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Existence and Uniqueness of Solutions for Nonlinear Impulsive Differential Equations with Three-Point and Integral Boundary Conditions

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Abstract. The aim of this paper is to investigate the solution of the system of nonlinear impulsive differential equations with three-point and integral boundary conditions. The Green function is constructed and the original problem is reduced to the equivalent integral equations. Sufficient conditions are found for the existence and uniqueness of solutions to the boundary value problems for the system of first order nonlinear impulsive ordinary differential equations with three-point and integral boundary conditions. Banach's fixed point theorem is used to prove the uniqueness and Schaefer's fixed point theorem is used to prove the existence of a solution of the considered problem.

Key Words and Phrases: three-point and integral boundary conditions, existence, uniqueness, fixed point, impulsive differential equations.

2010 Mathematics Subject Classifications: 34B10, 34B37, 34A37

1. Introduction

In recent years, impulsive differential equations have become an active area of research due to their applications in various fields of science and engineering. These equations are widely used in modeling impulsive problems in physics, biology, medicine, population dynamics, biotechnology, industrial robotics, scenarios involving automatic control systems, etc. (see [9,11,16,17,21,25,31,39,40] and the references therein). Some physical systems characterized by sharp changes in the state of the system are usually described by the impulsive differential equations. These changes occur at the fixed or non-fixed points in time over a short period of time. Many authors have studied various aspects of boundary value problems

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with nonlocal boundary conditions for the nonlinear differential equations in several branches of physics and applied mathematics, we refer the readers to [4-8, 10-15, 18, 20, 22-24, 26-30, 32-38, 41-46].

In some cases, the mathematical models of natural processes involve conditions in the form of integrals. It may be explained by the fact that in such cases it becomes impossible to measure the main parameters of the system, while the average value is known. Differential equations with integral boundary conditions have applications in numerous fields such as modeling and analyzing of many physical systems including blood flow problems, chemical engineering, thermoelasticity, underground water flow, population dynamics, etc. For more details of integral boundary conditions, see [1-3, 6-8, 10, 12-14, 18-20, 24, 26, 35, 38] and references therein.

In this paper we firstly consider first order impulsive differential equation with three point and integral boundary conditions. The problem we treat is a generalization of the known works on the considered topic. The existence and uniqueness of the solution of this problem is proved using the fixed point technique.

2. Problem statement

This paper deals with the existence and uniqueness of the system of nonlinear impulsive differential equations of the type

$$\dot{x}(t) = f(t, x(t)) \;\; for \;\; t \neq t_i, i = 1, 2, ..., p, t \in [0, T], \tag{1}$$

subject to impulsive conditions

$$x(t_i^+) - x(t_i) = I_i(x(t_i)), i = 1, 2, ..., p, t \in [0, T], \tag{2}$$

$$0 = t_0 < t_1 < ... < t_{p_1} < \tau < t_{p_{1+1}} < ... < t_p = t_{p+1} = T,$$

and three-point and integral boundary conditions

$$Ax(0) + Bx(\tau) + Cx(T) + \int_0^T n\,(t)\,x(t)dt = d, \tag{3}$$

where A, B, C are constant square matrices of order n such that $\det N \neq 0, N = (A+B+C+\int_0^T n\,(t)\,dt)$; $f : [0, T]\times R^n \to R^n$, $n : [0, T] \to R^{n\times n}$ and $I_i : R^n \to R^n$ are given functions; $\Delta x(t_i) = x(t_i^+)-x(t_i^-)$, where $x(t_i^+) = \lim\limits_{h\to 0+} x(t_i+h)$, $x(t_i^-) = \lim\limits_{h\to 0+} x(t_i - h) = x(t_i)$ are the right- and left-hand limits of $x(t)$ at point $t = t_i$, respectively.

Note that this work is a natural continuation of the works [35,36,47,48].

The present paper is organized as follows. Section 3 describes the necessary background. Section 4 introduces theorems related to existence and uniqueness of a solution of problem (1), (2) and (3) that are proved under some sufficient conditions on nonlinear terms.

3. Preliminaries

In this section, we recall notations, preliminary facts and basic definitions which are used throughout this paper. We denote by $C([0,T];R^n)$ the Banach space of all continuous vector functions $x(t)$ from $[0,T]$ into R^n with the norm

$$\|x\| = \max\{|x(t)| : t \in [0,T]\},$$

where $|\cdot|$ is the norm in the space R^n.

We consider the linear space

$PC([0,T],R^n) = \{x : [0,T] \to R^n : x(t) \in C((t_i, t_{i+1}], R^n), i = 0,1,...,p, x(t_i^-)$, $x(t_i^+)$ exist, $i = 1,...,p$ and $x(t_i^-) = x(t_i)\}$.

It is also clear that $PC([0,T],R^n)$ is a Banach space with the norm

$$\|x\|_{PC} = \max\left\{\|x\|_{(t_i,t_{i+1}]}, i = 0,1,...,p\right\}.$$

Now let's give the definition of solution of the problem (1)-(3).

Definition 1. *A function $x \in PC([0,T],R^n)$ is said to be a solution of the problem* (1)-(3) *if $\dot{x}(t) = f(t,x(t))$ for each $t \neq t_i, i = 1,2,...,p, t \in [0,T]$; $x(t_i^+) - x(t_i) = I_i(x(t_i))$ for each $i = 1,2,...,p, t \in [0,T]$, $0 = t_0 < t_1 < ... < t_{p_1} < \tau < t_{p_1+1} < ... < t_p < t_{p+1} = T$, and boundary conditions* (3) *are satisfied.*

We need the following lemmas.

Lemma 1. *Let $y \in C([0,T],R^n)$ and $I_i(x(t_i)) \in R^n, i = 1,2,...,p$. Then the unique solution $x(t) \in PC([0,T],R^n)$ of the boundary value problem for impulsive differential equation*

$$\dot{x}(t) = y(t), t \in [0,T], t \neq t_i, i = 1,2,...,p, \tag{4}$$

$$x(t_i^+) - x(t_i) = I_i, i = 1,2,...,p,$$

$$0 < t_1 < ... < t_{p_1} < \tau < t_{p_1+1} < ... < t_p = T, \tag{5}$$

$$Ax(0) + Bx(\tau) + Cx(T) + \int_0^T n(t)\, x(t)dt = d, \tag{6}$$

is given by

$$x(t) = D + \int_0^T G(t,s)y(s)ds + \sum_{0<t_k<T} G(t_i,t_k)a_k$$

for $t \in (t_i, t_{i+1}], i = 0, 1, ..., p,\ \ D = N^{-1}d$, *where*

$$G(t,\tau) = \begin{cases} G_1(t,s), & 0 < s < \tau \\ G_2(t,s), & \tau < s \le T \end{cases}$$

with

$$G_1(t,s) = \begin{cases} N^{-1}\left(A + \int_0^s n(\xi)d\xi\right), & 0 \le s \le t, \\ -N^{-1}\left(B + C + \int_s^T n(\xi)d\xi\right), & t < s \le \tau, \\ -N^{-1}\left(C + \int_s^T n(\xi)d\xi\right), & \tau < s \le T, \end{cases}$$

and

$$G_2(t,s) = \begin{cases} N^{-1}\left(A + \int_0^s n(\xi)d\xi\right), & 0 \le s \le \tau, \\ N^{-1}\left(A + B + \int_0^s n(\xi)d\xi\right), & \tau < s \le t, \\ -N^{-1}\left(C + \int_s^T n(\xi)d\xi\right), & t < s \le T. \end{cases}$$

Proof. Assume that $x(t)$ is a solution of boundary value problem (1)-(3). Then integrating equation (1) for $t \in (0, t_{j+1})$ we obtain

$$\int_0^t y(s)ds = \int_0^t x'(s)ds = \left[x(t_1) - x(0^+)\right] + \left[x(t_2) - x(t_1^+)\right] + ... + \left[x(t) - x(t_j^+)\right]$$

$$= -x(0) - \left[x(t_1^+) - x(t_1)\right] - \left[x(t_2^+ - x(t_2)\right] - ... - \left[x(t_j^+) - x(t_j)\right] + x(t).$$

Using above formula and condition (5), we can write

$$x(t) = x(0) + \int_0^t y(s)ds + \sum_{0<\tau_j<t} a_j. \tag{7}$$

Since the solution defined by (7) satisfies condition (6) we get

$$Nx(0) = d - B\int_0^\tau y\left(s\right)ds - B\sum_{0<\tau_j<\tau} a_j - C\int_0^T y(s)ds -$$

$$-C\sum_{0<\tau_j<T} a_j - \int_0^T n\left(t\right)\int_0^t y\left(s\right)dsdt - \int_0^T n\left(t\right)\sum_{0<t_j<t} a_j dt. \tag{8}$$

Since $detN \neq 0$, from (8) we obtain

$$x(0) = D - N^{-1}B\int_0^{\tau} y(s)\,ds - N^{-1}\int_0^T n(t)\int_0^t y(s)\,dsdt-$$

$$-N^{-1}\int_0^T n(t)\sum_{0<t_j<t} a_j dt - N^{-1}B\sum_{0<\tau_j<\tau} a_j-$$

$$-N^{-1}C\int_0^T y(s)ds - N^{-1}C\sum_{0<\tau_j<T} a_j. \tag{9}$$

Since the equalities

$$\int_0^T n(t)\int_0^t y(s)\,dsdt = \int_0^T\int_t^T n(s)\,dsy(t)\,dt,$$

$$\int_0^T n(t)\sum_{0<t_i<t} a_i dt = \sum_{0<t_i<T}\int_{t_i}^T n(t)\,dta_i,$$

are satisfied, from (9) we get

$$x(0) = D - N^{-1}B\int_0^{\tau} y(s)\,ds - N^{-1}\int_0^T\int_t^T n(s)\,dsy(t)\,dt-$$

$$-N^{-1}\sum_{0<t_i<T}\int_{t_i}^T n(t)\,dta_i - N^{-1}B\sum_{0<\tau_j<\tau} a_j-$$

$$-N^{-1}C\int_0^T y(s)ds - N^{-1}C\sum_{0<\tau_j<T} a_j. \tag{10}$$

Now taking into account the value $x(0)$ determined from equality (10) in (7), we get

$$x(t) = D - N^{-1}B\int_0^{\tau} y(s)\,ds - N^{-1}\int_0^T\int_t^T n(s)\,dsy(t)\,dt-$$

$$N^{-1}\sum_{0<t_i<T}\int_{t_i}^T n(t)\,dta_i - N^{-1}B\sum_{0<\tau_j<\tau} a_j - N^{-1}C\int_0^T y(s)ds-$$

$$-N^{-1}C\sum_{0<\tau_j<T} a_j + \int_0^t y(s)ds + \sum_{0<\tau_j<t} a_j. \tag{11}$$

Now consider $t \in [t_j, t_{j+1}]$, $t_{j+1} < \tau$. Then we can rewrite equality (11) as follows:

$$x(t) = D - N^{-1}B\int_0^t y(s)ds - N^{-1}B\int_t^\tau y(s)ds - N^{-1}B\sum_{0<\tau_j<t} a_j -$$

$$-N^{-1}B\sum_{t<\tau_j<\tau} a_j - x(t) = D - N^{-1}B\int_0^t y(s)ds - N^{-1}B\int_t^\tau y(s)ds -$$

$$-N^{-1}B\sum_{0<\tau_j<t} a_j - N^{-1}B\sum_{t<\tau_j<\tau} a_j - N^{-1}C\int_0^t y(s)ds - N^{-1}C\int_t^\tau y(s)ds$$

$$-N^{-1}C\int_\tau^T y(s)ds - N^{-1}C\sum_{0<\tau_j<t} a_j - N^{-1}C\sum_{t<\tau_j<\tau} a_j -$$

$$-N^{-1}\int_0^t\int_s^T n(\alpha)\,d\alpha y(s)\,ds - N^{-1}\int_t^\tau\int_s^T n(\alpha)\,d\alpha y(s)\,ds -$$

$$-N^{-1}\int_\tau^T\int_s^T n(\alpha)\,d\alpha y(s)\,ds -$$

$$-\sum_{\tau<t_j<T} a_j - N^{-1}\sum_{0<t_j<t}\int_{t_j}^T n(t)\,dt a_j - N^{-1}\sum_{t<t_j<\tau}\int_{t_j}^T n(t)\,dt a_j -$$

$$-N^{-1}\sum_{\tau<t_j<T}\int_{t_j}^T n(t)\,dt a_j + \int_0^t y(s)ds + \sum_{0<\tau_j<t} a_j.$$

Grouping like terms, and then simplifying we get

$$x(t) = D + N^{-1}\int_0^t\left(A + \int_0^s n(\alpha)\,d\alpha\right)y(s)ds -$$

$$-N^{-1}\int_t^\tau (B + C + \int_s^T n(\alpha)\,d\alpha)y(s)ds - -N^{-1}\int_\tau^T\left(C + \int_s^T n(\alpha)\,d\alpha\right)y(s)ds +$$

$$+\sum_{0<\tau_j<t}\left(N^{-1}A + \int_0^{t_j} n(\alpha)\,d\alpha\right)a_j - \sum_{t<\tau_j<\tau} N^{-1}(B + C +$$

$$+\int_{t_j}^T n(\alpha)\,d\alpha)a_j - \left(N^{-1}C + \int_s^T n(\alpha)\,d\alpha\right)\sum_{\tau<\tau_j<T} a_j. \tag{12}$$

Let us introduce the following function:

$$G_1(t,s)=\begin{cases} N^{-1}\left(A+\int_0^s n(\xi)d\xi\right), & 0\le s\le t,\\ -N^{-1}\left(B+C+\int_s^T n(\xi)d\xi\right), & t<s\le\tau,\\ -N^{-1}\left(C+\int_s^T n(\xi)d\xi\right), & \tau<s\le T,\end{cases}$$

Using above equality as in (12), we obtain

$$x(t)=D+\int_0^T G_1(t,s)y(\tau)d\tau+\sum_{0<t_k<T}G_1(t_j,t_k)a_k.$$

For the case $t\in(t_j,t_{j+1}]$, $t_j>\tau$ we can rewrite equality (11) as follows:

$$x(t)=D-N^{-1}B\int_0^\tau y(s)ds-N^{-1}B\sum_{0<t_j<\tau}a_j-$$

$$-N^{-1}C\int_0^\tau y(s)ds-N^{-1}C\int_\tau^t y(s)ds-$$

$$-N^{-1}C\int_t^T y(s)ds-N^{-1}C\sum_{0<t_j<\tau}a_j-N^{-1}C\sum_{\tau<t_j<t}a_j-N^{-1}C\sum_{t<t_j<T}a_j+$$

$$+\int_0^\tau y(s)ds+\int_\tau^t y(s)ds+\sum_{0<t_j<\tau}a_j+\sum_{\tau<\tau_j<t}a_j-$$

$$-N^{-1}\int_0^\tau\int_s^T n(\alpha)\,d\alpha y(s)\,ds-N^{-1}\int_\tau^t\int_s^T n(\alpha)\,d\alpha y(s)\,ds-$$

$$-N^{-1}\int_t^T\int_s^T n(\alpha)\,d\alpha y(s)\,ds-N^{-1}\sum_{0<t_j<t}\int_{t_j}^T n(t)\,dta_j-$$

$$-N^{-1}\sum_{t<t_j<\tau}\int_{t_j}^T n(t)\,dta_j-N^{-1}\sum_{\tau<t_j<T}\int_{t_j}^T n(t)\,dta_j=$$

$$=D+N^{-1}\int_0^\tau\left(A+\int_0^s n(\alpha)\,d\alpha\right)y(s)ds+N^{-1}\int_\tau^t(A+B+\int_0^s n(\alpha)\,d\alpha)y(s)ds-$$

$$-N^{-1}\int_t^T\left(C+\int_s^T n(\alpha)\,d\alpha\right)y(s)ds+N^{-1}\sum_{0<t_j<\tau}\left(A+\int_0^{t_j}n(\alpha)\,d\alpha\right)a_j+$$

$$+N^{-1}\sum_{\tau<t_j<t}(A+B+\int_0^{t_j}n(\alpha)\,d\alpha)a_j-N^{-1}\sum_{t<t_j<T}\left(C+\int_{t_j}^T n(\alpha)\,d\alpha\right)a_j.$$

Let's introduce the following function:

$$G_2(t,s)=\begin{cases} N^{-1}\left(A+\int_0^s n(\xi)d\xi\right), & 0\le s\le\tau,\\ N^{-1}\left(A+B+\int_0^s n(\xi)d\xi\right), & \tau<s\le t,\\ -N^{-1}\left(C+\int_s^T n(\xi)d\xi\right), & t<s\le T.\end{cases}$$

Hence, for each $t\in(t_j,t_{j+1}]$, we have

$$x(t)=D+\int_0^T G_2(t,\tau)y(\tau)d\tau+\sum_{0<\tau_k<T} G_2(t_j,t_k)a_k.$$

Finally, we derive the formula for the solution of boundary-value problem (1)-(3)

$$x(t)=D+\int_0^T G(t,s)y(s)ds+\sum_{0<t_k<T} G(t_j,t_k)a_k. \qquad (13)$$

The lemma is proved. ◄

Remark 1. *Note that for the solution* (13) *we have*

1. $x(t)=D$ *is the solution of* $\dot{x}(t)=0$ *with nonlocal boundary condition* $Ax(0)+Bx(t_1)+Cx(T)+\int_0^T n(t)\,x(t)\,dt=d$;
2. *The function* $x(t)=\int_0^T G(t,\tau)y(\tau)d\tau$ *is the solution of* $\dot{x}(t)=y(t)$ *with nonlocal boundary condition* $Ax(0)+Bx(t_1)+Cx(T)+\int_0^T n(t)\,x(t)\,dt=0$. *Here* $G(t,\tau)$ *is Green's function of problem* (4)-(6)*;*
3. *The functions* $\sum_{0<t_k<T} G(t_i,t_k)a_j, i=1,2,...,p$ *are solutions of* $\dot{x}(t)=0$ *with nonlocal boundary condition* $Ax(0)+Bx(t_1)+Cx(T)+\int_0^T n(t)\,x(t)\,dt$ $=0$ *and condition (5).*

Lemma 2. *Assume that* $f\in C([0,T]\times R^n,R^n)$ *and* $I_k(x)\in C(R^n)$, $k=1,2,...,p$. *Then the function* $x(t)$ *is a solution of impulsive boundary value problem* (1)-(3) *if and only if* $x(t)$ *is a solution of the impulsive integral equation*

$$x(t)=D+\int_0^T G(t,s)f(s,x(s))ds+\sum_{k=1}^{p} G(t_i,t_k)I_k(x(t_k))$$

for

$$t\in(t_i,t_{i+1}],i=0,1,...,p. \qquad (14)$$

Proof. Let $x(t)$ be a solution of boundary-value problem (1)-(3). In a similar way as in Lemma 1 we can prove that $x(t)$ is also a solution of impulsive integral equation (14). It is obvious that, the solution of impulsive integral equation (14) satisfies boundary-value problem (1)-(3). Lemma 2 is proved. ◄

4. Main results

Here we deal with the uniqueness result for problem (1)-(3). Our strategy to treat this problem is based on Banach's fixed point theorem.

Theorem 1. *Assume that:*

(H1) There exists a constant $M > 0$ such that

$$|f(t,x) - f(t,y)| \le M\,|x-y|$$

for any $t \in [0,T]$ and all $x, y \in R^n$.

(H2) There exist constants $l_i > 0, i = 1,2,...,p$ such that

$$|I_i(x) - I_i(y)| \le l_i\,|x-y|$$

for all $x, y \in R^n$.

If

$$L = S(MT + \sum_{k=1}^{p} l_k) < 1, \tag{15}$$

then boundary value problem (1)-(3) *has a unique solution on* $[0,T]$. *Here*

$$S = \max_{[0,T]\times[0,T]} \|G(t,s)\|\,.$$

Proof. Transform problem (1)-(3) into a fixed point problem. Let the operator

$$F : PC([0,T], R^n) \to PC([0,T], R^n)$$

be defined as

$$F(x)(t) = D + \int_0^T G(t,s) f(s, x(s))ds + \sum_{k=1}^{p} G(t_i, t_k) I_k\,(x(t_k)) \tag{16}$$

for $t \in (t_i, t_{i+1}], i = 0, 1, ..., p$.

We will show that F is a contraction. Consider $x, y \in PC([0,T], R^n)$. Then for each $t \in (t_i, t_{i+1}]$,

$$|F(x)(t) - F(y)(t)| \le \int_0^T |G(t,s)|\,|f\,((s,x(s)) - f\,((s,y(s))|\,ds +$$

$$+ \sum_{k=1}^{p} |G(t_i, t_k|\,|I_k\,(x(t_k)) - I_k\,(y(t_k))| \le$$

$$\leq SMT \|x-y\| + S\sum_{k=1}^{p} l_k |x(t_k) - y(t_k)| \leq$$

$$\leq \left[S\left(MT + \sum_{k=1}^{p} l_k \right) \right] \|x-y\|_{PC}.$$

Then

$$\|F(x)(t) - F(y)(t)\|_{PC} \leq L \|x-y\|_{PC},$$

hence F is contraction, and thus it has a unique fixed point. So, the operator F is a solution of the problem (1)-(3).

Now the uniqueness result for the problem (1)-(3) follows from the Banach fixed point principle.

Theorem 1 is proved. ◄

Next we state the existence result for problem (1)-(3) by using Schaefer's fixed point theorem.

Theorem 2. *Assume that:*

(H3) The function $f : [0,T] \times R^n \to R^n$ *is continuous.*

(H4) There exists a constant $M_1 > 0$ *such that* $|f(t,x)| \leq M_1$ *for any* $t \in [0,T]$ *and all* $x \in R^n$.

(H5) The functions $I_k(x), x \in R^n, k = 1,2,...,p$ *are continuous and there exists a constant* $M_2 > 0$ *such that*

$$\max_{k\in\{1,\ldots,p\}} |I_k(x)| \leq M_2$$

for all $x \in R^n$.

Then the boundary value problem (1)-(3) *has at least one solution on* $[0,T]$.

Proof. We show that under the assumptions of the theorem, the operator F has a fixed point. The proof is divided into several steps.

Step 1. Here we show that F is continuous.

Let $\{x_n\}$ be a sequence such that $x_n \to x$ in $PC([0,T], R^n)$. Then for any

$$t \in (t_i, t_{i+1}], i = 0,1,...,p,$$

we have

$$|F(x_n)(t) - F(x)(t)| \leq$$

$$\leq \int_0^T |G(t,s)| \, |f(s, x_n(s)) - f(s, x(s))| \, ds +$$

$$+\sum_{k=1}^{p} |G(t_i,t_k| \, |I_k(x_n(t_k)) - I_k(x(t_k))| \le$$

$$\le ST \max_{s\in[0,T]} |f(s,x_n(s)) - f(s,x(s))| +$$

$$+S\sum_{k=1}^{p} |I_k(x_n(t_k)) - I_k(x(t_k))| \; .$$

Since f, and $I_k, k = 1, 2, ..., p$ are continuous functions, we have

$$\|F(x_n)(t) - F(x)(t)\|_{PC} \to 0$$

as $n \to \propto$. Thus, F is continuous.

Step 2. We prove that the operator F maps bounded sets in $PC([0,T], R^n)$. To do so, it is enough to show that for any $\eta > 0$ there exists a positive constant l such that for any $\{x \in B_\tau = PC([0,T], R^n) : \|x\| \le \eta\}$ we have $\|F(x(\cdot))\| \le l$. By triangle inequality, assumptions (H4) and (H5) for $t \in (t_i, t_{i+1}]$, we get

$$|F(x)(t)| \le |D| + \int_0^T |G(t,s)| \, |f(s,x(s))| \, ds + \sum_{k=1}^{p} |G(t_i,\tau_k)| \, |I_k(t_k)|$$

for any $t \in [0,T]$.

Hence,

$$|F(x)(t)| \le |D| + STM_1 + SpM_2.$$

So,

$$\|F(x)(t)\|_{PC} \le |D| + S[TM_1 + pM_2] = l.$$

Step 3. We show that the operator F maps bounded sets into equicontinuous sets of $PC([0,T], R^n)$.

Let $\xi_1, \xi_2 \in [0,T]$, $\xi_1 < \zeta_2$, B_η be a bounded set of $PC([0,T], R^n)$ as in Step 2, and assume that $x \in B_\eta$.

Case 1: $\xi_1, \xi_2 \in [t_j, t_{j+1}]$, $t_{j+1} < \tau$. Then

$$|F(x)(\xi_2) - F(x)(\xi_1)| =$$

$$= \left| N^{-1}A\int_{\zeta_1}^{\xi_2} f(s,x(s))\,ds + N^{-1}(B+C)\int_{\xi_1}^{\xi_2} f(s,x(s))\,ds \right| \le \int_{\xi_1}^{\xi_2} |f(s,x(s))|\,ds.$$

Case 2: $\xi_1 \in [t_p, \tau], \xi_2 \in [\tau, t_{p_1+1}]$. Then

$$|F(x)(\xi_2) - F(x)(\xi_1)| = \left| \int_{\xi_1}^{\xi_2} f(s,x(s))\,ds \right| \le \int_{\xi_1}^{\xi_2} |f(s,x(s))| \, .$$

Case 3: $\xi_1, \xi_2 \in [t_j, t_{j+1}], t_j > \tau$.

Case 3 is proved similar to Case 1.

The right-hand side of the above inequalities for all three cases 1-3 tends to zero as $\xi_1 \to \xi_2$. As a consequence of Steps 1, 2 and 3 and the Arzela-Ascoli theorem, we can conclude that the operator $F : PC([0,T], R^n) \to PC([0,T], R^n)$ is completely continuous.

Step 4. Now it remains to show that the set $\Delta = \{x \in PC([0,T], R^n) : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}$ is bounded. Assume that $x = \lambda(Fx)$ for some $0 < \lambda < 1$. Then, for any $t \in (t_i, t_{i+1}], i = 0, 1, ..., p$ we have

$$x(t) = \lambda \left[D + \int_0^T G(t,s) f(s, x(s))\, ds + \sum_{k=1}^{p} G(t_i, \tau_k) I_k(x(t_k)) \right].$$

In view of assumptions (H4) and (H5), we get for any $t \in [0,T]$

$$|F(x)(t)| \le |D| + [M_1 T + pM_2]S.$$

So, for every $\in [0,T]$, we obtain the following estimate:

$$\|x\|_{PC} \le |D| + [M_1 T + pM_2]S = R,$$

which means that x is a solution of (1)-(3).

Theorem 2 is proved. ◄

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