is not unprecedented, it has already been studied before with other binary recurrences. See, for example, [3] where the authors found all Fibonacci numbers which are products of two Pell numbers, and all Pell numbers which are product of two Fibonacci numbers. Note that finding the common terms of two homogenous linear recurrences is one important question within number theory. Our result is the following.

Theorem 1. *The only positive integer triple* (k, ℓ, m) *which satisfies*

$$(2^k - 1)(3^\ell - 1) = 5^m - 1 \tag{1}$$

is $(k, \ell, m) = (2, 2, 2)$.

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Now we describe here the bird's eye view of the proof. First we use Baker method (Matveev's theorem, see Theorem 2) to have the upper bound $\min\{k, \ell\} < 0.26 \cdot 10^{12}(1 + \log x)$, where $x = \max\{k, \ell, m\}$. A second application gives the bound $3 \cdot 10^{26}$ on both k and ℓ. Then we apply the LLL algorithm for reduction, and obtain $\min\{k, \ell\} \leq 193$. This relatively small number via Matveev's theorem implies $k, \ell < 2 \cdot 10^{15}$. A repetition of LLL procedure now returns with $\min\{k, \ell\} \leq 113$, and then the Baker-Davenport method (Theorem 3) with $m \leq 256$. Finally, $k \leq 591$ and $\ell \leq 302$ follow, and a computer search completes the proof.

Assume that the positive integers $2 \leq a \leq b$ and c are fixed. The equation

$$(a^k - 1)(b^\ell - 1) = c^m - 1 \tag{2}$$

sometimes surely has infinitely many solutions, for instance if $a = 2$, $c = b^2$. Computer search shows that most often there is probably 0 or 1 solution, while $(2^k - 1)(5^\ell - 1) = 13^m - 1$ possesses at least two triples: $(k, \ell, m) = (2, 1, 1)$ and $(3, 2, 2)$. It is apparent from the proof of our theorem that the method is more general and usually able to handle equation (2) with a, b, c fixed.

2. Lemmas and linear forms in logarithms

In this section we introduce some notations and lemmas. We begin with

Lemma 1. *If equation (1) holds with $k, \ell \geq 2$, then*

$$5^m > 2\max\{2^k, 3^\ell\} > 2^k + 3^\ell.$$

Proof. The statement is obvious. ◄

Lemma 2. *If $k \geq 4, \ell \geq 3$ and m satisfy (1), then*

$$0.43k + 0.68\ell - 0.2 < m < 0.44k + 0.69\ell.$$

Proof. Clearly,

$$2^{k-0.1}3^{\ell-0.1} < 5^m < 2^k 3^\ell$$

holds, the left hand side is being valid for $k \geq 4$ and $\ell \geq 3$. Taking logarithms in the extreme sides of the above inequality and approximating both $(\log 2)/(\log 5)$ and $(\log 3)/(\log 5)$, we get the conclusion of the lemma. ◄

Lemma 3. *If the positive integers k, ℓ and m satisfy (1), then*

$$k \equiv \ell \equiv 2 \pmod 4, \qquad m \equiv 0 \pmod 2.$$

Proof. Consider (1) modulo 3. If k is odd, then the left hand side of (1) (denoted by LHS) is congruent to -1 modulo 3 which contradicts the right hand side (denoted by RHS). If k is even, then LHS is congruent to 0 modulo 3, and so is RHS. Consequently, m is even. Knowing that k is even, and considering the title equation modulo 5, we obtain the statement of the lemma. ◀

Lemma 4. *The only solution to (1) with $k=2$ or $\ell=2$ is $k=\ell=2$.*

Proof. For $k=2$, we get $3^{\ell+1}-5^m=2$. Since m is even, this reduces to the elliptic equation $3^r x^3-y^2=2$, where $r\in\{0,1,2\}$ is the residue class of $\ell+1$ modulo 3 and $(x,y)=(3^{(\ell+1-r)/3},5^{m/2})$.

For $\ell=2$, we get $2^{k+3}-7=5^m=(5^{m/2})^2$, a particular case of a Ramanujan's famous equation $2^n-7=y^2$, whose largest solution is $n=15$. ◀

We also need some results from the theory of lower bounds in non-zero linear forms in logarithms of algebraic numbers. We start by recalling Theorem 9.4 of [1], which is a modified version of a result of Matveev [5]. Let $\mathbb{L}$ be an algebraic number field of degree $d_{\mathbb{L}}$. Let $\eta_1,\eta_2,\ldots,\eta_l\in\mathbb{L}$ be not 0 or 1 and $d_1,\ldots,d_l$ be non-zero integers. We put

$$D=\max\{|d_1|,\ldots,|d_l|,3\},$$

and put

$$\Gamma=\prod_{i=1}^{l}\eta_i^{d_i}-1.$$

Let $A_1,\ldots,A_l$ be positive integers such that

$$A_j\geq h'(\eta_j):=\max\{d_{\mathbb{L}}h(\eta_j),|\log\eta_j|,0.16\},\quad \text{for}\quad j=1,\ldots l,$$

where for an algebraic number η of minimal polynomial

$$f(X)=a_0(X-\eta^{(1)})\cdots(X-\eta^{(k)})\in\mathbb{C}[X]$$

over the integers with positive a_0, we write $h(\eta)$ for its Weil height given by

$$h(\eta)=\frac{1}{k}\left(\log a_0+\sum_{j=1}^{k}\max\{0,\log|\eta^{(j)}|\}\right).$$

The following consequence of Matveev's theorem is Theorem 9.4 in [1].

Theorem 2. *If $\Gamma\neq 0$ and $\mathbb{L}\subseteq\mathbb{R}$, then*

$$\log|\Gamma|>-1.4\cdot 30^{l+3}l^{4.5}d_{\mathbb{L}}^2(1+\log d_{\mathbb{L}})(1+\log D)A_1A_2\cdots A_l.$$

We recall the Baker-Davenport reduction method (see [4, Lemma 5a]), which is useful to reduce the bounds arising from applying Theorem 2.

Theorem 3. *Let $\kappa \neq 0$ and μ be real numbers. Assume that M is a positive integer. Let P/Q be the convergent of the continued fraction expansion of κ such that $Q > 6M$, and put*

$$\xi = \|\mu Q\| - M \cdot \|\kappa Q\|,$$

where $\| \cdot \|$ denotes the distance from the nearest integer. If $\xi > 0$, then there is no solution of the inequality

$$0 < |m\kappa - n + \mu| < AB^{-k}$$

for positive integers m, n and k with

$$\frac{\log (AQ/\xi)}{\log B} \leq k \qquad \text{and} \qquad m \leq M.$$

3. The proof of Theorem 1

3.1. First bound on $\min\{k, \ell\}$, and then on k and ℓ

The case $k = 2$ or $\ell = 2$ has been treated in Lemma 4. From now on, by Lemma 3, we assume that $k \geq 6$ and $\ell \geq 6$. And we need to show that there is no such solution. Equation (1) is equivalent to

$$2^k 3^\ell = 5^m + 2^k + 3^\ell - 2, \tag{3}$$

and then to

$$\frac{2^k 3^\ell}{5^m} - 1 = \frac{2^k + 3^\ell - 2}{5^m}. \tag{4}$$

Clearly,

$$0 < \frac{2^k + 3^\ell - 2}{5^m} < \frac{2 \max\{2^k, 3^\ell\}}{5^m} < \frac{4}{2^{\min\{k,\ell\}}}. \tag{5}$$

The last inequality can be seen if one extends the middle fraction by $\min\{2^k, 3^\ell\}$, and then by (3) and Lemma 1 we have

$$\begin{aligned} \frac{2 \max\{2^k, 3^\ell\} \min\{2^k, 3^\ell\}}{5^m \min\{2^k, 3^\ell\}} &< \frac{2(5^m + 2^k + 3^\ell)}{5^m \min\{2^k, 3^\ell\}} < \frac{2(5^m + 2 \max\{2^k, 3^\ell\})}{5^m \min\{2^k, 3^\ell\}} \\ &< \frac{4 \cdot 5^m}{5^m \min\{2^k, 3^\ell\}}. \end{aligned}$$

Thus (5) follows.

Put $x=\max\{k,\ell,m\}$. Now apply Theorem 2 to the left hand side of (4) (which is positive) with $\eta_1=2$, $\eta_2=3$, $\eta_3=5$, and with $A_1=\log 2$, $A_2=\log 3$, $A_3=\log 5$. Thus,

$$\log\left(\frac{2^k3^\ell}{5^m}-1\right) > -1.4\cdot 30^6\cdot 3^{4.5}(1+\log x)\cdot\log 2\cdot\log 3\cdot\log 5.$$

Then we obtain

$$\frac{2^k3^\ell}{5^m}-1>\exp(-0.18\cdot 10^{12}(1+\log x)).$$

Combining this with (4) and (5), we obtain

$$\min\{k,\ell\}<c_1(1+\log x),$$

where $c_1:=0.26\cdot 10^{12}$.

Now let us distinguish two cases. First suppose $k\le\ell$, and consider the left hand side of

$$\frac{(2^k-1)3^\ell}{5^m}-1<\frac{2^k}{5^m}<\frac{2}{3^\ell}. \tag{6}$$

Matveev's theorem with the parameters

$\eta_1=2^k-1$, $\eta_2=3$, $\eta_3=5$; $b_1=1$, $h(\eta_1)<k\log 2<\log 2\cdot 0.26\cdot 10^{12}(1+\log x)$,

and $A_1=0.19\cdot 10^{12}(1+\log x))$, $A_2=\log 3$, $A_3=\log 5$ implies

$$\log\left(\frac{(2^k-1)3^\ell}{5^m}-1\right)>-0.49\cdot 10^{23}(1+\log x)^2. \tag{7}$$

Now Lemma 2 provides

$$x=\max\{\ell,m\}<\max\{\ell,0.44k+0.69\ell\}<1.13\ell.$$

Combining it with (6) and (7), we see

$$\ell\log 3-\log 2<0.49\cdot 10^{23}(1+\log(1.13\ell))^2.$$

Thus, $\ell<2\cdot 10^{26}$.

Assume now that $\ell\le k$. In a way similar to the previous arguments, from the inequality

$$\frac{(3^\ell-1)2^k}{5^m}-1<\frac{3^\ell}{5^m}<\frac{2}{2^k}, \tag{8}$$

applying Theorem 2, we obtain

$$\log\left(\frac{(3^\ell-1)2^k}{5^m}-1\right)>-0.47\cdot 10^{23}(1+\log x)^2.$$

Finally,

$$k \log 2 - \log 2 < 0.47 \cdot 10^{23}(1 + \log(1.13k))^2$$

holds, which yields $k < 3 \cdot 10^{26}$.

Now we record the results we have proved so far.

Proposition 1. *If (1) holds, then* $k, \ell < 3 \cdot 10^{26}$. *Furthermore,* $\min\{k, \ell\} < c_1(1 + \log x)$. *We recall that* $c_1 = 0.26 \cdot 10^{12}$, *and* $x = \max\{k, \ell, m\}$.

3.2. Reduction of the bounds and final computations

In this section, we reduce the bounds of Proposition 1. In order to apply the LLL algorithm for

$$\Gamma := k \log 2 + \ell \log 3 - m \log 5,$$

first we note that

$$0 < \frac{2^k 3^\ell}{5^m} - 1 < \frac{4}{2^{\min\{k,\ell\}}} < \frac{1}{2}$$

holds if $\min\{k, \ell\} \geq 3$. Thus, $0 < \Gamma < 1/2$. Hence,

$$\Gamma < \frac{8}{2^{\min\{k,\ell\}}}.$$

For the computational aspects of the application of LLL algorithm we refer to the book of H. Cohen [2], pp. 58-63, and the `LLL(lvect, integer)` command of the package IntegerRelation in Maple. We implemented the computations in Maple by following Cohen's approach. Recall the upper bound of Proposition 1 for k and ℓ. Furthermore, $m < 1.13 \cdot 3 \cdot 10^{26} < 3.5 \cdot 10^{26}$. We specified $X_1 = X_2 = X_3 = 3.5 \cdot 10^{26}$ (see [2] and the notation therein), and then $C = 10^{86}$ was fixed. Thus $Q = 2.45 \cdot 10^{53}$, $T = 5.25 \cdot 10^{26}$. Here we introduce $\lfloor w \rceil$ for denoting the nearest integer of $w \in \mathbb{R}$. The LLL algorithm uses the initial matrix

$$\mathcal{B} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \lfloor C \log 2 \rceil & \lfloor C \log 3 \rceil & \lfloor C \log 5 \rceil \end{bmatrix},$$

and returns with $\mathcal{B}_L = [b_{ij}] \in \mathbb{Z}^{3\times 3}$, where

$$\begin{aligned} b_{11} &= 113377537508635389408894400 67, \\ b_{12} &= 2330422670523603185158133 4341, \\ b_{13} &= -62676558316087526307960333326; \\ b_{21} &= -33043600810744979419960450935, \end{aligned}$$

$$
\begin{aligned}
b_{22} &= -171837152188761985549674576 52,\\
b_{23} &= -311355622710142363261686833 14;\\
b_{31} &= -879276295721788661178088532 4,\\
b_{32} &= 533348445148351080191083447 93,\\
b_{33} &= 274788653658398788520294474 75.
\end{aligned}
$$

The approximate number of the entries of $\mathcal{B}_L$ is 10^{28}. Then, by the Gram-Schmidt orthogonalization, we obtain $|\Gamma| > 3.5\cdot 10^{-58}$. This bound provides

$$\min\{k,\ell\} \le 193.$$

At this point we go back to (6) (and then to (8)), and use the upper bound $h(\eta_1) < 193\log 2$ (and then $193\log 3$) instead of $\log 2\cdot c_1(1+\log x)$ (and then instead of $\log 3\cdot c_1(1+\log x)$). In summary, the application of Matveev's theorem provides the improvement of Proposition 1 as follows.

Proposition 2. *If (1) holds, then $k,\ell < 2\cdot 10^{15}$.*

Using this new bound, LLL algorithm (which we do not detail here) can reduce the upper bound for $\min\{k,\ell\}$. More precisely, we have the following

Proposition 3. *If $(2^k-1)(3^\ell-1)=5^m-1$ holds, then $\min\{k,\ell\}\le 113$.*

Suppose again that $k\le\ell$, and recall (6). Assuming $m>113$, we get $5^{m/2} > 5^{k/2} > 2^k$. Thus,

$$0 < \Gamma_1 := \frac{(2^k-1)3^\ell}{5^m} - 1 < \frac{2^k}{5^m} < \frac{1}{5^{m/2}} < \frac{1}{2}.$$

Hence,

$$\varepsilon_1 := \ell\log 3 - m\log 5 + \log(2^k-1) < \frac{2}{5^{m/2}}.$$

Now we apply Theorem 3 to the inequality

$$\left|\ell\frac{\log 3}{\log 5} - m + \frac{\log(2^k-1)}{\log 5}\right| < \frac{2}{5^{m/2}\log 5} < \frac{2}{5^{m/2}}$$

with the notation $A=2$, $B=5$, $\kappa=(\log 3)/(\log 5)$, $\mu=(\log(2^k-1))/(\log 5)$. The possible values for k are $6,\ 10,\ \ldots,\ 110$. Moreover, by Lemma 2,

$$m < 0.44k + 0.69\ell < 1.13\cdot 2\cdot 10^{15} < 2.5\cdot 10^{15} =: M.$$

For each fixed value of k (≤ 110) we found $m\le 131$. Then the left hand side of Lemma 2 yields $\ell\le 302$.

A similar machinery works for $\ell \leq k$ with

$$0 < \Gamma_2 := \frac{(3^\ell - 1)2^k}{5^m} - 1 < \frac{3^\ell}{5^m} < \frac{1}{5^{m/4}}$$

if $\ell = 6,\ 10,\ \ldots,\ 110$. Now we obtain $m \leq 256$, and then $k \leq 591$.

Comparing the two branches, we conclude the following result.

Theorem 4. *If* $(2^k - 1)(3^\ell - 1) = 5^m - 1$ *holds and* $k \geq 6$, $\ell \geq 6$, *then* $k \leq 591$, $\ell \leq 302$, *and* $m \leq 256$.

These bounds are relatively small, so the remaining cases can be verified by computer. No new solution is found. Thus the proof is complete. ◄

Acknowledgments

The work on this paper started when the second author visited School of Mathematics of the Wits University. He thanks this Institution for support. F. L. was supported in part by NRF (South Africa) Grants CPRR160325161141, an A-rated researcher award, and the Focus Area Number Theory grant from CoEMaSS at Wits, and by CGA (Czech Republic) Grant 17-02804S. For L. Sz. the research was supported by Hungarian National Foundation for Scientific Research Grant No. 128088. This presentation has been made also in the frame of the "Efop-3.6.1-16-2016-00018 – Improving the role of the research + development + innovation in the higher education through institutional developments assisting intelligent specialization in Sopron and Szombathely".

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Received 27 November 2018
Accepted 27 October 2019