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On the Density of a Sum of Algebras in the Space of Continuous Functions

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Abstract. In this paper, we obtain a necessary condition for the density of a sum of finitely many algebras in the space of continuous functions. This condition complements the known sufficient condition of Sproston and Straus [22]. In case of two algebras and under a suitable restriction, our necessary condition turns out to be also sufficient.

Key Words and Phrases: Hausdorff space, quotient space, cycle, bolt of lightning.

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1. Introduction

Let X be a compact Hausdorff space and $C(X)$ be the space of continuous real-valued functions on X endowed with the topology of uniform convergence. Assume we are given a finite number of closed subalgebras $A_1, ..., A_k$ of $C(X)$ that contain the constants. This paper is dedicated to the following problem. What conditions imposed on $A_1, ..., A_k$ are necessary and/or sufficient for the density of $\overline{A_1 + \cdots + A_k} = C(X)$? In order to introduce the problem and its history to the reader, we recall some notions associated with the algebras A_i, $i = 1, ..., k$. First consider the equivalence relation R_i, $i = 1, ..., k$, between elements in X defined as follows

$$a \overset{R_i}{\sim} b \text{ if } f(a) = f(b) \text{ for all } f \in A_i. \tag{1}$$

Obviously, for each $i = 1, ..., k$, the quotient space $X_i = X/R_i$ with respect to the relation R_i, equipped with the quotient space topology, is compact. In addition, the natural projections $s_i : X \to X_i$ are continuous. Note that the quotient spaces X_i are not only compact but also Hausdorff (see, e.g., [15, p.54]).

In view of the Stone-Weierstrass theorem, we can write

$$A_i = \{f(s_i(x)) : \ f \in C(X_i)\}, \ i = 1, ..., k. \tag{2}$$

Such a representation of algebras was used in many works (see, e.g., [3, 4, 15, 22]). In [22], Sproston and Straus considered the set functions

$$\tau_i(Z) = \{x \in Z : \ |s_i^{-1}(s_i(x)) \bigcap Z| \geq 2\}, \quad Z \subset X, \ i = 1, \dots, k,$$

where $|Y|$ denotes the cardinality of a considered set Y. Define $\tau(Z)$ to be $\bigcap_{i=1}^{k} \tau_i(Z)$ and define $\tau^2(Z) = \tau(\tau(Z))$, $\tau^3(Z) = \tau(\tau^2(Z))$ and so on inductively. One of the main results of [22] says that $A_1 + \cdots + A_k = C(X)$ provided that $\tau^n(X) = \emptyset$ for some positive integer n. It should be noted that this condition first appeared in the work of Sternfeld [23], where he proved that $\tau^n(X) = \emptyset$ (for some n) guarantees the representation if X is a compact metric space. Sproston and Straus proved the last statement for X being a compact Hausdorff space. For $k = 2$, the condition is also necessary for the representation, but not in general if $k > 2$ (see the counterexample in [22]). Note that the above condition $\tau^n(X) = \emptyset$ is geometric in nature. It holds if points of X are of a certain geometrical structure. This is easily seen in the case of two subalgebras. For $k = 2$, the condition $\tau^n(X) = \emptyset$ can be expressed in terms of sets of points in X which are geometrically explicit. In special case of the algebras $U = \{u(x)\}$, $V = \{v(y)\}$, these points were introduced in the literature under different names such as "permissible lines" [7] "bolts of lightning" [1, 9, 13, 14, 15, 20], "trips" [21], "paths" [8, 10, 12, 18, 19], "links" [6, 17], etc. The term bolt of lightning is the most common and is due to Arnold [1]. It first appeared in his solution of Hilbert's thirteenth problem. Note that a *bolt of lightning* is a finite ordered subset $L = \{p_1, p_2, \cdots p_n\}$ in $\mathbb{R}^2$ such that $p_i \neq p_{i+1}$, each line segment $p_i p_{i+1}$ (unit of the bolt) is parallel to the coordinate axis x or y, and two adjacent units $p_i p_{i+1}$ and $p_{i+1} p_{i+2}$ are perpendicular. A bolt of lightning L is said to be closed if $p_n p_1 \perp p_1 p_2$ (in this case, n is an even number). For a compact set $X \subset \mathbb{R}^2$ and the algebras $U = \{u(x)\}$, $V = \{v(y)\}$, it is not difficult to prove that $\tau^n(X) = \emptyset$ if and only if there are no closed bolts in X and the lengths (number of points) of all bolts are uniformly bounded (see [15]).

In [22], Sproston and Straus obtained also a sufficient condition for the density of $A_1 + \cdots + A_k$ in $C(X)$. We formulate their result in the next section.

The purpose of this paper is to obtain a necessary condition for the density of $\overline{A_1 + \cdots + A_k} = C(X)$. We hope that our condition complements the above mentioned sufficient condition of Sproston and Straus. In case of two algebras and under a suitable restriction, we will see that our necessary condition turns out to be also sufficient.

2. Cycles with respect to algebras and the main result

We begin this section with the definition of objects, which are essential for the further analysis of the considered density problem. Assume, as above, X is a compact Hausdorff space, $C(X)$ is the space of continuous real-valued functions on X and A_i, $i = 1, ..., k$, are closed subalgebras of $C(X)$ that contain the constants. As it is shown above these algebras can be written in the form (2).

Cycles with respect to the algebras A_i, $i = 1, ..., k$, are defined as follows.

Definition 1. *(see [2]) A set of points $l = (x_1, \dots, x_n) \subset X$ is called a cycle with respect to the algebras A_i, $i = 1, ..., k$, if there exists a vector $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{Z}^n$ with the nonzero integer coordinates λ_j such that*

$$\sum_{j=1}^{n} \lambda_j \delta_{s_i(x_j)} = 0, \quad for\ all\ i = 1, \dots, k.$$

Here δ_a is a characteristic function of the single point set $\{a\}$.

For example, the set $l = \{(0,0,0),\ (0,0,1),\ (0,1,0),\ (1,0,0),\ (1,1,1)\}$ is a cycle in $\mathbb{I}^3$, $\mathbb{I} = [0,1]$, with respect to the algebras $A_i = \{p(z_i) : p \in C[0,1]\},\ i = 1, 2, 3$. The vector λ in Definition 1 can be taken as $(-2, 1, 1, 1, -1)$.

The idea of cycles with respect to k directions in $\mathbb{R}^d$ was first implemented by Braess and Pinkus [5] in a work dedicated to ridge function interpolation. Klopotowski, Nadkarni, Rao [16] defined cycles of minimal lengths with respect to canonical projections and called them *loops*. In Ismailov [11], these objects (under the name of *closed paths*) have been generalized to those having association with k arbitrary functions. In these works, it was proven that nonexistence of cycles with respect to k directions, canonical projections in $\mathbb{R}^k$ and k arbitrary functions is both necessary and sufficient for interpolation by ridge functions (see [5]), representation of multivariate functions by sums of univariate functions (see [16]) and representation by linear superpositions (see [11]), respectively.

In Marshall and O'Farrell [20], a finite sequence $(x_1, ..., x_n)$ in X with $x_i \neq x_{i+1}$ satisfying either $s_1(x_1) = s_1(x_2)$, $s_2(x_2) = s_2(x_3)$, $s_1(x_3) = s_1(x_4), ...,$ or $s_2(x_1) = s_2(x_2)$, $s_1(x_2) = s_1(x_3)$, $s_2(x_3) = s_2(x_4), ...,$ is called a *bolt* with respect to (A_1, A_2) (see also [3, 4]). If $(x_1, ..., x_n, x_1)$ is a bolt and n is an even number, then the bolt $(x_1, ..., x_n)$ is called closed. These objects are straightforward generalization of classical bolts (see Introduction) and appeared in several results concerning the density of $A_1 + A_2$ in $C(X)$. Bolts with respect to (A_1, A_2) are essential for description of regular Borel measures orthogonal to $A_1 + A_2$ (see

[20]). Note that a cycle with respect to two algebras A_1 and A_2 is a union of closed bolts with respect to (A_1, A_2) (see [2]).

Each cycle $l = (x_1, \dots, x_n)$ and an associated vector $\lambda = (\lambda_1, \dots, \lambda_n)$ generate the following functional

$$F_{l,\lambda}(f) = \sum_{j=1}^{n} \lambda_j f(x_j), \quad f \in C(X).$$

From Definition 1 it follows that for each function $g_i \in A_i$, $i = 1, \dots, k$,

$$F_{l,\lambda}(g_i) = \sum_{j=1}^{n} \lambda_j g_i(x_j) = 0.$$

Hence, $F_{l,\lambda}(g) = 0$, for any $g \in A_1 + \cdots + A_k$. That is, the functional $F_{l,\lambda}$ belongs to the annihilator of the space $A_1 + \cdots + A_k$.

Assume for $i = 1, ..., k$, A_i is a subalgebra of $C(X)$ generated by one element $w_i \in A_i$. Following Khavinson, we say that an algebra $A \subset C(X)$ is generated by an element $w \in A$ if $A = \{h(w(x) : h \in C(\mathbb{R})\}$ (see [15, p. 33]). Note that $a \overset{R_i}{\sim} b$ if and only if $w_i(a) = w_i(b)$; thus any cycle with respect to the algebras A_i is a cycle with respect to the real-valued functions w_i and vice versa. In [11], Ismailov proved that if $F_{l,\lambda}(f) = 0$, for any cycle $l \subset X$, then $f = \sum_{i=1}^{k} h_i \circ w_i$, where $h_i : \mathbb{R} \to \mathbb{R}$ are some functions (generally discontinuous) depending on f. It follows that f is decomposed into the sum $\sum_{i=1}^{k} f_i \circ s_i$, where $s_i : X \to X_i$, $i = 1, ..., k$, are the natural projections defined above and $f_i : X_i \to \mathbb{R}$. But this does not mean that we can always choose f_i continuous on X_i (see [2]).

Note that the relation $a \overset{R_i}{\sim} b$ when a and b belong to some bolt in a given compact Hausdorff space X defines an equivalence relation. The equivalence classes are called orbits (see [20, 21]).

The following theorem is valid.

Theorem 1. *Let all orbits in X be topologically closed. Then $A_1 + A_2$ is dense in $C(X)$ if and only if X contains no closed bolts with respect to (A_1, A_2).*

Proof. Necessity. Let for each $i = 1, 2$ X_i be the quotient space of X equipped with equivalence relation (1), and s_i be appropriate natural projections. It is clear that if X has a closed bolt $b = (b_1, ..., b_{2n})$ with respect to (A_1, A_2), then b contains a closed bolt $b' = (b'_1, ..., b'_{2m})$ with respect to (A_1, A_2), such that all points $b'_1, ..., b'_{2m}$ are distinct. By Urysohn's great lemma, there exists a continuous function $f = f(x)$ on X such that $f(b'_i) = 1$, $i = 1, 3, ..., 2m - 1$, $f(b'_i) = -1$, $i = 2, 4, ..., 2m$ and $-1 < f(x) < 1$ at all other points x of the set X. Consider the measure

$$\mu_{b'} = \frac{1}{2m}\sum_{i=1}^{2m}(-1)^{i-1}\delta_{b'_i}\ ,$$

where $\delta_{b'_i}$ is a point mass at b'_i. For this measure, $\int\limits_X fd\mu_{b'} = 1$ and $\int\limits_X gd\mu_{b'} = 0$ for all functions $g \in A_1 + A_2$. Thus the set $A_1 + A_2$ cannot be dense in $C(X)$.

Sufficiency. We are going to prove that the only annihilating regular Borel measure for $A_1 + A_2$ is the zero measure. Suppose, contrary to this assumption, there exists a nonzero annihilating measure on X for $A_1 + A_2$. The class of such measures with total variation not more than 1 we denote by M. Clearly, M is weak-* compact and convex. By the Krein-Milman theorem, there exists an extreme measure μ in M. Since the orbits are closed, μ must be supported on a single orbit. Denote this orbit by T.

For a fixed point $z_1 \in T$ set $T_1 = \{z_1\}$, $T_2 = s_1^{-1}(s_1T_1)$, $T_3 = s_2^{-1}(s_2T_2)$, $T_4 = s_1^{-1}(s_1T_3)$, ... Obviously, $T_1 \subset T_2 \subset T_3 \subset \cdots$. Therefore, for some $k \in \mathbb{N}$, $|\mu|(T_{2k}) > 0$, where $|\mu|$ is a total variation measure of μ. Since μ is orthogonal to any function from A_1, we have $\mu(T_{2k}) = 0$. From the Hahn-Jordan decomposition $\mu = \mu^+ - \mu^-$ it follows that $\mu^+(T_{2k}) = \mu^-(T_{2k}) > 0$.

Fix a Borel subset $D_0 \subset T_{2k}$ such that $\mu^+(D_0) > 0$ and $\mu^-(D_0) = 0$. Since μ is orthogonal to any function from A_2, we have $\mu(s_2^{-1}(s_2D_0)) = 0$. Therefore, one can choose a Borel set D_1 such that $D_1 \subset s_2^{-1}(s_2D_0) \subset T_{2k+1}$, $D_1 \cap D_0 = \varnothing$, $\mu^+(D_1) = 0$, $\mu^-(D_1) \geqslant \mu^+(D_0)$. By the same way one can choose a Borel set D_2 such that $D_2 \subset s_1^{-1}(s_1D_1) \subset T_{2k+2}$, $D_2 \cap D_1 = \varnothing$, $\mu^-(D_2) = 0$, $\mu^+(D_2) \geqslant \mu^-(D_1)$, and so on.

The sets $D_0, D_1, D_2, ...,$ are pairwise disjoint. For otherwise, there would exist positive integers n and m, with $n < m$ and a bolt $(y_n, y_{n+1}, ..., y_m)$ such that $y_i \in D_i$ for $i = n, ..., m$ and $y_m \in D_m \cap D_n$. But then there would exist bolts $(z_1, z_2, ..., z_{n-1}, y_n)$ and $(z_1, z_2', ..., z_{n-1}', y_m)$ with z_i and z_i' in T_i for $i = 2, ..., n-1$. Hence, the set

$$\{z_1, z_2, ..., z_{n-1}, y_n, y_{n+1}, ..., y_m, z_{n-1}', ..., z_2', z_1\}$$

would contain a closed bolt. This would be contrary to the condition of the theorem.

Now, since the sets $D_0, D_1, D_2, ...,$ are pairwise disjoint, and $|\mu|(D_i) \geqslant \mu^+(D_0) > 0$, for each $i = 1, 2, ...,$ it follows that the total variation of μ is infinite. This contradiction completes the proof. ◄

Remark 1. *In [21], the above theorem was formulated without proof. All essential ideas of the proof belong to Marshall and O'Farrell.*

In general case of k algebras, for the density of the sum $A_1 + \cdots + A_k$ in $C(X)$, we consider below one sufficient and one necessary condition. The sufficient condition belongs to Sproston and Straus [22]. Their result uses the set function $\tau(X)$ (see Introduction).

Theorem 2. *(see [22]) Let* $\bigcap_{n=1}^{\infty} \tau(X) = \emptyset$. *Then* $\overline{A_1 + \cdots + A_k} = C(X)$.

The main result of this paper is the following theorem, which establishes a necessary condition for the density of $A_1 + \cdots + A_k$ in the space $C(X)$.

Theorem 3. *If* $A_1 + \cdots + A_k$ *is dense in* $C(X)$, *then the set* X *does not contain a cycle with respect to the algebras* A_i, $i = 1, ..., k$.

Proof. Suppose the contrary. Suppose that the set X contains cycles. As it is noted above, each cycle $l = (x_1, \ldots, x_n)$ and the associated vector $\lambda = (\lambda_1, \ldots, \lambda_n)$ generate the functional

$$F_{l,\lambda} : C(X) \to \mathbb{R},\ F_{l,\lambda}(f) = \sum_{j=1}^{n} \lambda_j f(x_j).$$

Clearly, $F_{l,\lambda}$ is a linear and continuous functional with norm $\sum_{j=1}^{n} |\lambda_j|$. It is not difficult to verify that $F_{l,\lambda}(g) = 0$ for all functions $g \in A_1 + \cdots + A_k$. Let f_0 be a continuous function such that $f_0(x_j) = 1$ if $\lambda_j > 0$ and $f_0(x_j) = -1$ if $\lambda_j < 0$, $j = 1, \ldots, n$. For this function, $F_{l,\lambda}(f_0) \neq 0$. Thus, we have constructed a nonzero linear functional which belongs to the annihilator of the manifold $A_1 + \cdots + A_2$. This means that the sum $A_1 + \cdots + A_2$ is not dense in $C(X)$. The obtained contradiction proves the theorem. ◄

Remark 2. *Theorem 3 complements the above theorem of Sproston and Straus.*

Remark 3. *Note that if* $k = 2$ *and all orbits are closed, the condition of Theorem 3 is also sufficient. This assertion follows from Theorem 1 and the fact that a cycle with respect to two algebras is a union of closed bolts (see [2]).*

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