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Multidimensional Bilinear Hardy Inequalities

N. Bilgiçli, R.Ch. Mustafayev*, T. Ünver

Abstract. Our goal in this paper is to find a characterization of n-dimensional bilinear Hardy inequalities

$$\bigg\| \int_{B(0,\cdot)} f \cdot \int_{B(0,\cdot)} g \bigg\|_{q,u,(0,\infty)} \le C \, \|f\|_{p_1,v_1,\mathbb{R}^n} \, \|g\|_{p_2,v_2,\mathbb{R}^n}, \quad f,\, g \in \mathfrak{M}^+(\mathbb{R}^n),$$

$$\bigg\| \int_{{}^{\mathfrak{c}}B(0,\cdot)} f \cdot \int_{{}^{\mathfrak{c}}B(0,\cdot)} g \bigg\|_{q,u,(0,\infty)} \le C \, \|f\|_{p_1,v_1,\mathbb{R}^n} \, \|g\|_{p_2,v_2,\mathbb{R}^n}, \quad f,\, g \in \mathfrak{M}^+(\mathbb{R}^n),$$

when $0 < q \le \infty$, $1 \le p_1,\, p_2 \le \infty$ and u and v_1, v_2 are weight functions on $(0,\infty)$ and $\mathbb{R}^n$, respectively. Obtained results are new when $p_i = 1$ or $p_i = \infty$, $i = 1, 2$, or $0 < q \le 1$ even in 1-dimensional case.
Since the solution of the first inequality can be obtained from the characterization of the second one by usual change of variables we concentrate our attention on characterization of the latter. The characterization of this inequality is easily obtained for $p_1 \le q$ using the characterizations of multidimensional weighted Hardy-type inequalities while in the case $q < p_1$ the problem is reduced to the solution of multidimensional weighted iterated Hardy-type inequality.
To achieve our goal, we characterize the validity of multidimensional weighted iterated Hardy-type inequality

$$\bigg\| \bigg\| \int_{{}^{\mathfrak{c}}B(0,s)} h(z)dz \bigg\|_{p,u,(0,t)} \bigg\|_{q,\mu,(0,\infty)} \le c\|h\|_{\theta,v,(0,\infty)}, \; h \in \mathfrak{M}^+(\mathbb{R}^n)$$

where $0 < p,\, q < \infty$, $1 \le \theta \le \infty$, $u \in \mathcal{W}(0,\infty)$, $v \in \mathcal{W}(\mathbb{R}^n)$ and μ is a non-negative Borel measure on $(0,\infty)$. We are able to obtain the characterization under the additional condition that the measure μ is non-degenerate with respect to $U^{q/p}$.
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*Corresponding author.

1. Introduction

The aim of this paper is to study the boundedness of n-dimensional bilinear Hardy operators $H_2^n : L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^q(u)$ and $\big(H_2^n\big)^* : L^{p_1}(w_1) \times L^{p_2}(w_2) \to L^q(u)$, defined for all $f_1,\, f_2 \in \mathfrak{M}^+(\mathbb{R}^n)$ by

$$H_2^n(f_1, f_2)(t) : = \int_{B(0,t)} f_1(x)\,dx \cdot \int_{B(0,t)} f_2(x)\,dx, \quad t > 0,$$

and

$$\big(H_2^n\big)^*(f_1, f_2)(t) : = \int_{{}^{\mathfrak{c}}B(0,t)} f_1(x)\,dx \cdot \int_{{}^{\mathfrak{c}}B(0,t)} f_2(x)\,dx, \quad t > 0,$$

that is, to investigate the validity of n-dimensional bilinear Hardy inequalities

$$\bigg\| \int_{B(0,\cdot)} f \cdot \int_{B(0,\cdot)} g \bigg\|_{q,u,(0,\infty)} \le C\, \|f\|_{p_1,v_1,\mathbb{R}^n}\, \|g\|_{p_2,v_2,\mathbb{R}^n},\ f,\, g \in \mathfrak{M}^+(\mathbb{R}^n), \quad (1)$$

and

$$\bigg\| \int_{{}^{\mathfrak{c}}B(0,\cdot)} f \cdot \int_{{}^{\mathfrak{c}}B(0,\cdot)} g \bigg\|_{q,u,(0,\infty)} \le C\, \|f\|_{p_1,v_1,\mathbb{R}^n}\, \|g\|_{p_2,v_2,\mathbb{R}^n},\ f,\, g \in \mathfrak{M}^+(\mathbb{R}^n). \quad (2)$$

The motivation for the investigation of n-dimensional m-linear Hardy ineqalities can be explained, for instance, by the paper [19], where a weight theory has been developed for a new multi(sub)linear maximal function

$$\mathcal{M}(f_1, \cdots, f_m)(x) := \sup_{Q \ni x} \prod_{i=1}^m \frac{1}{|Q|} \int_Q |f_i(y_i)|\,dy_i, \quad x \in \mathbb{R}^n,$$

where the supremum is taken over all cubes in $\mathbb{R}^n$ containing x with sides parallel to the coordinate axes, introduced in order to control the multilinear Calderón-Zygmund operators. Recall that, this operator is strictly smaller that the m-fold product of M, that is, the operator $\prod_{i=1}^m Mf_j$, where M is the Hardy-Littlewood maximal operator. Drawing paralells between linear and m-linear theories, in our opinion, it will be useful to have a characterization of weight functions for which n-dimensional m-linear Hardy operator

$$H_m^n(f_1, \cdots, f_m)(t) := \int_{B(0,t)} f_1(x)\,dx \cdots \int_{B(0,t)} f_m(x)\,dx, \quad t > 0$$

is bounded from $L^{p_1}(w_1) \times \cdots \times L^{p_m}(w_m)$ into $L^p(u)$, that is, the inequality

$$\|H_m^n(f_1, \cdots, f_m)\|_{L^p(u)} \leq C\|f_1\|_{L^{p_1}(w_1)} \cdots \|f_m\|_{L^{p_m}(w_m)}$$

holds.

In one-dimensional case, the bilinear Hardy operator $H_2 \equiv H_2^1$, acting on $\mathfrak{M}^+(0,\infty) \times \mathfrak{M}^+(0,\infty)$, is defined by

$$H_2(f,g)(x) = \int_0^x f(t)\,dt \cdot \int_0^x g(t)\,dt.$$

As far as we know, the boundedness of $H_2 : \mathfrak{M}^+(0,\infty) \times \mathfrak{M}^+(0,\infty) \to L^q(u)$, that is, the bilinear Hardy inequality

$$\left(\int_0^\infty \left(\int_0^x f \cdot \int_0^x g \right)^q u(x)\,dx \right)^{1/q} \leq$$

$$\leq C \left(\int_0^\infty f^{p_1} v_1 \right)^{1/p_1} \left(\int_0^\infty g^{p_2} v_2 \right)^{1/p_2}, \quad f,\, g \in \mathfrak{M}^+(0,\infty) \tag{3}$$

has not been considered previously in the literature, apart from [4] and [16], where general bilinear operators were considered and their boundedness was characterized in the case $1/q \geq 1/p_1 + 1/p_2$, by means of a Schur-type criterion. The boundedness of $H_2 : L^{p_1}(v_1) \times L^{p_2}(v_2) \to L^q(u)$ was characterized recently in [1] via the discretization method, and in [17, 2, 24] using the iteration method. The range of exponents in all papers was $1 < p_1,\, p_2,\, q < \infty$.

As in 1-dimensional case (cf. [17]), the characterization of n-dimensional bilinear Hardy inequalities can be easily obtained using the characterizations of multidimensional weighted Hardy-type inequalites, when $p_1 \leq q$ (see Theorems 3, 5 and 6). In the most difficult case where $q < p_1$, interchanging the suprema and applying the multidimensional weighted Hardy-type inequalities, by integrating by parts, we get that inequality (2) is equivalent to the inequality

$$\left(\int_0^\infty \left(\int_0^x \left(\int_{{}^{\mathfrak{c}}B(0,t)} g \right)^q u(t)\,dt \right)^{r_1/q} d\left[-\|v_1^{-1/p_1}\|_{p_1', {}^{\mathfrak{c}}B(0,x)}^{r_1} \right] \right)^{1/r_1} \leq$$

$$\leq C\|g\|_{p_2, v_2, \mathbb{R}^n}, \quad g \in \mathfrak{M}^+(\mathbb{R}^n)$$

with $1/r_1 = 1/q - 1/p_1$ (see Theorem 4).

In this paper we characterize the validity of the multidimensional weighted iterated Hardy-type inequality

$$\left\| \left\| \int_{{}^{\mathfrak{c}}B(0,s)} h(z)dz \right\|_{p,u,(0,t)} \right\|_{q,\mu,(0,\infty)} \leq c\|h\|_{\theta,v,\mathbb{R}^n}, \; h \in \mathfrak{M}^+(\mathbb{R}^n), \tag{4}$$

where $0 < p, q < \infty$, $1 \le \theta \le \infty$, $u \in \mathcal{W}(0,\infty)$, $v \in \mathcal{W}(\mathbb{R}^n)$ and μ is a non-negative Borel measure on $(0,\infty)$ (see Theorem 2). We are able to obtain the characterization under the additional condition that the measure μ is non-degenerate with respect to $U^{q/p}$, that is, conditions (14) are satisfied.

In 1-dimensional case there exist different solutions of iterated Hardy-type inequalities

$$\left\| \left\| \int_t^\infty h(\tau)\,d\tau \right\|_{p,u,(0,\cdot)} \right\|_{q,w,(0,\infty)} \le C\,\|h\|_{\theta,v,(0,\infty)},\ h \in \mathfrak{M}^+(0,\infty), \tag{5}$$

where $0 < p$, $q \le \infty$, $1 \le \theta \le \infty$ and $u, w, v \in \mathcal{W}(0,\infty)$.

Note that the inequality (5) have been considered in the case $p = 1$ in [6] (see also [7]), where the result was presented without proof, in the case $p = \infty$ in [13] and in the case $\theta = 1$ in [8] and [25], where the special type of weight function v was considered. Recall that the inequality has been completely characterized in [9] and [10] in the case $0 < p < \infty$, $0 < q \le \infty$, $1 \le \theta \le \infty$ by using discretization and anti-discretization methods. Another approach to get the characterization of inequalities (5) was presented in [23]. But these characterizations involve auxiliary functions, which make conditions more complicated. The characterization of the inequality can be reduced to the characterization of the weighted Hardy inequality on the cones of non-increasing functions (see, [11, 12]). Different approach to solve iterated Hardy-type inequalities has been given in [20]. In order to characterize inequality (4) we will use the technique of [9] and [10].

It should be noted that none of the above would ever have existed if it wasn't for the (now classical) well-known characterizations of weights for which the Hardy inequality holds. This subject, which is, incidentally, exactly one hundred years old, is absolutely indispensable in this area of mathematics (cf. [22, 18]). In our proof below multidimensional analogues of such results from [3, 5, 21] will be heavily used.

The paper is organized as follows. We start with some notations and preliminaries in Section 2. The discretization and anti-discretization methods for solution of inequalities (4) are given in Sections 3 and 4, respectively. Finally, the solutions of multidimensional bilinear Hardy inequalities are presented in Section 5.

2. Notations and Preliminaries

Throughout the paper, we denote by c or C a positive constant, which is independent of main parameters, but may vary from line to line. However a constant with subscript such as c_1 does not change in different occurrences. By

$a \lesssim b$ ($b \gtrsim a$) we mean that $a \le \lambda b$, where $\lambda > 0$ depends only on inessential parameters. If $a \lesssim b$ and $b \lesssim a$, we write $a \approx b$ and say that a and b are equivalent. Throughout the paper we use the abbreviation LHS($*$) (RHS($*$)) for the left (right) hand side of the relation ($*$). By χ_Q we denote the characteristic function of a set Q. Unless a special remark is made, the differential element dx is omitted when the integrals under consideration are the Lebesgue integrals.

For $x \in \mathbb{R}^n$ and $r > 0$, let $B(x,r) := \{y \in \mathbb{R}^n : |x-y| < r\}$ be the open ball centered at x of radius r and ${}^{\mathfrak{c}}B(x,r) := \mathbb{R}^n \backslash B(x,r)$. We define $S[a,b) := \{x \in \mathbb{R}^n : a \le |x| < b\} = {}^{\mathfrak{c}}B(0,a) \backslash {}^{\mathfrak{c}}B(0,b)$, where $0 \le a < b < \infty$.

Let μ be a non-negative measure on $A \subset \mathbb{R}^n$, $n \ge 1$. By $\mathfrak{M}(A,\mu)$ we denote the set of all μ-measurable functions on A. The symbol $\mathfrak{M}^+(A,\mu)$ stands for the collection of all $f \in \mathfrak{M}(A,\mu)$ which are non-negative on A. The family of all weight functions (also called just weights) on A, that is, non-negative functions locally integrable with respect to measure μ on A, is given by $\mathcal{W}(A,\mu)$. If the measure μ is the Lebesgue measure on A, then we omit the symbol μ in the notation.

For $p \in (0,\infty]$ and $w \in \mathfrak{M}^+(A,\mu)$, we define the functional $\|\cdot\|_{p,w,A,\mu}$ on $\mathfrak{M}(A,\mu)$ by

$$\|f\|_{p,w,A,\mu} := \begin{cases} \left(\int_A |f(x)|^p w(x)\, d\mu(x)\right)^{1/p} & \text{if} \quad p < \infty, \\ \mu - \operatorname{ess\,sup}_A |f(x)| w(x) & \text{if} \quad p = \infty. \end{cases}$$

If, in addition, $w \in \mathcal{W}(A,\mu)$, then the weighted Lebesgue space $L^p(w,A,\mu)$ is given by

$$L^p(w,A,\mu) = \{f \in \mathfrak{M}(A,\mu) : \ \|f\|_{p,w,A,\mu} < \infty\}$$

and it is equipped with the quasi-norm $\|\cdot\|_{p,w,A,\mu}$.

When $w \equiv 1$ on A, we write simply $L^p(A)$ and $\|\cdot\|_{p,A}$ instead of $L^p(w,A)$ and $\|\cdot\|_{p,w,A}$, respectively.

For $u \in \mathcal{W}(0,\infty)$, $v \in \mathcal{W}(\mathbb{R}^n)$ and $1 \le \theta \le \infty$, we denote

$$U(t) := \int_0^t u(s)ds, \quad V_\theta(t) := \begin{cases} \left\|v^{-1/\theta}\right\|_{\theta', {}^{\mathfrak{c}}B(0,t)}, & \text{when } \theta < \infty, \\ \left\|v^{-1}\right\|_{1, {}^{\mathfrak{c}}B(0,t)}, & \text{when } \theta = \infty, \end{cases} \quad t \in (0,\infty),$$

and assume that $U(t) > 0$, $t \in (0,\infty)$.

We adopt the following usual conventions.

Convention 1. (i) *Throughout the paper we put* $0 \cdot \infty = 0$, $\infty/\infty = 0$ *and* $0/0 = 0$.

(ii) *If* $\theta \in [1,+\infty]$, *we define* θ' *by* $1/\theta + 1/\theta' = 1$.

(iii) *If* $I = (a,b) \subseteq \mathbb{R}$ *and* g *is a monotone function on* I, *then by* $g(a)$ *and* $g(b)$ *we mean the limits* $\lim_{x\to a+} g(x)$ *and* $\lim_{x \to b-} g(x)$, *respectively.*

Let us now recall some definitions and basic facts concerning discretization and anti-discretization which can be found in [14], [15] and [8].

Definition 1. *Let $\{a_k\}$ be a sequence of positive real numbers. We say that $\{a_k\}$ is geometrically increasing or geometrically decreasing and write $a_k \uparrow\uparrow$ or $a_k \downarrow\downarrow$ when*

$$\inf_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} > 1 \quad or \quad \sup_{k\in\mathbb{Z}} \frac{a_{k+1}}{a_k} < 1,$$

respectively.

Definition 2. *Let u be a continuous strictly increasing function on $[0,\infty)$ such that $u(0)=0$ and $\lim_{t\to\infty} u(t) = \infty$. Then we say that u is* admissible*.*

Definition 3. *Let u be an admissible function. A function h is called u-quasiconcave if h is equivalent to a non-decreasing function on $(0,\infty)$ and h/u is equivalent to a non-increasing function on $(0,\infty)$.*

Definition 4. *A u-quasiconcave function h is called* non-degenerate *if*

$$\lim_{t\to 0+} h(t) = \lim_{t\to\infty} \frac{1}{h(t)} = \lim_{t\to\infty} \frac{h(t)}{u(t)} = \lim_{t\to 0+} \frac{u(t)}{h(t)} = 0.$$

The family of non-degenerate u-quasiconcave functions will be denoted by Ω_u.

In view of [14, Lemma 2.7], we give the definition of discretizing sequence as follows:

Let u be an admissible function and $h\in\Omega_u$. Let $\alpha > 4$. A sequence $\{x_k\}$ defined by $x_0 = 1$ and

$$x_{k+1} := \inf\left\{t;\, \min\left\{\frac{h(t)}{h(x_k)}, \frac{h(x_k)u(t)}{u(x_k)h(t)}\right\} = \alpha\right\}, \quad \text{when} \quad k\ge 0;$$

$$x_{k-1} := \inf\left\{t;\, \min\left\{\frac{h(x_k)}{h(t)}, \frac{h(t)u(x_k)}{u(t)h(x_k)}\right\} = \alpha\right\}, \quad \text{when} \quad k\le 0,$$

is called a discretizing sequence for h with respect to u.

Let $\mathbb{Z}_1, \mathbb{Z}_2$ be defined by

$$\mathbb{Z}_1 := \{\, k\in\mathbb{Z};\ \alpha\, h(x_k) = h(x_{k+1})\,\}, \quad \mathbb{Z}_2 := \mathbb{Z}\setminus\mathbb{Z}_1. \tag{6}$$

Then

(i) $u(x_k)\uparrow\uparrow$;

(ii) $h(x_k)\uparrow\uparrow$ and $\frac{h(x_k)}{u(x_k)}\downarrow\downarrow$;

(iii) $\mathbb{Z} = \mathbb{Z}_1\cup\mathbb{Z}_2$, $\mathbb{Z}_1\cap\mathbb{Z}_2 = \emptyset$ and for every $t\in[x_k, x_{k+1}]$

$$h(x_k)\approx h(t) \quad \text{if} \quad k\in\mathbb{Z}_1,$$

$$\frac{h(x_k)}{u(x_k)} \approx \frac{h(t)}{u(t)} \quad \text{if} \quad k\in\mathbb{Z}_2.$$

Definition 5. *[14, Definition 2.9] Let u be an admissible function and let μ be a non-negative Borel measure on $[0,\infty)$. We say that the function h defined as*

$$h(x) = u(x) \int_{[0,\infty)} \frac{d\mu(t)}{u(t)+u(x)}, \qquad x \in (0,\infty),$$

is the fundamental function of the measure μ with respect to u. We will also say that μ is a representation measure of h with respect to u.

We say that μ is non-degenerate with respect to u if the following conditions are satisfied:

$$\int_{[0,\infty)} \frac{d\mu(t)}{u(t)+u(x)} < \infty, \quad x \in (0,\infty), \qquad \int_{[0,1]} \frac{d\mu(t)}{u(t)} = \int_{[1,\infty)} d\mu(t) = \infty.$$

Let u be an admissible function and let μ be a non-negative Borel measure on $[0,\infty)$. Assume that μ is non-degenerate with respect to u. Let h be the fundamental function of μ with respect to u. Then $h \in \Omega_u$ (see [14, Remark 2.10]).

We recall some known results from [14]. Our formulations of the following statements, which are more convenient for our future applications, are not exactly the same as in the mentioned paper. But by following the proofs of these theorems in [14], it is not difficult to see that such formulations are also true.

Theorem 1. *[14, Theorem 2.11] Let $p,q,r \in (0,\infty)$. Let u be an admissible function. Let μ be a non-negative Borel measure on $[0,\infty)$ non-degenerate with respect to u^q, and let h be the fundamental function of μ with respect to u^q. Let σ be u^p-quasiconcave. Let $\{x_k\}_{k\in\mathbb{Z}}$ be a discretizing sequence for h with respect to u^q. Then*

$$\int_{[0,\infty)} \frac{h(t)^{r/q-1}}{\sigma(t)^{r/p}}\, d\mu(t) \approx \sum_{k\in\mathbb{Z}} \frac{h(x_k)^{r/q}}{\sigma(x_k)^{r/p}}.$$

Corollary 1. *[14, Corollary 2.13] Let $q \in (0,\infty)$, let u be an admissible function, let f be u-quasiconcave, let μ be a non-negative Borel measure on $[0,\infty)$ non-degenerate with respect tot u^q, and let h be the fundamental function of μ with respect to u^q. Let $\{x_k\}$ be a discretizing sequence for h with respect to u^q. Then*

$$\left(\int_{[0,\infty)} \left(\frac{f(t)}{u(t)} \right)^q d\mu(t) \right)^{1/q} \approx \left(\sum_{k\in\mathbb{Z}} \left(\frac{f(x_k)}{u(x_k)} \right)^q h(x_k) \right)^{1/q}.$$

Lemma 1. *[14, Lemma 3.5] Let $p,q,r \in (0,\infty)$. Let u be an admissible function. Let $h \in \Omega_{u^q}$ and g be u^p-quasiconcave function on $(0,\infty)$. Let $\{x_k\}$ be a discretizing sequence of h with respect to u^q. Then*

$$\sup_{t\in(0,\infty)} \frac{h(t)^{1/q}}{g(t)^{1/p}} \approx \sup_{k\in\mathbb{Z}} \frac{h(x_k)^{1/q}}{g(x_k)^{1/p}}.$$

If $q \in (0, +\infty]$ and $\{w_k\} = \{w_k\}_{k\in\mathbb{Z}}$ is a sequence of positive numbers, we denote by $\ell^q(\{w_k\}, \mathbb{Z})$ the following discrete analogue of a weighted Lebesgue space: if $0 < q < +\infty$, then

$$\ell^q(\{w_k\}, \mathbb{Z}) = \left\{ \{a_k\}_{k\in\mathbb{Z}} : \ \|a_k\|_{\ell^q(\{w_k\},\mathbb{Z})} := \left(\sum_{k\in\mathbb{Z}} |a_k w_k|^q \right)^{1/q} < +\infty \right\}$$

and

$$\ell^\infty(\{w_k\}, \mathbb{Z}) = \left\{ \{a_k\}_{k\in\mathbb{Z}} : \ \|a_k\|_{\ell^\infty(\{w_k\},\mathbb{Z})} := \sup_{k\in\mathbb{Z}} |a_k w_k| < +\infty \right\}.$$

If $w_k = 1$ for all $k \in \mathbb{Z}$, we write simply $\ell^q(\mathbb{Z})$ instead of $\ell^q(\{w_k\}, \mathbb{Z})$.

We quote some known results (see, for instance, [14, Lemma 3.1 and 3.2]).

Lemma 2. *Let $q \in (0, +\infty]$. If $\{\tau_k\}_{k\in\mathbb{Z}}$ is a geometrically decreasing sequence, then*

$$\left\| \tau_k \sum_{m\le k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathbb{Z})}$$

and

$$\left\| \tau_k \sup_{m\le k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\tau_k a_k\|_{\ell^q(\mathbb{Z})}$$

for all non-negative sequences $\{a_k\}_{k\in\mathbb{Z}}$.

Let $\{\sigma_k\}_{k\in\mathbb{Z}}$ be a geometrically increasing sequence. Then

$$\left\| \sigma_k \sum_{m\ge k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\sigma_k a_k\|_{\ell^q(\mathbb{Z})}$$

and

$$\left\| \sigma_k \sup_{m\ge k} a_m \right\|_{\ell^q(\mathbb{Z})} \approx \|\sigma_k a_k\|_{\ell^q(\mathbb{Z})}$$

for all non-negative sequences $\{a_k\}_{k\in\mathbb{Z}}$.

Given two (quasi-)Banach spaces X and Y, we write $X \hookrightarrow Y$ if $X \subset Y$ and if the natural embedding of X in Y is continuous.

The following statement is a discrete version of the classical Landau resonance theorem. Proof can be found, for example, in [14].

Proposition 1. ([14, Proposition 4.1]) *Let* $0 < \theta$, $q \le +\infty$ *and let* $\{v_k\}_{k\in\mathbb{Z}}$ *and* $\{w_k\}_{k\in\mathbb{Z}}$ *be two sequences of positive numbers. Assume that*

$$\ell^\theta(\{v_k\},\mathbb{Z}) \hookrightarrow \ell^q(\{w_k\},\mathbb{Z}). \tag{7}$$

Then

$$\big\|\{w_k v_k^{-1}\}\big\|_{\ell^\rho(\mathbb{Z})} \le C,$$

where $1/\rho := (1/q - 1/\theta)_+$ † *and* C *stands for the norm of embedding* (7).

We shall use the following inequality, which is a simple consequence of the discrete Hölder inequality:

$$\|\{a_k b_k\}\|_{\ell^q(\mathbb{Z})} \le \|\{a_k\}\|_{\ell^\rho(\mathbb{Z})}\|\{b_k\}\|_{\ell^\theta(\mathbb{Z})}. \tag{8}$$

Finally, we recall the following "an integration by parts" formula.

Proposition 2. *Let* g *be a non-negative function on* $(0,\infty)$ *such that* $0 < \int_0^x g(t)\,dt < \infty$, $x > 0$. *Assume that* f *is a non-negative non-increasing right-continuous function on* $(0,\infty)$ *such that* $\lim_{x\to\infty} f(x) = 0$. *Let* $\alpha > 0$. *Then*

$$\int_0^\infty \left(\int_0^x g\right)^\alpha g(x)f(x)\,dx < \infty \quad\iff\quad \int_{(0,\infty)} \left(\int_0^x g\right)^{\alpha+1} d[-f(x)] < \infty.$$

Moreover,

$$\int_0^\infty \left(\int_0^x g\right)^\alpha g(x)f(x)\,dx \approx \int_{(0,\infty)} \left(\int_0^x g\right)^{\alpha+1} d[-f(x)].$$

3. Discretization of Inequality (4)

In this section we discretize the inequality

$$\left(\int_{(0,\infty)} \left(\frac{1}{U(t)}\int_0^t \left(\int_{{}^cB(0,y)} h(z)dz\right)^p u(y)dy\right)^{q/p} d\mu(t)\right)^{1/q} \le c\|h\|_{\theta,v,\mathbb{R}^n}. \tag{9}$$

At first we do the following remarks.

†For any $a \in \mathbb{R}$ we denote $a_+ = a$ when $a > 0$ and $a_+ = 0$ when $a \le 0$.

Remark 1. *Recall that, if F is a non-negative non-increasing function on $(0,\infty)$, then*

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{\tau\in(0,t)} G(\tau); \tag{10}$$

likewise, when F is a non-negative non-decreasing function on $(0,\infty)$, then

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} F(t)G(t) = \operatorname*{ess\,sup}_{t\in(0,\infty)} F(t) \operatorname*{ess\,sup}_{\tau\in(t,\infty)} G(\tau) \tag{11}$$

(see, for instance, [15, p. 85]).

Given a non-negative non-decreasing function b on $(0,\infty)$, denote

$$\mathcal{B}(x,t) := \frac{b(x)}{b(x)+b(t)} \qquad (x>0,\ t>0).$$

Observe that

$$\mathcal{B}(x,t) \approx \min\left\{1, \frac{b(x)}{b(t)}\right\}.$$

It is easy to see that $\mathcal{B}(x,t)$ is a b-quasiconcave function of x for any fixed $t>0$.

It has been shown in [15, p. 85] that the relation

$$\begin{aligned}
\operatorname*{ess\,sup}_{t\in(0,\infty)} \mathcal{B}(x,t)g(t) &\approx \operatorname*{ess\,sup}_{t\in(0,\infty)} g(t)\min\left\{1,\frac{b(x)}{b(t)}\right\} \\
&= \operatorname*{ess\,sup}_{t\in(0,x)} b(t) \operatorname*{ess\,sup}_{\tau\in(t,\infty)} \frac{g(\tau)}{b(\tau)} \\
&= b(x) \operatorname*{ess\,sup}_{t\in(x,\infty)} \frac{1}{b(t)} \operatorname*{ess\,sup}_{\tau\in(0,t)} g(\tau)
\end{aligned} \tag{12}$$

holds for any $g\in\mathfrak{M}^+(0,\infty)$. Consequently, $\operatorname{ess\,sup}_{t\in(0,\infty)} \mathcal{B}(x,t)g(t)$ is a b-quasiconcave function.

Lemma 3. *Let $0<p,\ q<\infty$ and let $u\in\mathcal{W}(0,\infty)$, $v\in\mathcal{W}(\mathbb{R}^n)$. Assume that u is such that U is admissible. Suppose that non-negative Borel measure μ on $(0,\infty)$ is non-degenerate with respect to $U^{q/p}$. Let $\{x_k\}$ be any discretizing sequence for the fundamental function φ of μ with respect to $U^{q/p}$. Then*

$$\mathrm{LHS}\,(9) \approx \left\| \left\{ \left\| \int_{S[y,x_k)} h(z)dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})} +$$

$$+ \left\| \left\{ \varphi(x_k)^{1/q} \int_{S[x_k,x_{k+1})} h(z)dz \right\} \right\|_{\ell^q(\mathbb{Z})}.$$

Proof. Let $0 < p,\, q < \infty$. Suppose that U is admissible on $(0,\infty)$. Assume that μ is a non-negative Borel measure on $(0,\infty)$ and φ is the fundamental function of μ with respect to $U^{q/p}$, that is,

$$\varphi(x) := \int_{(0,\infty)} \mathcal{U}(x,t)^{q/p} d\mu(t) \qquad \text{for all} \qquad x \in (0,\infty), \tag{13}$$

where

$$\mathcal{U}(x,t) := \frac{U(x)}{U(t)+U(x)}.$$

Assume that the measure μ is non-degenerate with respect to $U^{q/p}$:

$$\begin{gathered}\int_{(0,\infty)} \frac{d\mu(t)}{U(t)^{q/p}+U(x)^{q/p}} < \infty,\ x \in (0,\infty), \\ \int_{(0,1]} \frac{d\mu(t)}{U(t)^{q/p}} = \int_{[1,\infty)} d\mu(t) = \infty.\end{gathered} \tag{14}$$

Then $\varphi \in \Omega_{U^{q/p}}$, and there exists a discretizing sequence for φ with respect to $U^{q/p}$. Let $\{x_k\}$ be one such sequence. Then $\varphi(x_k) \uparrow\uparrow$ and $\varphi(x_k)U^{-q/p}(x_k) \downarrow\downarrow$. Furthermore, there is a decomposition $\mathbb{Z} = \mathbb{Z}_1 \cup \mathbb{Z}_2$, $\mathbb{Z}_1 \cap \mathbb{Z}_2 = \emptyset$ such that for every $k \in \mathbb{Z}_1$ and $t \in [x_k, x_{k+1}]$, $\varphi(x_k) \approx \varphi(t)$ and for every $k \in \mathbb{Z}_2$ and $t \in [x_k, x_{k+1}]$, $\varphi(x_k)U(x_k)^{-q/p} \approx \varphi(t)U(t)^{-q/p}$.

Applying Corollary 1 to the U-quasiconcave function

$$f(t) = \int_0^t \left(\int_{{}^{c}B(0,y)} h(z)dz \right)^p u(y)dy,$$

we get

$$\text{LHS}\,(9) \approx \left\| \left\{ \left\| \int_{{}^{c}B(0,y)} h(z)dz \right\|_{p,u,(0,x_k)} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})}.$$

Using Lemma 2

$$\begin{aligned}\text{LHS}\,(9) &\approx \left\| \left\{ \left\| \int_{S[y,\infty)} h(z)dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})} \\ &\approx \left\| \left\{ \left\| \int_{S[y,x_k)} h(z)dz + \int_{S[x_k,\infty)} h(z)dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})}\end{aligned}$$

$$\approx \left\| \left\{ \left\| \int_{S[y,x_k)} h(z)dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})} + \left\| \left\{ \left\| \int_{S[x_k,\infty)} h(z)dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})},$$

where $I_k := [x_{k-1}, x_k)$, $k \in \mathbb{Z}$. Since $\|1\|_{p,u,I_k}^p \approx U(x_k)$, we obtain

$$\text{LHS}\,(9) \approx \left\| \left\{ \left\| \int_{S[y,x_k)} h(z)dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{1/q} \int_{S[x_k,\infty)} h(z)dz \right\} \right\|_{\ell^q(\mathbb{Z})}.$$

By using Lemma 2 on the second term, we arrive at

$$\text{LHS}\,(9) \approx \left\| \left\{ \left\| \int_{S[y,x_k)} h(z)dz \right\|_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{1/q} \int_{S[x_k,x_{k+1})} h(z)dz \right\} \right\|_{\ell^q(\mathbb{Z})}.$$

◄

Lemma 4. *Let $0 < p, q < \infty$, $1 \le \theta \le \infty$, $1/\rho = (1/q - 1/\theta)_+$, and let $u \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(\mathbb{R}^n)$ be such that U is admissible and $V_\theta(t) < \infty$, $t \in (0,\infty)$ with $\lim_{t\to\infty} V_\theta(t) = 0$. Suppose that non-negative Borel measure μ on $(0,\infty)$ is non-degenerate with respect to $U^{q/p}$. Let $\{x_k\}$ be any discretizing sequence for the fundamental function φ of the measure μ with respect to $U^{q/p}$. Then inequality (9) holds for every $h \in \mathfrak{M}^+(\mathbb{R}^n)$ if and only if*

$$A := \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} B(x_{k-1}, x_k) \right\} \right\|_{\ell^\rho(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{1/q} C(x_k, x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} < \infty, \tag{15}$$

where

$$B(x_{k-1}, x_k) := \sup_{h \in \mathfrak{M}^+(S[x_{k-1},x_k))} \frac{\left\| \int_{S[t,x_k)} h(z)dz \right\|_{p,u,[x_{k-1},x_k)}}{\|h\|_{\theta,v,S[x_{k-1},x_k)}}, \tag{16}$$

and

$$C(x_k, x_{k+1}) := \sup_{h\in\mathfrak{M}^+(S[x_k,x_{k+1}))} \frac{\|h\|_{1,S[x_k,x_{k+1})}}{\|h\|_{\theta,v,S[x_k,x_{k+1})}}. \tag{17}$$

Moreover, the best constant in inequality (9) *satisfies* $c \approx A$.

Proof. **Sufficiency**. In view of (16) and inequality (8), we have

$$\begin{aligned}
&\left\|\left\{\left\|\int_{S[y,x_k)} h(z)dz\right\|_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\right\}\right\|_{\ell^q(\mathbb{Z})} \\
&\le \left\|\left\{B(x_{k-1},x_k)\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\|h\|_{\theta,v,S[x_{k-1},x_k)}\right\}\right\|_{\ell^q(\mathbb{Z})} \\
&\le \left\|\left\{B(x_{k-1},x_k)\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\right\}\right\|_{\ell^\rho(\mathbb{Z})} \left\|\{\|h\|_{\theta,v,S[x_{k-1},x_k)}\}\right\|_{\ell^\theta(\mathbb{Z})} \\
&= \left\|\left\{B(x_{k-1},x_k)\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\right\}\right\|_{\ell^\rho(\mathbb{Z})} \|h\|_{\theta,v,\mathbb{R}^n}.
\end{aligned} \tag{18}$$

By (17) and (8), we get

$$\begin{aligned}
&\left\|\left\{\varphi(x_k)^{1/q}\int_{S[x_k,x_{k+1})} h(z)dz\right\}\right\|_{\ell^q(\mathbb{Z})} \\
&\le \left\|\left\{\varphi(x_k)^{1/q}C(x_k,x_{k+1})\|h\|_{\theta,v,S[x_k,x_{k+1})}\right\}\right\|_{\ell^q(\mathbb{Z})} \\
&\lesssim \left\|\left\{\varphi(x_k)^{1/q}C(x_k,x_{k+1})\right\}\right\|_{\ell^\rho(\mathbb{Z})} \|h\|_{\theta,v,\mathbb{R}^n}.
\end{aligned} \tag{19}$$

By Lemma 3, using (18) and (19), we obtain

$$\begin{aligned}
\mathrm{LHS}\,(9) \lesssim &\left(\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}B(x_{k-1},x_k)\right\}\right\|_{\ell^\rho(\mathbb{Z})}\right. \\
&\left.+\left\|\left\{\varphi(x_k)^{1/q}C(x_k,x_{k+1})\right\}\right\|_{\ell^\rho(\mathbb{Z})}\right)\|h\|_{\theta,v,\mathbb{R}^n} = A\,\|h\|_{\theta,v,\mathbb{R}^n}.
\end{aligned}$$

Consequently, (9) holds provided that $A < \infty$ and $c \le A$.

Necessity. Assume that the inequality (9) holds with $c < \infty$. By (16), there are $h_k \in \mathfrak{M}^+(\mathbb{R}^n)$, $k \in \mathbb{Z}$, such that $\operatorname{supp} h_k \subset S[x_{k-1}, x_k)$,

$$\|h_k\|_{\theta,v,S[x_{k-1},x_k)} = 1, \left\| \int_{S[y,x_k)} h_k(z)dz \right\|_{p,u,I_k} \geq \frac{1}{2} B(x_{k-1}, x_k) \text{ for all } k \in \mathbb{Z}. \quad (20)$$

Define

$$h = \sum_{m \in \mathbb{Z}} a_m h_m, \quad (21)$$

where $\{a_k\}_{k \in \mathbb{Z}}$ is any sequence of positive numbers. Then, by Lemma 3, we have

$$\text{LHS}\,(9) \gtrsim \left\| \left\{ \left\| \int_{S[y,x_k)} \sum_{m \in \mathbb{Z}} a_m h_m \right\|_{p,u,I_k} \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})}$$

$$\gtrsim \left\| \left\{ a_k B(x_{k-1}, x_k) \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})}. \quad (22)$$

Moreover,

$$\text{RHS}\,(9) = c \left\| \sum_{m \in \mathbb{Z}} a_m h_m \right\|_{\theta,v,\mathbb{R}^n} = c \, \|\{a_k\}\|_{\ell^\theta(\mathbb{Z})}. \quad (23)$$

By (9), (22) and (23), we obtain

$$\left\| \left\{ a_k B(x_{k-1}, x_k) \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \right\} \right\|_{\ell^q(\mathbb{Z})} \lesssim c \, \|\{a_k\}\|_{\ell^\theta(\mathbb{Z})}. \quad (24)$$

Then, by Proposition 1 we arrive at

$$\left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} B(x_{k-1}, x_k) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \lesssim c. \quad (25)$$

On the other hand, by (17), there are $\psi_k \in \mathfrak{M}^+(\mathbb{R}^n)$, $k \in \mathbb{Z}$, such that $\operatorname{supp} \psi_k \subset S[x_k, x_{k+1})$,

$$\|\psi_k\|_{\theta,v,S[x_k,x_{k+1})} = 1 \text{ and } \|\psi_k\|_{1,S[x_k,x_{k+1})} \geq \frac{1}{2} C(x_k, x_{k+1}) \text{ for all } k \in \mathbb{Z}. \quad (26)$$

Define

$$h = \sum_{m\in\mathbb{Z}} b_m \psi_m, \tag{27}$$

where $\{b_k\}_{k\in\mathbb{Z}}$ is any sequence of positive numbers. Then, by Lemma 3, we have

$$\text{LHS}\,(9) \gtrsim \left\| \left\{ \varphi(x_k)^{1/q} \int_{S[x_k,x_{k+1})} \sum_{m\in\mathbb{Z}} b_m \psi_m \right\} \right\|_{\ell^q(\mathbb{Z})} \gtrsim$$

$$\gtrsim \left\| \left\{ b_k \varphi(x_k)^{1/q} C(x_k,x_{k+1}) \right\} \right\|_{\ell^q(\mathbb{Z})}.$$

We also have

$$\text{RHS}\,(9) = c \left\| \sum_{m\in\mathbb{Z}} b_m \psi_m \right\|_{\theta,v,\mathbb{R}^n} = c \left\| \{b_k\} \right\|_{\ell^\theta(\mathbb{Z})}.$$

Consequently

$$\left\| \left\{ b_k \varphi(x_k)^{1/q} C(x_k,x_{k+1}) \right\} \right\|_{\ell^q(\mathbb{Z})} \lesssim c \left\| \{b_k\} \right\|_{\ell^\theta(\mathbb{Z})}.$$

Then, applying Proposition 1, we get

$$\left\| \left\{ \varphi(x_k)^{1/q} C(x_k,x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \lesssim c. \tag{28}$$

Combining (25) and (28), we arrive at $A \lesssim c$. ◄

Remark 2. *Let $1 \le \theta \le \infty$. Note that*

$$C(x_k,x_{k+1}) = \begin{cases} \left\| v^{-1/\theta} \right\|_{\theta',S[x_k,x_{k+1})}, & \textit{when } \theta < \infty, \\ \left\| v^{-1} \right\|_{1,S[x_k,x_{k+1})}, & \textit{when } \theta = \infty, \end{cases} \qquad k \in \mathbb{Z}. \tag{29}$$

If $\theta < \infty$, then, in view of Lemma 2, it is evident that

$$\begin{aligned} \left\| \left\{ \varphi(x_k)^{1/q} C(x_k,x_{k+1}) \right\} \right\|_{\ell^\rho(\mathbb{Z})} &= \left\| \left\{ \varphi(x_k)^{1/q} \left\| v^{-1/\theta} \right\|_{\theta',S[x_k,x_{k+1})} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &\approx \left\| \left\{ \varphi(x_k)^{1/q} \left\| v^{-1/\theta} \right\|_{\theta',S[x_k,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})}. \end{aligned}$$

Monotonicity of $\left\| v^{-1/\theta} \right\|_{\theta',S[t,\infty)}$ implies that

$$\left\| \left\{ \varphi(x_k)^{1/q} \left\| v^{-1/\theta} \right\|_{\theta',S[x_k,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \ge \left\| \left\{ \varphi(x_k)^{1/q} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \lim_{t\to\infty} \left\| v^{-1/\theta} \right\|_{\theta',S[t,\infty)}.$$

Since $\{\varphi(x_k)^{1/q}\}$ *is geometrically increasing, we obtain*

$$\left\|\left\{\varphi(x_k)^{1/q}\|v^{-1/\theta}\|_{\theta',S[x_k,\infty)}\right\}\right\|_{\ell^\rho(\mathbb{Z})} \geq \varphi(\infty)^{1/q}\lim_{t\to\infty}\|v^{-1/\theta}\|_{\theta',S[t,\infty)}.$$

This inequality shows that $\lim_{t\to\infty}\|v^{-1/\theta}\|_{\theta',S[t,\infty)}$ *must be equal to* 0, *because* $\varphi(\infty)$ *is always equal to* ∞ *by our assumptions on the function* φ.

Similarly, $\lim_{t\to\infty}\|v^{-1}\|_{1,S[t,\infty)}$ *must be equal to* 0, *when* $\theta = \infty$.

Therefore, throughout the paper we consider weight functions v *such that* $\lim_{t\to\infty} V_\theta(t) = 0$.

Note also that the condition $V_\theta(t) < \infty$, $t \in (0,\infty)$ *implies* $\lim_{t\to\infty} V_\theta(t) = 0$, *when* $1 < \theta \leq \infty$.

4. Anti-discretization of conditions

In this section we anti-discretize the conditions obtained in Lemma 4.

Lemma 5. *Let* $0 < p, q < \infty$, $1 \leq \theta \leq \infty$, $1/\rho = (1/q - 1/\theta)_+$ *and let* $u \in \mathcal{W}(0,\infty)$ *and* $v \in \mathcal{W}(\mathbb{R}^n)$ *be such that* U *is admissible and* $V_\theta(t) < \infty$, $t \in (0,\infty)$ *with* $\lim_{t\to\infty} V_\theta(t) = 0$. *Suppose that non-negative Borel measure* μ *on* $(0,\infty)$ *is non-degenerate with respect to* $U^{q/p}$. *Let* $\{x_k\}$ *be any discretizing sequence for the fundamental function* φ *of the measure* μ *with respect to* $U^{q/p}$.

(a) *If* $\theta \leq p$, *then* $A \approx A^*$, *where*

$$A^* := \left\|\left\{\varphi(x_k)^{1/q}\left(\sup_{t\in(0,\infty)}\mathcal{U}(t,x_k)^{1/p}V_\theta(t)\right)\right\}\right\|_{\ell^\rho(\mathbb{Z})}.$$

(b) *If* $p < \theta$ *and* $1/r = 1/p - 1/\theta$, *then*

$$A \approx B^*,$$

where

$$B^* := \left\|\left\{\varphi(x_k)^{1/q}\left(\int_{(0,\infty)}\mathcal{U}(t,x_k)^{r/p}d\left(-V_\theta(t-)^r\right)\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}.$$

Here

$$V_\theta(t-) := \lim_{\tau\to t-} V_\theta(\tau).$$

Proof. (a) By [21, Theorem 2.2, (a) and (f)], from Lemma 4 we have

$$A \approx \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \sup_{t\in I_k} \left(\int_{x_{k-1}}^{t} u(s)ds \right)^{1/p} \left\|v^{-1/\theta}\right\|_{\theta',S[t,x_k)} \right\} \right\|_{\ell^\rho(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{1/q} \left\|v^{-1/\theta}\right\|_{\theta',S[x_k,x_{k+1})} \right\} \right\|_{\ell^\rho(\mathbb{Z})}.$$

By Lemma 2, we get

$$\begin{aligned} A &\lesssim \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \sup_{t\in I_k} U(t)^{1/p} \left\|v^{-1/\theta}\right\|_{\theta',S[t,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{1/q} \left\|v^{-1/\theta}\right\|_{\theta',S[x_k,x_{k+1})} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &\approx \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \sup_{t\in(0,x_k)} U(t)^{1/p} \left\|v^{-1/\theta}\right\|_{\theta',S[t,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{1/q} \left\|v^{-1/\theta}\right\|_{\theta',S[x_k,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &= \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \sup_{t\in(0,x_k)} U(t)^{1/p} \left\|v^{-1/\theta}\right\|_{\theta',S[t,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{1/q} \left(\sup_{t\in[x_k,\infty)} \left\|v^{-1/\theta}\right\|_{\theta',S[t,\infty)} \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &\approx \left\| \left\{ \varphi(x_k)^{1/q} \left(\sup_{t\in(0,\infty)} \mathcal{U}(t,x_k)^{1/p} \left\|v^{-1/\theta}\right\|_{\theta',S[t,\infty)} \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} = A^*. \end{aligned}$$

We now prove the reverse estimate. We have

$$\begin{aligned} A^* &\approx \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \sup_{t\in I_k} U(t)^{1/p} \left\|v^{-1/\theta}\right\|_{\theta',S[t,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \varphi(x_k)^{1/q} \left\|v^{-1/\theta}\right\|_{\theta',S[x_k,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &\lesssim \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\sup_{t\in I_k} \left(\int_{x_{k-1}}^{t} u(s)ds \right)^{1/p} \left\|v^{-1/\theta}\right\|_{\theta',S[t,\infty)} \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})} \\ &+ \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} U(x_{k-1})^{1/p} \left\|v^{-1/\theta}\right\|_{\theta',S[x_{k-1},\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \end{aligned}$$

$$+ \left\| \left\{ \varphi(x_k)^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_k,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$\lesssim \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\sup_{t \in I_k} \left(\int_{x_{k-1}}^t u(s) ds \right)^{1/p} \|v^{-1/\theta}\|_{\theta', S[t,\infty)} \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \varphi(x_{k-1})^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_{k-1},\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \varphi(x_k)^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_k,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$\approx \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\sup_{t \in I_k} \left(\int_{x_{k-1}}^t u(s) ds \right)^{1/p} \|v^{-1/\theta}\|_{\theta', S[t,\infty)} \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \varphi(x_k)^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_k,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$\lesssim \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\sup_{t \in I_k} \left(\int_{x_{k-1}}^t u(s) ds \right)^{1/p} \|v^{-1/\theta}\|_{\theta', S[t,x_k)} \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \varphi(x_k)^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_k,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$\approx \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\sup_{t \in I_k} \left(\int_{x_{k-1}}^t u(s) ds \right)^{1/p} \|v^{-1/\theta}\|_{\theta', S[t,x_k)} \right) \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \varphi(x_k)^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_k,x_{k+1})} \right\} \right\|_{\ell^\rho(\mathbb{Z})} \approx A.$$

(b) Assume that $\theta < \infty$. By [21, Theorem 2.2, (b) and (g)], and (29), from Lemma 4, we have

$$A \approx \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^t u(s)\, ds \right)^{r/\theta} u(t) \|v^{-1/\theta}\|^r_{\theta', S[t,x_k)}\, dt \right)^{1/r} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \varphi(x_k)^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_k,x_{k+1})} \right\} \right\|_{\ell^\rho(\mathbb{Z})}.$$

Since

$$\left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^t u(s) ds \right)^{r/\theta} u(t) dt \right)^{1/r} \approx U(x_k)^{1/p},$$

it is easy to see that

$$A \approx \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\int_{x_{k-1}}^{x_k} \left(\int_{x_{k-1}}^t u(s)\, ds \right)^{r/\theta} u(t) \|v^{-1/\theta}\|^r_{\theta', S[t,x_k)}\, dt \right)^{1/r} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t}u(s)ds\right)^{r/\theta}u(t)dt\right)^{1/r}\|v^{-1/\theta}\|_{\theta',S[x_k,x_{k+1})}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$

$$\lesssim\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^{t}u(s)\,ds\right)^{r/\theta}u(t)\|v^{-1/\theta}\|^r_{\theta',S[t,\infty)}\,dt\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$

$$\le\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k}U(t)^{r/\theta}u(t)\|v^{-1/\theta}\|^r_{\theta',S[t,\infty)}\,dt\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$

$$\approx\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k}\|v^{-1/\theta}\|^r_{\theta',S[t,\infty)}\,d\Big(U(t)^{r/p}\Big)\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}.$$

Integrating by parts, we arrive at

$$A\lesssim\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\left(\int_{[x_{k-1},x_k)}U(t)^{r/p}\,d\Big(-\|v^{-1/\theta}\|^r_{\theta',S[t-,\infty)}\Big)\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$+\left\|\left\{\varphi(x_k)^{1/q}\|v^{-1/\theta}\|_{\theta',S[x_k-,\infty)}\right\}\right\|_{\ell^\rho(\mathbb{Z})} \tag{30}$$

By Lemma 2, in view of Remark 2, we have

$$\left\|\left\{\varphi(x_k)^{1/q}\|v^{-1/\theta}\|_{\theta',S[x_k-,\infty)}\right\}\right\|_{\ell^\rho(\mathbb{Z})}=$$

$$=\left\|\left\{\varphi(x_k)^{1/q}\left(\|v^{-1/\theta}\|^r_{\theta',S[x_k-,\infty)}-\lim_{m\to\infty}\|v^{-1/\theta}\|^r_{\theta',S[x_m-,\infty)}\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$

$$\approx\left\|\left\{\varphi(x_k)^{1/q}\left(\sum_{i=k}^{\infty}\Big(\|v^{-1/\theta}\|^r_{\theta',S[x_i-,\infty)}-\|v^{-1/\theta}\|^r_{\theta',S[x_{i+1}-,\infty)}\Big)\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$

$$\approx\left\|\left\{\varphi(x_k)^{1/q}\left(\|v^{-1/\theta}\|^r_{\theta',S[x_k-,\infty)}-\|v^{-1/\theta}\|^r_{\theta',S[x_{k+1}-,\infty)}\right)\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$

$$\approx\left\|\left\{\varphi(x_k)^{1/q}\left(\int_{[x_k,x_{k+1})}d\Big(-\|v^{-1/\theta}\|^r_{\theta',S[t-,\infty)}\Big)\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}.$$

Using the latter in (30) and applying Lemma 2, we arrive at

$$A \lesssim \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\int_{(0,x_k)} U(t)^{r/p}\, d\Big(- \|v^{-1/\theta}\|^r_{\theta', S[t-,\infty)} \Big) \right)^{1/r} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \varphi(x_k)^{1/q} \left(\int_{[x_k,\infty)} d\Big(- \|v^{-1/\theta}\|^r_{\theta', S[t-,\infty)} \Big) \right)^{1/r} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$\approx \left\| \left\{ \varphi(x_k)^{1/q} \left(\int_{(0,\infty)} \mathcal{U}(t,x_k)^{r/p}\, d\Big(- \|v^{-1/\theta}\|^r_{\theta', S[t-,\infty)} \Big) \right)^{1/r} \right\} \right\|_{\ell^\rho(\mathbb{Z})} = B^*.$$

Consequently, $A \lesssim B^*$.

Conversely, by Lemma 2, in view of Remark 2, we have

$$B^* \approx \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\int_{[x_{k-1},x_k)} U(t)^{r/p} d\Big(- \|v^{-1/\theta}\|^r_{\theta', S[t-,\infty)} \Big) \right)^{1/r} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \varphi(x_k)^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_k-,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$\lesssim \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\int_{[x_{k-1},x_k)} \left(\int_{x_{k-1}}^t u(s)ds \right)^{r/p} d\Big(- \|v^{-1/\theta}\|^r_{\theta', S[t-,\infty)} \Big) \right)^{1/r} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} U(x_{k-1})^{1/p} \|v^{-1/\theta}\|_{\theta', S[x_{k-1}-,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \varphi(x_k)^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_k-,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$\lesssim \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\int_{[x_{k-1},x_k)} \left(\int_{x_{k-1}}^t u(s)ds \right)^{r/p} d\Big(- \|v^{-1/\theta}\|^r_{\theta', S[t-,\infty)} \Big) \right)^{1/r} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+ \left\| \left\{ \varphi(x_{k-1})^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_{k-1}-,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})} + \left\| \left\{ \varphi(x_k)^{1/q} \|v^{-1/\theta}\|_{\theta', S[x_k-,\infty)} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$\approx \left\| \left\{ \frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}} \left(\int_{[x_{k-1},x_k)} \left(\int_{x_{k-1}}^t u(s)ds \right)^{r/p} d\Big(- \|v^{-1/\theta}\|^r_{\theta', S[t-,\infty)} \Big) \right)^{1/r} \right\} \right\|_{\ell^\rho(\mathbb{Z})}$$

$$+\left\|\left\{\varphi(x_k)^{1/q}\|v^{-1/\theta}\|_{\theta',S[x_k-,\infty)}\right\}\right\|_{\ell^\rho(\mathbb{Z})}.$$

Integrating by parts yields

$$B^*\lesssim\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\left(\int_{[x_{k-1},x_k)}\|v^{-1/\theta}\|^r_{\theta',S[t,\infty)}d\left(\left(\int_{x_{k-1}}^t u(s)ds\right)^{r/p}\right)\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$+\left\|\left\{\varphi(x_k)^{1/q}\|v^{-1/\theta}\|_{\theta',S[x_k-,\infty)}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$\approx\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\left(\int_{[x_{k-1},x_k)}\left(\int_{x_{k-1}}^t u(s)ds\right)^{r/\theta}u(t)\|v^{-1/\theta}\|^r_{\theta',S[t,\infty)}dt\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$+\left\|\left\{\varphi(x_k)^{1/q}\|v^{-1/\theta}\|_{\theta',S[x_k-,\infty)}\right\}\right\|_{\ell^\rho(\mathbb{Z})}.$$

Since

$$\left\|\left\{\varphi(x_k)^{1/q}\|v^{-1/\theta}\|_{\theta',S[x_k-,\infty)}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$=\left\|\left\{\varphi(x_{k-1})^{1/q}\|v^{-1/\theta}\|_{\theta',S[x_{k-1}-,\infty)}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$\approx\left\|\left\{\frac{\varphi(x_{k-1})^{1/q}}{U(x_{k-1})^{1/p}}\|v^{-1/\theta}\|_{\theta',S[x_{k-1}-,\infty)}\left(\int_{x_{k-2}}^{x_{k-1}}\left(\int_{x_{k-2}}^t u(s)ds\right)^{r/\theta}u(t)dt\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$\le\left\|\left\{\frac{\varphi(x_{k-1})^{1/q}}{U(x_{k-1})^{1/p}}\left(\int_{x_{k-2}}^{x_{k-1}}\left(\int_{x_{k-2}}^t u(s)ds\right)^{r/\theta}u(t)\|v^{-1/\theta}\|^r_{\theta',S[t-,\infty)}dt\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$=\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^t u(s)ds\right)^{r/\theta}u(t)\|v^{-1/\theta}\|^r_{\theta',S[t-,\infty)}dt\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})},$$

we arrive at

$$B^*\lesssim\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^t u(s)ds\right)^{r/\theta}u(t)\|v^{-1/\theta}\|^r_{\theta',S[t,\infty)}dt\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$\approx\left\|\left\{\frac{\varphi(x_k)^{1/q}}{U(x_k)^{1/p}}\left(\int_{x_{k-1}}^{x_k}\left(\int_{x_{k-1}}^t u(s)ds\right)^{r/\theta}u(t)\|v^{-1/\theta}\|^r_{\theta',S[t,x_k)}dt\right)^{1/r}\right\}\right\|_{\ell^\rho(\mathbb{Z})}$$
$$+\left\|\left\{\varphi(x_k)^{1/q}\|v^{-1/\theta}\|_{\theta',S[x_k,\infty)}\right\}\right\|_{\ell^\rho(\mathbb{Z})}\approx A.$$

Now assume that $\theta = \infty$. In this case the proof can be done in the same line and we leave it to the reader. The only difference is that one should apply [21, Theorem 2.2, (e)] and take into account that $C(x_k, x_{k+1}) = \|v^{-1}\|_{1,S[x_k,x_{k+1})}$, $k \in \mathbb{Z}$. ◀

We are now in a position to characterize inequality (9).

Theorem 2. *Let $0 < p, q < \infty$, $1 \le \theta \le \infty$, $1/\rho = (1/q - 1/\theta)_+$ and let $u \in \mathcal{W}(0,\infty)$ and $v \in \mathcal{W}(\mathbb{R}^n)$ be such that U is admissible and $V_\theta(t) < \infty$, $t \in (0,\infty)$ with $\lim_{t\to\infty} V_\theta(t) = 0$. Suppose that non-negative Borel measure μ on $(0,\infty)$ is non-degenerate with respect to $U^{q/p}$. Then the inequality (9) holds for every measurable function on $\mathbb{R}^n$ if and only if*

(i) $\theta \le \min\{p,q\}$ *and*

$$N_1 := \sup_{x\in(0,\infty)} \bigg(\int_{(0,\infty)} \mathcal{U}(x,t)^{q/p}\, d\mu(t) \bigg)^{1/q} \sup_{t\in(0,\infty)} \mathcal{U}(t,x)^{1/p} V_\theta(t) < \infty.$$

Moreover, the best constant in (9) *satisfies* $c \approx N_1$.

(ii) $q < \theta < p$ *and*

$$N_2 := \Bigg(\int_{(0,\infty)} \bigg(\int_{(0,\infty)} \mathcal{U}(x,t)^{q/p}\, d\mu(t) \bigg)^{\rho/\theta} \times$$

$$\times \bigg(\sup_{t\in(0,\infty)} \mathcal{U}(t,x)^{1/p} V_\theta(t) \bigg)^{\rho} d\mu(x) \Bigg)^{1/\rho} < \infty.$$

Moreover, the best constant in (9) *satisfies* $c \approx N_2$.

(iii) $p < \theta \le q$, $r = \theta p/(\theta - p)$ *and*

$$N_3 := \sup_{x\in(0,\infty)} \bigg(\int_{(0,\infty)} \mathcal{U}(x,t)^{q/p}\, d\mu(t) \bigg)^{1/q} \times$$

$$\times \bigg(\int_{(0,\infty)} \mathcal{U}(t,x)^{r/p}\, d\big(- V_\theta(t-)^r \big) \bigg)^{1/r} < \infty.$$

Moreover, the best constant in (9) *satisfies* $c \approx N_3$.

(iv) $\max\{p,q\} < \theta$, $r = \theta p/(\theta - p)$ *and*

$$N_4 := \Bigg(\int_{(0,\infty)} \bigg(\int_{(0,\infty)} \mathcal{U}(x,t)^{q/p}\, d\mu(t) \bigg)^{\rho/\theta} \times$$

$$\times \left(\int_{(0,\infty)} \mathcal{U}(t,x)^{r/p} \, d\big(- V_\theta(t-)^r \big) \right)^{\rho/r} d\mu(x) \Bigg)^{1/\rho} < \infty.$$

Moreover, the best constant in (9) *satisfies* $c \approx N_4$.

(v) $\theta = \infty$ *and*

$$N_5 := \left(\int_{(0,\infty)} \left(\int_{(0,\infty)} \mathcal{U}(t,x) \, d\big(- V_\infty(t)^p \big) \right)^{q/p} d\mu(x) \right)^{1/q} < \infty.$$

Moreover, the best constant in (9) *satisfies* $c \approx N_5$.

Proof.

(i) The proof follows by Lemma 4, Lemma 5, (a), and Lemma 1.

(ii) The proof follows by Lemma 4, Lemma 5, (a) and Theorem 1.

(iii) The proof follows by Lemma 4, Lemma 5, (b), and Lemma 1.

(iv) The proof follows by Lemma 4, Lemma 5, (b), and Theorem 1.

(v) The proof follows by Lemma 4, Lemma 5, (b), and Theorem 1. ◀

5. Characterization of n-dimensional bilinear Hardy inequalities

In this section we give characterization of n-dimensional bilinear Hardy inequalities (1) and (2).

The following note allows us to concentrate our attention only on characterization of (2).

Remark 3. *Note that the inequality*

$$\left(\int_0^\infty \left(\int_{B(0,t)} f \cdot \int_{B(0,t)} g \right)^q u(t) dt \right)^{1/q} \le C \left(\int_{\mathbb{R}^n} f^{p_1} v_1 \right)^{1/p_1} \left(\int_{\mathbb{R}^n} g^{p_2} v_2 \right)^{1/p_2} \tag{31}$$

is equivalent to the inequality

$$\left(\int_0^\infty \left(\int_{{}^c B(0,t)} f \cdot \int_{{}^c B(0,t)} g \right)^q \tilde{u}(t) dt \right)^{1/q}$$

$$\leq C\Big(\int_{\mathbb{R}^n} f^{p_1}\tilde{v}_1\Big)^{1/p_1}\Big(\int_{\mathbb{R}^n} g^{p_2}\tilde{v}_2\Big)^{1/p_2}, \tag{32}$$

where

$$\tilde{u}(t) = u\big(t^{-1}\big)t^{-2}, \tilde{v}_1(x) = v_1\big(|x|^{-2}x\big)|x|^{-2n(1-p_1)}, \tilde{v}_2(x) = v_2\big(|x|^{-2}x\big)|x|^{-2n(1-p_2)}.$$

Indeed: Since any $f \in \mathfrak{M}^+(\mathbb{R}^n)$ *can be uniquely represented as* $f(x) = g\big(|x|^{-2}x\big)|x|^{-2n}$, $g \in \mathfrak{M}^+(\mathbb{R}^n)$, *then inequality* (31) *is equivalent to the following inequality:*

$$\bigg(\int_0^\infty \bigg(\int_{B(0,t)} f\big(|y|^{-2}y\big)|y|^{-2n}\,dy \int_{B(0,t)} g\big(|y|^{-2}y\big)|y|^{-2n}\,dy\bigg)^q u(t)dt\bigg)^{1/q} \leq$$

$$\leq C\bigg(\int_{\mathbb{R}^n} \big(f\big(|y|^{-2}y\big)\big)^{p_1} v_1(y)|y|^{-2np_1}dy\bigg)^{1/p_1} \times$$

$$\times\bigg(\int_{\mathbb{R}^n} \big(g\big(|y|^{-2}y\big)\big)^{p_2} v_2(y)|y|^{-2np_2}\,dy\bigg)^{1/p_2}. \tag{33}$$

Using the substitution $x = |y|^{-2}y$ *in multidimensional integrals, we see that* (33) *is equivalent to the inequality*

$$\bigg(\int_0^\infty \bigg(\int_{{}^{\mathfrak{c}}B(0,1/t)} f(x)dx \cdot \int_{{}^{\mathfrak{c}}B(0,1/t)} g(x)dx\bigg)^q u(t)dt\bigg)^{1/q} \leq$$

$$\leq C\bigg(\int_{\mathbb{R}^n} f(x)^{p_1} v_1\big(|x|^{-2}x\big)|x|^{-2n(1-p_1)}dx\bigg)^{1/p_1} \times$$

$$\times\bigg(\int_{\mathbb{R}^n} g(x)^{p_2} v_2\big(|x|^{-2}x\big)|x|^{-2n(1-p_2)}\,dx\bigg)^{1/p_2},$$

and finally applying $\tau = 1/t$, *we conclude that the latter is equivalent to*

$$\bigg(\int_0^\infty \bigg(\int_{{}^{\mathfrak{c}}B(0,\tau)} f(x)dx \cdot \int_{{}^{\mathfrak{c}}B(0,\tau)} g(x)dx\bigg)^q u\big(\tau^{-1}\big)\tau^{-2}\,d\tau\bigg)^{1/q} \leq$$

$$\leq C\bigg(\int_{\mathbb{R}^n} f(x)^{p_1} v_1\big(|x|^{-2}x\big)|x|^{-2n(1-p_1)}dx\bigg)^{1/p_1} \times$$

$$\times\bigg(\int_{\mathbb{R}^n} g(x)^{p_2} v_2\big(|x|^{-2}x\big)|x|^{-2n(1-p_2)}\,dx\bigg)^{1/p_2}.$$

Now we present and prove our main results.

Theorem 3. *Let $0 < q < \infty$, $1 \le p_1, p_2 \le \infty$, $p_1 \le q$, and let $u \in \mathcal{W}(0,\infty)$, $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$. Then inequality* (2) *holds for all $f, g \in \mathfrak{M}^+(\mathbb{R}^n)$ if and only if:*

(a) $1 \le p_2 \le q < \infty$, *and*

$$B_1 := \sup_{t\in(0,\infty)} U(t)^{1/q} \big\|v_1^{-1/p_1}\big\|_{p_1',\,{}^{\mathrm{c}}B(0,t)} \big\|v_2^{-1/p_2}\big\|_{p_2',\,{}^{\mathrm{c}}B(0,t)} < \infty.$$

Moreover, the best constant C in (2) *satisfies $C \approx B_1$.*

(b) $1 \le p_2 < \infty$, $0 < q < p_2$, $1/r_2 = 1/q - 1/p_2$, *and*

$$B_2 := \sup_{t\in(0,\infty)} \big\|v_1^{-1/p_1}\big\|_{p_1',\,{}^{\mathrm{c}}B(0,t)} \bigg(\int_0^t U(y)^{r_2/p_2} u(y) \big\|v_2^{-1/p_2}\big\|^{r_2}_{p_2',\,{}^{\mathrm{c}}B(0,y)}\,dy\bigg)^{1/r_2} < \infty.$$

Moreover, the best constant C in (2) *satisfies $C \approx B_2$.*

(c) $p_2 = \infty$, *and*

$$B_3 := \sup_{t\in(0,\infty)} \big\|v_1^{-1/p_1}\big\|_{p_1',\,{}^{\mathrm{c}}B(0,t)} \bigg(\int_0^t u(y) \big\|v_2^{-1}\big\|^{q}_{1,\,{}^{\mathrm{c}}B(0,y)}\,dy\bigg)^{1/q} < \infty.$$

Moreover, the best constant C in (2) *satisfies $C \approx B_3$.*

Proof. Interchanging the suprema, we obtain

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\Big\| \int_{{}^{\mathrm{c}}B(0,\cdot)} f \cdot \int_{{}^{\mathrm{c}}B(0,\cdot)} g \Big\|_{q,u,(0,\infty)}}{\|f\|_{p_1,v_1,\mathbb{R}^n}\, \|g\|_{p_2,v_2,\mathbb{R}^n}}$$
$$= \sup_{g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{1}{\|g\|_{p_2,v_2,\mathbb{R}^n}} \sup_{f\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\Big\| \int_{{}^{\mathrm{c}}B(0,\cdot)} f \cdot \int_{{}^{\mathrm{c}}B(0,\cdot)} g \Big\|_{q,u,(0,\infty)}}{\|f\|_{p_1,v_1,\mathbb{R}^n}}. \tag{34}$$

By [21, Theorem 2.2, (a) and (f)], we get

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\Big\| \int_{{}^{\mathrm{c}}B(0,\cdot)} f \cdot \int_{{}^{\mathrm{c}}B(0,\cdot)} g \Big\|_{q,u,(0,\infty)}}{\|f\|_{p_1,v_1,\mathbb{R}^n}\, \|g\|_{p_2,v_2,\mathbb{R}^n}}$$
$$\approx \sup_{g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{1}{\|g\|_{p_2,v_2,\mathbb{R}^n}} \sup_{t\in(0,\infty)} \bigg(\int_0^t \bigg(\int_{{}^{\mathrm{c}}B(0,\tau)} g\bigg)^q u(\tau)\,d\tau\bigg)^{1/q} \big\|v_1^{-1/p_1}\big\|_{p_1',\,{}^{\mathrm{c}}B(0,t)}.$$

(a) Let $1 \le p_2 \le q < \infty$. Again, interchanging the suprema, by [21, Theorem 2.2, (a) and (f)], on using (10), we get

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\left\| \int_{{}^{\mathfrak{c}}B(0,\cdot)} f \cdot \int_{{}^{\mathfrak{c}}B(0,\cdot)} g \right\|_{q,u,(0,\infty)}}{\|f\|_{p_1,v_1,\mathbb{R}^n} \|g\|_{p_2,v_2,\mathbb{R}^n}}$$

$$= \sup_{t\in(0,\infty)} \big\|v_1^{-1/p_1}\big\|_{p_1',{}^{\mathfrak{c}}B(0,t)} \sup_{g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{{}^{\mathfrak{c}}B(0,\tau)} g\right)^q \chi_{(0,t)}(\tau) u(\tau)\, d\tau \right)^{1/q}}{\|g\|_{p_2,v_2,\mathbb{R}^n}}$$

$$\approx \sup_{t\in(0,\infty)} \big\|v_1^{-1/p_1}\big\|_{p_1',{}^{\mathfrak{c}}B(0,t)} \sup_{y\in(0,\infty)} \left(\int_0^y \chi_{(0,t)}(\tau) u(\tau)\, d\tau \right)^{1/q} \big\|v_2^{-1/p_2}\big\|_{p_2',{}^{\mathfrak{c}}B(0,y)}$$

$$= \sup_{t\in(0,\infty)} \big\|v_1^{-1/p_1}\big\|_{p_1',{}^{\mathfrak{c}}B(0,t)} \sup_{y\in(0,t)} \left(\int_0^y u \right)^{1/q} \big\|v_2^{-1/p_2}\big\|_{p_2',{}^{\mathfrak{c}}B(0,y)}$$

$$= \sup_{t\in(0,\infty)} U(t)^{1/q} \big\|v_1^{-1/p_1}\big\|_{p_1',{}^{\mathfrak{c}}B(0,t)} \big\|v_2^{-1/p_2}\big\|_{p_2',{}^{\mathfrak{c}}B(0,t)}.$$

(b) Let $1 \le p_2 < \infty$, $0 < q < p_2$ and $1/r_2 = 1/q - 1/p_2$. Interchanging the suprema, by [21, Theorem 2.2, (b) and (g)], we obtain

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\left\| \int_{{}^{\mathfrak{c}}B(0,\cdot)} f \cdot \int_{{}^{\mathfrak{c}}B(0,\cdot)} g \right\|_{q,u,(0,\infty)}}{\|f\|_{p_1,v_1,\mathbb{R}^n} \|g\|_{p_2,v_2,\mathbb{R}^n}}$$

$$\approx \sup_{t\in(0,\infty)} \big\|v_1^{-1/p_1}\big\|_{p_1',{}^{\mathfrak{c}}B(0,t)} \sup_{g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{{}^{\mathfrak{c}}B(0,\tau)} g\right)^q \chi_{(0,t)}(\tau) u(\tau)\, d\tau \right)^{1/q}}{\|g\|_{p_2,v_2,\mathbb{R}^n}}$$

$$\approx \sup_{t\in(0,\infty)} \big\|v_1^{-1/p_1}\big\|_{p_1',{}^{\mathfrak{c}}B(0,t)} \times$$

$$\times \left(\int_0^\infty \left(\int_0^y \chi_{(0,t)}(\tau) u(\tau)\, d\tau \right)^{r_2/p_2} \chi_{(0,t)}(y) u(y) \big\|v_2^{-1/p_2}\big\|_{p_2',{}^{\mathfrak{c}}B(0,y)}^{r_2}\, dy \right)^{1/r_2}$$

$$= \sup_{t\in(0,\infty)} \big\|v_1^{-1/p_1}\big\|_{p_1',{}^{\mathfrak{c}}B(0,t)} \left(\int_0^t U(y)^{r_2/p_2} u(y) \big\|v_2^{-1/p_2}\big\|_{p_2',{}^{\mathfrak{c}}B(0,y)}^{r_2}\, dy \right)^{1/r_2}.$$

(c) Let $p_2 = \infty$. Interchanging the suprema, by [21, Theorem 2.2, (e)], we obtain

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{{}^{\mathsf{c}}B(0,t)} f \cdot \int_{{}^{\mathsf{c}}B(0,t)} g\right)^q u(t)\,dt\right)^{1/q}}{\|f\|_{p_1,v_1,\mathbb{R}^n}\,\|g\|_{p_2,v_2,\mathbb{R}^n}}$$

$$\approx \sup_{t\in(0,\infty)} \left\|v_1^{-1/p_1}\right\|_{p_1',{}^{\mathsf{c}}B(0,t)} \sup_{g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{{}^{\mathsf{c}}B(0,\tau)} g\right)^q \chi_{(0,t)}(\tau)u(\tau)\,d\tau\right)^{1/q}}{\|g\|_{p_2,v_2,\mathbb{R}^n}}$$

$$\approx \sup_{t\in(0,\infty)} \left\|v_1^{-1/p_1}\right\|_{p_1',{}^{\mathsf{c}}B(0,t)} \left(\int_0^t u(y)\left\|v_2^{-1}\right\|_{1,{}^{\mathsf{c}}B(0,y)}^q\,dy\right)^{1/q}.$$

◀

Theorem 4. *Let $1 \le p_1, p_2 < \infty$, $0 < q < p_1$, $1/r_1 = 1/q - 1/p_1$. Suppose that v_1, $v_2 \in \mathcal{W}(\mathbb{R}^n)$ are such that $\|v_i^{-1/p_i}\|_{p_i',{}^{\mathsf{c}}B(0,t)} < \infty$, $t \in (0,\infty)$ with $\lim_{t\to\infty} \|v_i^{-1/p_i}\|_{p_i',{}^{\mathsf{c}}B(0,t)} = 0$, $i = 1,2$. Assume that $u \in \mathcal{W}(0,\infty)$ is such that $U^{r_1/q}$ is admissible and the fundamental function of the measure*

$$d\mu(t) = U(t)^{r_1/q} d\left[-\left\|v_1^{-1/p_1}\right\|_{p_1',{}^{\mathsf{c}}B(0,t)}^{r_1/p_1'}\right]$$

is non-degenerate with respect to $U^{r_1/q}$, that is, $\varphi \in \Omega_{U^{r_1/q}}$, where

$$\varphi(x) = \int_{(0,\infty)} \left(\frac{U(x)U(t)}{U(x)+U(t)}\right)^{r_1/q} d\left[-\left\|v_1^{-1/p_1}\right\|_{p_1',{}^{\mathsf{c}}B(0,t)}^{r_1}\right], \quad x \in (0,\infty).$$

Then inequality (2) holds for all f, $g \in \mathfrak{M}^+(\mathbb{R}^n)$ if and only if:

(i) *$p_2 \le q$, and*

$$A_1 := \sup_{x\in(0,\infty)} \left(\int_{(0,\infty)} \mathcal{U}(x,t)^{r_1/q}\, U(t)^{r_1/q}\, d\left[-\left\|v_1^{-1/p_1}\right\|_{p_1',{}^{\mathsf{c}}B(0,x)}^{r_1}\right]\right)^{1/r_1}$$
$$\times \sup_{t\in(0,\infty)} \mathcal{U}(t,x)^{1/q}\left\|v_2^{-1/p_2}\right\|_{p_2',({}^{\mathsf{c}}B(0,t))} < \infty.$$

Moreover, the best constant C in (2) satisfies $C \approx A_1$.

(ii) *$q < p_2 \le r_1$, $1/r_2 = 1/q - 1/p_2$, and*

$$A_2 := \sup_{x\in(0,\infty)} \left(\int_{(0,\infty)} \mathcal{U}(x,t)^{r_1/q}\, U(t)^{r_1/q} d\left[-\left\|v_1^{-1/p_1}\right\|_{p_1',{}^{\mathsf{c}}B(0,t)}^{r_1}\right]\right)^{1/r_1}$$

$$\times \left(\int_{(0,\infty)} \mathcal{U}(t,x)^{r_2/q}\, d\Big(- \big\|v^{-1/p_2}\big\|^{r_2}_{p_2',(t-,\infty)}\Big)\right)^{1/r_2} < \infty.$$

Moreover, the best constant C in (2) *satisfies $C \approx A_2$.*

(iii) $r_1 < p_2 < \infty$, $1/r_2 = 1/q - 1/p_2$, $1/l = 1/r_1 - 1/p_2$, *and*

$$\begin{aligned} A_3 :=& \Bigg(\int_{(0,\infty)} \Bigg(\int_{(0,\infty)} \mathcal{U}(x,t)^{r_1/q}\, U(t)^{r_1/q} d \left[- \big\|v_1^{-1/p_1}\big\|^{r_1}_{p_1',\,{}^{\mathfrak{c}}B(0,t)}\right]\Bigg)^{(l-r_1)/r_1} \\ &\times \left(\int_{(0,\infty)} \mathcal{U}(t,x)^{r_2/q}\, d\Big(- \big\|v^{-1/p_2}\big\|^{r_2}_{p_2',(t-,\infty)}\Big)\right)^{l/r_2} \\ &\times U(x)^{r_1/q}\, d \left[- \big\|v_1^{-1/p_1}\big\|^{r_1}_{p_1',\,{}^{\mathfrak{c}}B(0,x)}\right]\Bigg)^{1/l} < \infty. \end{aligned}$$

Moreover, the best constant C in (2) *satisfies $C \approx A_3$.*

(iv) $p_2 = \infty$, *and*

$$\begin{aligned} A_4 := \Bigg(\int_{(0,\infty)} &\left(\int_{(0,\infty)} \mathcal{U}(t,x)\, d\Big(- \big\|v_2^{-1}\big\|^{q}_{1,S[t,\infty)}\Big)\right)^{r_1/q} \\ &\times U(x)^{r_1/q} d \left[- \big\|v_1^{-1/p_1}\big\|^{r_1}_{p_1',\,{}^{\mathfrak{c}}B(0,x)}\right]\Bigg)^{1/r_1} < \infty. \end{aligned}$$

Moreover, the best constant C in (2) *satisfies $C \approx A_4$.*

Proof. Assume that $1 \le p_1,\, p_2 < \infty$, $0 < q < p_1$ and $1/r_1 = 1/q - 1/p_1$. By [21, Theorem 2.2, (b) and (g)], (34) yields

$$\begin{aligned} &\sup_{f,g \in \mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{{}^{\mathfrak{c}}B(0,t)} f \cdot \int_{{}^{\mathfrak{c}}B(0,t)} g\right)^q u(t)\, dt\right)^{1/q}}{\left(\int_{\mathbb{R}^n} f^{p_1} v_1\right)^{1/p_1} \left(\int_{\mathbb{R}^n} g^{p_2} v_2\right)^{1/p_2}} \\ &\qquad = \sup_{g \in \mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_0^x \Psi(t)\, dt\right)^{r_1/p_1} \Psi(x)\, \big\|v_1^{-1/p_1}\big\|^{r_1}_{p_1',\,{}^{\mathfrak{c}}B(0,x)}\, dx\right)^{1/r_1}}{\left(\int_{\mathbb{R}^n} g^{p_2} v_2\right)^{1/p_2}}, \end{aligned}$$

where

$$\Psi(t) := \left(\int_{{}^{\mathfrak{c}}B(0,t)} g\right)^q u(t).$$

By Proposition 2, in view of $\lim_{t\to\infty} \|v_1^{-1/p_1}\|_{p_1',{}^{\mathsf{c}}B(0,t)} = 0$, we have

$$\left(\int_0^\infty \left(\int_0^x \Psi(t)\,dt \right)^{r_1/p_1} \Psi(x) \left\| v_1^{-1/p_1} \right\|_{p_1',{}^{\mathsf{c}}B(0,x)}^{r_1} dx \right)^{1/r_1}$$

$$\approx \left(\int_{(0,\infty)} \left(\int_0^x \Psi(t)\,dt \right)^{r_1/q} d\left[- \left\| v_1^{-1/p_1} \right\|_{p_1',{}^{\mathsf{c}}B(0,x)}^{r_1} \right] \right)^{1/r_1}.$$

Thus

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{{}^{\mathsf{c}}B(0,t)} f \cdot \int_{{}^{\mathsf{c}}B(0,t)} g \right)^q u(t)\,dt \right)^{1/q}}{\left(\int_{\mathbb{R}^n} f^{p_1} v_1 \right)^{1/p_1} \left(\int_{\mathbb{R}^n} g^{p_2} v_2 \right)^{1/p_2}}$$

$$\approx \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\left(\int_{(0,\infty)} \left(\int_0^x \Psi(t)\,dt \right)^{r_1/q} d\left[- \left\| v_1^{-1/p_1} \right\|_{p_1',{}^{\mathsf{c}}B(0,x)}^{r_1} \right] \right)^{1/r_1}}{\left(\int_{\mathbb{R}^n} g^{p_2} v_2 \right)^{1/p_2}}$$

$$= \sup_{g\in\mathfrak{M}^+(0,\infty)} \frac{\left(\int_{(0,\infty)} \left(\frac{1}{U(x)} \int_0^x \Psi(t)\,dt \right)^{r_1/q} U(x)^{r_1/q} d\left[- \left\| v_1^{-1/p_1} \right\|_{p_1',{}^{\mathsf{c}}B(0,x)}^{r_1} \right] \right)^{1/r_1}}{\left(\int_{\mathbb{R}^n} g^{p_2} v_2 \right)^{1/p_2}}.$$

(i) The statement follows by Theorem 2, (i).

(ii) The statement follows by Theorem 2, (iii).

(iii) The statement follows by Theorem 2, (iv).

(iv) The statement follows by Theorem 2, (v). ◄

In the limiting case when $p_1 = \infty$ we obtain the following statement.

Theorem 5. *Let $1 \le p_2 \le \infty$, $0 < q < \infty$, and let $u \in \mathcal{W}(0,\infty)$, v_1, $v_2 \in \mathcal{W}(\mathbb{R}^n)$. Then inequality*

$$\left(\int_0^\infty \left(\int_{{}^{\mathsf{c}}B(0,t)} f \cdot \int_{{}^{\mathsf{c}}B(0,t)} g \right)^q u(t)\,dt \right)^{1/q} \le C \,\|f\|_{\infty,v_1,\mathbb{R}^n} \,\|g\|_{p_2,v_2,\mathbb{R}^n} \tag{35}$$

holds for all f, $g \in \mathfrak{M}^+(\mathbb{R}^n)$ if and only if:

(i) $p_2 \le q$, *and*

$$D_1 := \sup_{t\in(0,\infty)} \left(\int_0^t u(y)\big\|v_1^{-1}\big\|_{1,{}^{\mathfrak{c}}B(0,y)}^q \,dy\right)^{1/q} \big\|v_2^{-1/p_2}\big\|_{p_2',{}^{\mathfrak{c}}B(0,t)} < \infty.$$

Moreover, the best constant C in (35) *satisfies $C \approx D_1$.*

(ii) $q < p_2 < \infty$, $1/r_2 = 1/q - 1/p_2$, *and*

$$D_2 := \left(\int_0^\infty \left(\int_0^t u(\tau)\big\|v_1^{-1}\big\|_{1,{}^{\mathfrak{c}}B(0,\tau)}^q \,d\tau\right)^{\frac{r_2}{p_2}} \times \right.$$

$$\left. \times u(t)\,\big\|v_1^{-1}\big\|_{1,{}^{\mathfrak{c}}B(0,t)}^q \big\|v_2^{-1/p_2}\big\|_{p_2',{}^{\mathfrak{c}}B(0,t)}^{r_2} \,dt\right)^{\frac{1}{r_2}} < \infty.$$

Moreover, the best constant C in (35) *satisfies $C \approx D_2$.*

(iii) $p_2 = \infty$, *and*

$$D_3 := \left(\int_0^\infty u(t)\,\big\|v_1^{-1}\big\|_{1,{}^{\mathfrak{c}}B(0,t)}^q \big\|v_2^{-1}\big\|_{1,{}^{\mathfrak{c}}B(0,t)}^q \,dt\right)^{1/q} < \infty.$$

Moreover, the best constant C in (35) *satisfies $C \approx D_3$.*

Proof. By [21, Theorem 2.2, (e)], (34) yields

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{{}^{\mathfrak{c}}B(0,t)} f \cdot \int_{{}^{\mathfrak{c}}B(0,t)} g\right)^q u(t)\,dt\right)^{1/q}}{\|f\|_{\infty,v_1,\mathbb{R}^n}\,\|g\|_{p_2,v_2,\mathbb{R}^n}} =$$

$$= \sup_{g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\left(\int_0^\infty \left(\int_{{}^{\mathfrak{c}}B(0,t)} g\right)^q u(t)\big\|v_1^{-1}\big\|_{1,{}^{\mathfrak{c}}B(0,t)}^q \,dt\right)^{1/q}}{\|g\|_{p_2,v_2,\mathbb{R}^n}}.$$

The proof follows by application of [21, Theorem 2.2]. ◄

We have the following statement when $q = \infty$.

Theorem 6. *Let $1 \le p_1, p_2 \le \infty$, and let $u \in \mathcal{W}(0,\infty)$, $v_1, v_2 \in \mathcal{W}(\mathbb{R}^n)$. Then inequality*

$$\operatorname*{ess\,sup}_{t\in(0,\infty)} \left(\int_{{}^{\mathfrak{c}}B(0,t)} f \cdot \int_{{}^{\mathfrak{c}}B(0,t)} g\right) u(t) \le C\,\|f\|_{p_1,v_1,\mathbb{R}^n}\,\|g\|_{p_2,v_2,\mathbb{R}^n} \tag{36}$$

holds for all $f, g \in \mathfrak{M}^+(\mathbb{R}^n)$ if and only if:

(a) p_1, $p_2 < \infty$, *and*

$$E_1 := \operatorname*{ess\,sup}_{t\in(0,\infty)} u(t) \big\| v_1^{-1/p_1} \big\|_{p_1', {}^{\mathfrak{c}}B(0,t)} \big\| v_2^{-1/p_2} \big\|_{p_2', {}^{\mathfrak{c}}B(0,t)} < \infty.$$

Moreover, the best constant C in (36) *satisfies $C \approx E_1$.*

(b) $p_1 < \infty$, $p_2 = \infty$, *and*

$$E_2 := \operatorname*{ess\,sup}_{t\in(0,\infty)} u(t) \big\| v_1^{-1/p_1} \big\|_{p_1', {}^{\mathfrak{c}}B(0,t)} \big\| v_2^{-1} \big\|_{1, {}^{\mathfrak{c}}B(0,t)} < \infty.$$

Moreover, the best constant C in (36) *satisfies $C \approx E_2$.*

(c) $p_1 = p_2 = \infty$, *and*

$$E_3 := \operatorname*{ess\,sup}_{t\in(0,\infty)} u(t) \big\| v_1^{-1} \big\|_{1, {}^{\mathfrak{c}}B(0,t)} \big\| v_2^{-1} \big\|_{1, {}^{\mathfrak{c}}B(0,t)} < \infty.$$

Moreover, the best constant C in (36) *satisfies $C \approx E_3$.*

Proof. (a) and (b): Let $p_1 < \infty$. By [21, Theorem 2.2, (c) and (h)], (34) yields

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\operatorname{ess\,sup}_{t\in(0,\infty)} \Big(\int_{{}^{\mathfrak{c}}B(0,t)} f \cdot \int_{{}^{\mathfrak{c}}B(0,t)} g \Big) u(t)}{\|f\|_{p_1,v_1,\mathbb{R}^n} \, \|g\|_{p_2,v_2,\mathbb{R}^n}}$$

$$\approx \sup_{g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{1}{\|g\|_{p_2,v_2,\mathbb{R}^n}} \sup_{t\in(0,\infty)} \bigg(\operatorname*{ess\,sup}_{\tau\in(0,t)} \bigg(\int_{{}^{\mathfrak{c}}B(0,\tau)} g \bigg) u(\tau) \bigg) \big\| v_1^{-1/p_1} \big\|_{p_1', {}^{\mathfrak{c}}B(0,t)}.$$

Interchanging the suprema, by duality, on using (10), we arrive at

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\operatorname{ess\,sup}_{t\in(0,\infty)} \Big(\int_{{}^{\mathfrak{c}}B(0,t)} f \cdot \int_{{}^{\mathfrak{c}}B(0,t)} g \Big) u(t)}{\|f\|_{p_1,v_1,\mathbb{R}^n} \, \|g\|_{p_2,v_2,\mathbb{R}^n}}$$

$$\approx \sup_{t\in(0,\infty)} \big\| v_1^{-1/p_1} \big\|_{p_1', {}^{\mathfrak{c}}B(0,t)} \left(\operatorname*{ess\,sup}_{\tau\in(0,t)} u(\tau) \left(\sup_{g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\int_{{}^{\mathfrak{c}}B(0,\tau)} g}{\|g\|_{p_2,v_2,\mathbb{R}^n}} \right) \right)$$

$$= \sup_{t\in(0,\infty)} \big\| v_1^{-1/p_1} \big\|_{p_1', {}^{\mathfrak{c}}B(0,t)} \left(\operatorname*{ess\,sup}_{\tau\in(0,t)} u(\tau) \big\| v_2^{-1/p_2} \big\|_{p_2', {}^{\mathfrak{c}}B(0,\tau)} \right)$$

$$= \operatorname*{ess\,sup}_{t\in(0,\infty)} u(t) \big\| v_1^{-1/p_1} \big\|_{p_1', {}^{\mathfrak{c}}B(0,t)} \big\| v_2^{-1/p_2} \big\|_{p_2', {}^{\mathfrak{c}}B(0,t)},$$

when $p_2 < \infty$, and

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\operatorname{ess\,sup}_{t\in(0,\infty)} \left(\int_{{}^{\mathfrak{c}}B(0,t)} f \cdot \int_{{}^{\mathfrak{c}}B(0,t)} g \right) u(t)}{\|f\|_{p_1,v_1,\mathbb{R}^n} \|g\|_{\infty,v_2,\mathbb{R}^n}}$$

$$\approx \operatorname*{ess\,sup}_{t\in(0,\infty)} u(t) \big\| v_1^{-1/p_1} \big\|_{p_1', {}^{\mathfrak{c}}B(0,t)} \big\| v_2^{-1} \big\|_{1, {}^{\mathfrak{c}}B(0,t)},$$

when $p_2 = \infty$.

(c) Let $p_1 = p_2 = \infty$. By [21, Theorem 2.2, (d)], (34) yields

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\operatorname{ess\,sup}_{t\in(0,\infty)} \left(\int_{{}^{\mathfrak{c}}B(0,t)} f \cdot \int_{{}^{\mathfrak{c}}B(0,t)} g \right) u(t)}{\|f\|_{\infty,v_1,\mathbb{R}^n} \|g\|_{\infty,v_2,\mathbb{R}^n}}$$

$$\approx \sup_{g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{1}{\|g\|_{\infty,v_2,\mathbb{R}^n}} \sup_{t\in(0,\infty)} \left(\operatorname*{ess\,sup}_{\tau\in(0,t)} \left(\int_{{}^{\mathfrak{c}}B(0,\tau)} g \right) u(\tau) \right) \big\| v_1^{-1} \big\|_{1, {}^{\mathfrak{c}}B(0,t)}.$$

Interchanging the suprema, by duality, on using (10), we arrive at

$$\sup_{f,g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\operatorname{ess\,sup}_{t\in(0,\infty)} \left(\int_{{}^{\mathfrak{c}}B(0,t)} f \cdot \int_{{}^{\mathfrak{c}}B(0,t)} g \right) u(t)}{\|f\|_{\infty,v_1,\mathbb{R}^n} \|g\|_{\infty,v_2,\mathbb{R}^n}}$$

$$\approx \sup_{t\in(0,\infty)} \big\| v_1^{-1} \big\|_{1, {}^{\mathfrak{c}}B(0,t)} \left(\operatorname*{ess\,sup}_{\tau\in(0,t)} u(\tau) \left(\sup_{g\in\mathfrak{M}^+(\mathbb{R}^n)} \frac{\int_{{}^{\mathfrak{c}}B(0,\tau)} g}{\|g\|_{\infty,v_2,\mathbb{R}^n}} \right) \right)$$

$$= \sup_{t\in(0,\infty)} \big\| v_1^{-1} \big\|_{1, {}^{\mathfrak{c}}B(0,t)} \left(\operatorname*{ess\,sup}_{\tau\in(0,t)} u(\tau) \big\| v_2^{-1} \big\|_{1, {}^{\mathfrak{c}}B(0,\tau)} \right)$$

$$= \operatorname*{ess\,sup}_{t\in(0,\infty)} u(t) \big\| v_1^{-1} \big\|_{1, {}^{\mathfrak{c}}B(0,t)} \big\| v_2^{-1} \big\|_{1, {}^{\mathfrak{c}}B(0,t)}. \quad \blacktriangleleft$$

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Nevin Bilgiçli
Republic of Turkey Ministry of National Education,
Kırıkkale High School, 71100, Kırıkkale, Turkey
E-mail: nevinbilgicli@gmail.com

Rza Mustafayev
Department of Mathematics, Faculty of Science,
Karamanoglu Mehmetbey University, Karaman, 70100, Turkey
E-mail: rzamustafayev@gmail.com

Tuğçe Ünver
Department of Mathematics, Faculty of Science and
Arts, Kirikkale University, 71450 Yahsihan, Kirikkale, Turkey
E-mail: tugceunver@gmail.com

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