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On the Asymptotics of Solutions of Some Classes of Linear Differential Equations

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Abstract. Asymptotic behavior of solutions of ordinary differential equations in the case where the right part is a product of two differential expressions with almost constant coefficients is investigated.

Key Words and Phrases: quasiderivatives, quasidifferential equations, asymptotic behavior, spectral theory.

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1. Introduction

The goal of this paper is to obtain asymptotic formulas as $x \to +\infty$ for the fundamental system of solutions of the following differential equation:

$$l_2(l_1(y)) = \lambda y, \qquad x \in R_+ := [0, +\infty), \tag{1}$$

where λ is a complex parameter,

$$l_1(y) := y^{(n)} + (a_1 + p_1(x))y^{(n-1)} + (a_2 + p_2'(x))y^{(n-2)} + \ldots + (a_n + p_n'(x))y, \tag{2}$$

$$l_2(y) := y^{(m)} + (b_1 + q_1(x))y^{(m-1)} + (b_2 + q_2'(x))y^{(m-2)} + \ldots + (b_m + q_m'(x))y, \tag{3}$$

and all derivatives are understood in the sense of distributions.

Throughout this paper it is assumed that $a_1, a_2, \ldots, a_n, b_1, b_2, \ldots, b_m$ are complex numbers, $p_1, p_2, \ldots, p_n$, $q_1, q_2, \ldots, q_m$ are complex-valued measurable functions on R_+ such that

$$|p_1| + (1 + |p_2 - p_1|) \sum_{j=2}^{n} |p_j| \in L^1_{loc}(R_+)$$

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and

$$|q_1| + (1 + |q_2 - q_1|) \sum_{j=2}^{m} |q_j| \in L^1_{loc}(R_+).$$

More precise conditions on the functions p_i $(i = 1, 2, \ldots, n)$ and q_j $(j = 1, 2, \ldots, m)$ will be given in the formulations of theorems below.

The novelty of the obtained theorems in comparison with the well-known classical results is that we assume that derivatives are understood in the sense of distributions. In addition, the results are new for the functions p_i and q_j, differentiable a sufficient number of times.

2. The product of quasidifferential expressions

Let $a_1, a_2, \ldots, a_n$ be complex numbers and let $p_1, p_2, \ldots, p_n$ be measurable complex-valued functions on R_+ such that

$$|p_1| + (1 + |p_2 - p_1|) \sum_{j=2}^{n} |p_j| \in L^1_{loc}(R_+), \tag{4}$$

where, as usual, $L^1_{loc}(R_+)$ is the space of functions Lebesgue integrable on any segment $[\alpha, \beta] \subset R_+$.

Let $F := (f_{ij})$ be an n-dimensional square matrix of the form

$$F := \begin{pmatrix} 0 & 1 & 0 & . & . & . & 0 & 0 \\ 0 & 0 & 1 & . & . & . & 0 & 0 \\ . & . & . & . & . & . & . & . \\ 0 & 0 & 0 & . & . & . & 1 & 0 \\ f_{n-1,1} & f_{n-1,2} & f_{n-1,3} & . & . & . & f_{n-1,n-1} & 1 \\ f_{n1} & f_{n2} & f_{n3} & . & . & . & f_{n-1,n} & f_{nn} \end{pmatrix}.$$

The elements of the first $(n-2)$ rows of the matrix F are determined by the equalities $f_{ij} := 0$, if $j \neq i+1$ and $f_{i,i+1} := 1$. Define the elements $f_{n-1,j}$ of the penultimate row of this matrix, assuming

$$f_{n-1,j} := -p_{n+1-j}, \quad j = 1, 2, ...n-1, \qquad f_{n-1,n} := 1,$$

and the elements f_{nj} of the last row, assuming

$$f_{n1} := -(p_2 - p_1 - a_1)p_n - a_n, \qquad f_{nn} := p_2 - p_1 - a_1,$$

$$f_{nj} := p_{n+2-j} - (p_2 - p_1 - a_1)p_{n+1-j} - a_{n+1-j}, \quad j = 2, 3, ...n-1.$$

Following the generally accepted procedure (see, for example, [1, section I, p.8]), we define quasiderivatives $y_F^{[j]}$ $(j = 0, 1, ...n)$ of a given function y by means of the matrix F, assuming

$$y_F^{[j]} := y^{(j)}, \quad j = 0, 1, \ldots, n-2, \qquad y_F^{[n-1]} := y^{(n-1)} + \sum_{j=1}^{n-1} p_{n+1-j} y^{(j-1)},$$

$$y_F^{[n]} := (y_F^{[n-1]})' - \sum_{j=1}^{n} f_{nj} y_F^{[j-1]},$$

and the quasidifferential expression $\tau_F(y)$, assuming

$$\tau_F(y) := y_F^{[n]}.$$

The domain $D(\tau_F)$ of expression $\tau_F(y)$ is the set of all complex-valued functions y for which the quasiderivatives up to the order $(n-1)$ exist and are absolutely continuous on every segment $[\alpha, \beta] \subset R_+$. It is obvious that $\tau_F(y) \in L^1_{loc}(R_+)$ for any $y \in D(\tau_F)$.

In [2] it was shown that for all $y \in D(\tau_F)$ distribution $l_1(y)$ (see (2)) is a regular generalized function and

$$l_1(y) = \tau_F(y). \tag{5}$$

Further, let $b_1, b_2, \ldots, b_m$ be complex numbers, and $q_1, q_2, \ldots, q_m$ be complex-valued measurable functions on R_+ such that

$$|q_1| + (1 + |q_2 - q_1|) \sum_{j=2}^{m} |q_j| \in L^1_{loc}(R_+). \tag{6}$$

Denote by G an m-dimensional square matrix of the same structure as the matrix F but determined by the numbers b_j and functions q_j. The matrix G generates quasiderivatives $y_G^{[j]}$ $(j = 0, 1, \ldots, m)$ and quasidifferential expression $\tau_G(y)$. As before, for all $y \in D(\tau_G)$ distribution $l_2(y)$ (see (3)) is a regular generalized function and

$$l_2(y) = \tau_G(y). \tag{7}$$

Following [3] and [4], we define now the product of expressions $\tau_F(y)$ and $\tau_G(y)$. Let H be an $(n+m)$-dimensional square matrix of the form

$$H := \begin{pmatrix} F & M \\ O_{m \times n} & G \end{pmatrix},$$

where M is a matrix of dimension $n \times m$, all elements of which are equal to zero, except for the element in the lower left corner, equal to 1, and $O_{m\times n}$ is a zero matrix of dimension $m \times n$. Using the matrix H, we define quasi-derivatives $y_H^{[j]}$ $(j = 0, 1, ...n + m)$ of a given function y, assuming

$$y_H^{[j]} := y_F^{[j]}, \quad j = 0, 1, \dots, n,$$

$$y_H^{[n+j]} := (y_F^{[n]})_G^{[j]}, \quad j = 1, 2, \dots, m,$$

and quasidifferential expression, assuming

$$\tau_H(y) := (y_F^{[n]})_G^{[m]}.$$

It is well known that the domain $D(\tau_H)$ of the expression τ_H is given by

$$D(\tau_H) = \{y | y \in D(\tau_F) \,\text{and}\, \tau_F y \in D(\tau_G)\},$$

and, besides

$$\tau_H(y) = \tau_G(\tau_F(y))(:= \tau_G\tau_F(y)) \quad \text{for} \quad y \in D(\tau_H)(= D(\tau_G\tau_F)).$$

Now we consider the quasidifferential equation

$$\tau_G(\tau_F(y)) = \lambda y, \tag{8}$$

where λ is a complex parameter. We use the symbol $\mathbf{y}$ to denote the column vector $\mathbf{y} := (y_H^{[0]}, y_H^{[1]}, \dots, y_H^{[n+m-1]})^T$ (T is the transposition symbol), and consider the system of first order linear differential equations

$$\mathbf{y}' = (H + \Lambda)\mathbf{y}, \tag{9}$$

where $\Lambda := (\lambda_{ij})$ is a square matrix of dimension $(n + m)$, whose elements are determined by the equalities $\lambda_{n+m,1} := \lambda$ and $\lambda_{ij} := 0$ for all other values of i and j.

The conditions (4) and (6) imply that all elements of the matrix H are locally integrable on R_+ functions. Therefore, the conditions (4) and (6) ensure the validity of the existence and uniqueness theorem for the solution of the Cauchy problem for system (9) stated at an arbitrary point of the semi-axis R_+. On the other hand, the equation (8) is equivalent to the system (9) in the sense that if all quasiderivatives of y up to the order $(n + m - 1)$ are locally absolutely continuous on R_+ and y satisfies the equation (8) almost everywhere, then $\mathbf{y}$ is a solution of the system (9), and vice versa, if $\mathbf{y}$ is a solution of the system (9), then its coordinates are locally absolutely continuous on R_+ and the first coordinate $y(= y_H^{[0]})$ satisfies the equation (8) almost everywhere on R_+. Thus, the conditions (4) and (6) imply the validity of the existence and uniqueness theorem for the solution of the Cauchy problem for the equation (8) stated at an arbitrary point of the semi-axis R_+.

Formulas (5) and (7) allow treating the differential equation (1) with distribution coefficients as a quasidifferential equation (8), and this, in turn, allows asserting that the conditions (4) and (6) imply also the validity of the existence and uniqueness theorem for the solution of the Cauchy problem for the equation (1) stated at an arbitrary point of the semi-axis R_+.

3. Main results

We now formulate our main results. The symbol $o(1)$, as usual, will denote a function infinitely small as $x \to +\infty$.

Theorem 1. *Suppose that, the number λ in the equation (1), is different from the product $a_n b_m$ ($\lambda \neq a_n b_m$), and the functions $p_1, p_2, \ldots, p_n$ and $q_1, q_2, \ldots, q_m$ satisfy the conditions*

$$x^{\alpha-1}\Big(|p_1| + (1 + |p_2 - p_1|)\sum_{j=2}^{n}|p_j|\Big) \in L^1(R_+)$$

and

$$x^{\alpha-1}\Big(|q_1| + (1 + |q_2 - q_1|)\sum_{j=2}^{m}|q_j|\Big) \in L^1(R_+),$$

where α is the largest multiplicity of a root of the polynomial

$$\mathfrak{F}(z) = (z^n + a_1 z^{n-1} + \ldots + a_n)(z^m + b_1 z^{m-1} + \ldots + b_m) - \lambda.$$

Then the equation (1) has a fundamental system of solutions $y_1, y_2, \ldots, y_{n+m}$, such that if z_1 is a root of the polynomial $\mathfrak{F}(z)$ of multiplicity $l_1 \leqslant \alpha$, then the equation (1) has a subsystem $\{y_j\}$ $(j = 1, 2, \ldots, l_1)$ of fundamental solutions such that

$$y_j^{[s-1]}(x) = e^{z_1 x} x^{j-1}(z_1^{s-1} + o(1)), \quad s = 1, 2, \ldots, n, \tag{10}$$

$$y_j^{[n+s-1]}(x) = e^{z_1 x} x^{j-1}(z_1^{n+s-1} + a_1 z_1^{n+s-2} + \ldots + a_n z_1^{s-1} + o(1)), \tag{11}$$
$$s = 1, 2, \ldots, m,$$

as $x \to +\infty$.

Another subsystem of fundamental solutions $\{y_j\}$ $(j = l_1 + 1, \ldots l_1 + l_2)$ corresponding to the root z_2 of the polynomial $\mathfrak{F}(z)$ of multiplicity l_2 has the same asymptotics, and so on.

Proof. The quasidifferential equation (8) (and so the differential equation (1)) is equivalent to the system of equations (9). We write this system in the form

$$\mathbf{y}' = (A + R(x))\mathbf{y}. \tag{12}$$

Here, as before, $\mathbf{y}$ is the column vector $\mathbf{y} = (y_H^{[0]}, y_H^{[1]}, \ldots, y_H^{[n+m-1]})^T$, the nonzero elements a_{ij} of $(n+m)$-dimensional square matrix A are determined by the equalities

$$a_{j,j+1} = 1, \quad j = 1, 2, ..., n + m - 1, \qquad a_{nj} = -a_{n+1-j}, \quad j = 1, 2, ..., n,$$

$$a_{n+m,1} = \lambda, \qquad a_{n+m,n+j} = -b_{m+1-j}, \quad j = 1, 2, ..., m,$$

and the nonzero elements r_{ij} of the matrix function $R(x)$ are defined by the equalities

$$r_{n-1,j} = -p_{n+1-j},\ j = 1, 2, ...n - 1, \quad r_{n1} = -(p_2 - p_1 - a_1)p_n,$$

$$r_{nj} = p_{n+2-j} - (p_2 - p_1 - a_1)p_{n+1-j},\ j = 2, 3, ..., n - 1, \quad r_{nn} = p_2 - p_1,$$

$$r_{n+m-1,n+j} = -q_{m+1-j},\ j = 1, 2, ...m - 1, \quad r_{n+m,n+1} = -(q_2 - q_1 - b_1)p_m,$$

$$r_{n+m,n+j} = q_{m+2-j} - (q_2 - q_1 - b_1)q_{m+1-j},\ j = 2, 3, ..., m-1, \quad r_{n+m,n+m} = q_2 - q_1.$$

Note that the characteristic polynomial of the matrix A coincides with the polynomial $(-1)^{n+m}\mathfrak{F}(z)$. Let z_1 be the root of the equation $\mathfrak{F}(z) = 0$ of multiplicity l_1, i.e. the eigenvalue of the matrix A of algebraic multiplicity l_1. The structure of the matrix A is such that $a_{i,i+1} = 1$, if $1 \le i \le n + m - 1$, and $a_{ij} = 0$, if $2 \le i + 1 < j \le n + m$. Therefore, any eigenvector $\mathbf{c}$ of the matrix A corresponding to the eigenvalue z_1 has the form

$$\mathbf{c} = (1, z_1, z_1^2, \ldots, z_1^{n-1}, Q(z_1), z_1 Q(z_1), \ldots, z_1^{m-1} Q(z_1))^T,$$

where $Q(z) = z^n + a_1 z^{n-1} + \ldots + a_n$.

Thus, only one eigenvector corresponds to the eigenvalue z_1 of the matrix A, i.e. the geometric multiplicity of z_1 is equal to 1. In other words, each eigenvalue of the matrix A is associated with only one Jordan block in its canonical form. Therefore, the dimension of the Jordan block in the canonical form of the matrix A of greatest dimension coincides with the multiplicity of the eigenvalue of the matrix A of greatest multiplicity, i.e. with the number α from the condition of Theorem 1. Thus, in this situation the system of equations $\mathbf{y}' = A\mathbf{y}$ has solutions that can be represented as

$$e^{z_1 x}\mathbf{c} \quad \text{and} \quad e^{z_1 x} x^j \mathbf{c} + O(e^{z_1 x} x^{j-1}), \quad j = 1, 2, \ldots, l_1 - 1.$$

In addition, it follows from the conditions of Theorem 1 that all elements of the matrix $x^{\alpha-1}R(x)$ belong to the space $L^1(R_+)$. Applying the statement of problem 35 from [5, Ch. III, p. 120] (see also [6, Chapter IV, Remark to Theorem 4, p. 95]), we see that the system of equations (12) has a subsystem of solutions of $\{\mathbf{y}_j\}$, $j = 1, 2, \dots l_1$, representable in the form

$$\mathbf{y}_j = e^{z_1 x} x^{j-1}(\mathbf{c} + o(1)), \quad j = 1, 2, \dots, l_1, \text{ as } x \to +\infty.$$

A further look into the relationship between the solutions of the equation (1) and the solutions of the system (12), shows that the equation (1) has a subsystem of solutions $\{y_j\}$, $j = 1, 2, \dots l_1$, representable in the forms (10) and (11) as $x \to +\infty$.

Repeating these arguments for an eigenvalue z_2 of multiplicity l_2, we find that the equation (1) has a subsystem of solutions $\{y_j\}$, $j = l_1 + 1, \dots l_1 + l_2$, with an asymptotic form (10) and (11) (with z_1 replaced by z_2), and so on. From the above reasoning it also follows that, having considered all the roots of the polynomial $\mathfrak{F}(z)$, we obtain the asymptotics of some fundamental system of solutions $y_1, y_2, \dots, y_{n+m}$ of the equation (1). The theorem is proved. ◄

Remark 1. *The condition* $\lambda \neq a_n b_m$ *of Theorem 1 implies that the number* $z = 0$ *is not the root of the polynomial* $\mathfrak{F}(z)$*. This allows us to find not only the main term of the asymptotics of the solutions of the equation* (1)*, but also their quasiderivatives up to the* $(n+m-1)$*th order. In particular, asymptotic formulas for solutions of the equation* (1) *(formulas* (10) *with* $s = 1$*) can be differentiated up to the order* $(n-2)$*. If* $\lambda = a_n b_m$ *and the number* $z = 0$ *is the root of the polynomial* $\mathfrak{F}(z)$ *of multiplicity* l*, then the method of the proof of Theorem 1 allow us to state that the equation* (1) *has solutions of the form*

$$y_j = x^{j-1}(1 + o(1)), \quad j = 1, 2, \dots, l,$$

but for quasiderivatives of these solutions one can only say that

$$y_j^{[s]}(x) = x^{j-1} o(1), \quad s = 1, 2, \dots, n-1, n+1, \dots, m+n-1,$$

$$y_j^{[n]}(x) = x^{j-1}(a_n + o(1)).$$

Using more subtle methods for first-order linear differential systems, one can also study the situation when $\lambda = a_n b_m$. In particular, the following theorem holds.

Theorem 2. *Suppose that, in the equation* (1)*,*

$$a_1 = a_2 = \dots = a_n = b_1 = b_2 = \dots = b_m = \lambda = 0$$

and the functions $p_1, p_2, \ldots, p_n$ *and* $q_1, q_2, \ldots, q_m$ *satisfy the conditions*

$$|p_1| + (1 + x|p_2 - p_1|) \sum_{j=2}^{n} x^{j-2}|p_j| \in L^1(R_+)$$

and

$$|q_1| + (1 + x|q_2 - q_1|) \sum_{j=2}^{m} x^{j-2}|q_j| \in L^1(R_+).$$

Then the equation (1) *has a fundamental system of solutions* $\{y_j\}$ $(j = 1, 2, \ldots, n+m)$ *such that*

$$y_j^{[s]}(x) = \begin{cases} \dfrac{x^{j-1-s}}{(j-1-s)!}(1 + o(1)), & \text{if } s = 0, 1, \ldots, j-1, \\ x^{j-1-s}\, o(1), & \text{if } s = j, j+1, \ldots, n+m-1. \end{cases} \tag{13}$$

as $x \to +\infty$.

Proof. As we already noted in the proof of Theorem 1, the equation (8) (and so (1)) is equivalent to the system (12). Moreover, it follows from the conditions of Theorem 2 that the matrix A has a Jordan form containing one Jordan block with zero eigenvalue of multiplicity $(n + m)$. By the symbol D we denote a diagonal matrix

$$D := diag(1, x, \ldots, x^n, \ldots, x^{n+m-1}).$$

The calculations show that the nonzero elements $\tilde{r}_{ij}$ of the matrix $\tilde{R} := DRD^{-1}$ are determined by the equalities

$$\tilde{r}_{n-1,j} = -x^{n-j-1}p_{n+1-j}, \quad j = 1, 2, \ldots, n-1,$$

$$\tilde{r}_{n,j} = x^{n-j}(p_{n+2-j} + (p_1 - p_2)p_{n+1-j}), \quad j = 2, 3, \ldots, n-1,$$

$$\tilde{r}_{n,1} = x^{n-1}(p_1 - p_2)p_n, \quad \tilde{r}_{n,n} = p_2 - p_1,$$

$$\tilde{r}_{n+m-1,n+j} = -x^{m-j-1}q_{m+1-j}, \quad j = 1, 2, ...m-1,$$

$$\tilde{r}_{n+m,n+j} = x^{m-j}(q_{m+2-j} + (q_1 - q_2)q_{m+1-j}), \quad j = 2, 3, ..., m-1,$$

$$\tilde{r}_{n+m,n+1} = x^{m-1}(q_1 - q_2)p_m, \quad \tilde{r}_{n+m,n+m} = q_2 - q_1.$$

From these formulas and from the conditions of Theorem 2 it follows that all elements of the matrix $\tilde{R}$ belong to the space $L^1(R_+)$. Thus, the coefficients of the system (12) satisfy all the conditions of Theorem 1.10.1 from [7]. Applying the above theorem again, we see that the system (12) has solutions $\{\mathbf{y}_j\}$, $j = 1, 2, ..., n+m$, of the form

$$\mathbf{y}_j = \Big(\frac{x^{j-1}}{(j-1)!} + o(x^{j-1}), \frac{x^{j-2}}{(j-2)!} + o(x^{j-2}), ..., 1 + o(1), o(x^{-1}), ..., o(x^{-n-m+j})\Big)^T.$$

A further look into the relationship between the solutions of the equation (1) and the solutions of the system (12) shows that the equation (1) has a system of solutions y_j, $j = 1, 2, \dots n+m$, representable in the form (13) as $x \to +\infty$. The theorem is proved. ◄

It will be interesting to join the method of this paper with method of paper [8] for investigation of more complicated equations.

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