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Chlodowsky type (λ, q)-Bernstein-Stancu operators

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Abstract. In the present paper, we introduce Stancu-Chlodowsky type (λ, q)-Bernstein operators and investigate their approximation properties. We obtain convergence properties of these operators by using Korovkin's theorem and Voronovskaja type theorem for new operators. Finally, we generalize these operators and give some approximation properties.

Key Words and Phrases: Bernstein operators, q-Bernstein-Chlodowsky operators, (λ, q)-Bernstein operators, Korovkin type theorem, Voronovskaja type theorem.

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1. Introduction and preliminaries

In 1912, Bernstein [2] proposed the famous polynomials called nowadays Bernstein polynomials to prove the Weierstrass approximation theorem. He defined them as follows.

Let $B_n(f;x) : C[0,1] \to C[0,1]$ be defined for any $f \in C[0;1]$ by

$$B_n(f;x) = \sum_{k=0}^{n} \binom{n}{k} x^k (1-x)^{n-k} f\left(\frac{k}{n}\right), \qquad x \in [0,1]. \tag{1}$$

Later, it was found that these polynomials possess many remarkable properties and their various generalizations have been studied [14, 13, 6, 10, 11, 12, 15, 16, 17]. The importance of the Bernstein polynomials led to the discovery of their numerous generalizations aimed to provide appropriate tools for various areas of mathematics, such as approximation theory, computer-aided geometric design, and the statistical inference.

Let $q > 0$. Then for each non-negative integer n, the q-integer $[n]_q$, the q-factorial $[n]_q!$ and the q -binomial coefficients are defined by (see [8])

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$$[n]_q := \begin{cases} \frac{1-q^n}{1-q}, & \text{if } q \in \mathbb{R}^+ \setminus \{1\} \\ nn, & \text{if } q = 1, \end{cases} \quad \text{for } n \in \mathbb{N} \text{ and } [0]_q = 0,$$

$$[n]_q! := \begin{cases} [n]_q[n-1]_q \cdots [1]_q, & n \geq 1, \\ 1, & n = 0, \end{cases}$$

$$\left[\begin{array}{c} n \\ k \end{array} \right]_q := \frac{[n]_q!}{[k]_q![n-k]_q!}.$$

On the other hand, Karsli and Gupta [9] introduced q-Chlodowsky operators as follows:

$$B_n(f;x) = \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_q \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_q^{n-k} f\left(\frac{[k]_q}{[n]_q} b_n\right), \tag{2}$$

where

$$b_{n,k}(x;q) = \left[\begin{array}{c} n \\ k \end{array} \right]_q \left(\frac{x}{b_n}\right)^k \left(1 - \frac{x}{b_n}\right)_q^{n-k} \tag{3}$$

and $0 < q \leq 1$, $0 \leq x \leq b_n$ and b_n is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \dfrac{b_n}{[n]_q} = 0$.

Recently, the Chlodowsky variant of q-Bernstein-Schurer-Stancu operators was introduced in [18] as

$$B_{n,p,q}^{(\alpha,\beta)}(f;x) = \sum_{k=0}^{n+p} b_{n,p,k}(x;q) f\left(\frac{[k]_q + \alpha}{[n]_q + \beta} b_n\right) \tag{4}$$

where $n \in \mathbb{N}$, $p \in \mathbb{N}_0$, $0 < q \leq 1$, $0 \leq x \leq b_n$, $\alpha, \beta \in \mathbb{R}$ and $0 \leq \alpha \leq \beta$. For $p = 0$, $\alpha = \beta = 0$, we obtain the q-Bernstein-Chlodowsky polynomials.

In 2018, Cai and Zhou [5] introduced (λ, q)-Bernstein operator bases with shape parameter $\lambda \in [-1, 1]$ as

$$B_{n,\lambda}^q(f;x) = \sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) f\left(\frac{[k]_q}{[n]_q}\right), \tag{5}$$

where

$$\begin{cases} \widehat{b}_{n,0}(x;q) = b_{n,0}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,1}(x;q), \\ \widehat{b}_{n,k}(x;q) = b_{n,k}(x;q) + \lambda\Big(\frac{[n]_q-2[k]_q+1}{[n]_q^2-1} b_{n+1,k}(x;q) - \\ -\frac{[n]_q-2q[k]_q-1}{[n]_q^2-1} b_{n+1,k+1}(x;q)\Big), \\ \widehat{b}_{n,n}(x;q) = b_{n,n}(x;q) - \frac{\lambda}{[n]_q+1} b_{n+1,n}(x;q). \end{cases} \tag{6}$$

and $b_{n,k}(x;q) = \left[\begin{array}{c} n \\ k \end{array}\right]_q (x)^k (1-x)_q^{n-k}$, $\lambda \in [-1,1]$ is the shape parameter, $k = 1, 2, ..., n-1$, $n \geq 2$, $x \in [0,1]$ and $0 < q \leq 1$.

For more recent works on λ-Bernstein basis, we refer the readers to [1] and [4].

Lemma 1. *Let $e_r(t) = t^r$, $r \in \mathbb{N} \cup \{0\}$, $x \in [0,1]$, $0 < q \leq 1$ and $n > 1$. For the (λ, q)-Bernstein operators $B_{n,\lambda}^q(f;x)$, we have*

$$\begin{aligned}
(i) B_{n,\lambda}^q(e_0;x) &= 1 \\
(ii) B_{n,\lambda}^q(e_1;x) &= x + \frac{[n+1]_q x(1-x^n)}{[n]_q([n]_q-1)}\lambda - \frac{2[n+1]_q x\lambda}{[n]_q^2-1}\left[\frac{1-x^n}{[n]_q}\right. \\
&\left. + qx(1-x^{n-1})\right] + \frac{\lambda}{q[n]_q([n]_q+1)}\left[1-(1-x)_q^{n+1} - x^{n+1}\right. \\
&\left. -[n+1]_q x(1-x^n)\right] + \frac{\lambda}{([n]_q^2-1)}\left\{2[n+1]_q x^2(1-x^{n-1})\right. \\
&\left. -\frac{2[n+1]_q x}{q[n]_q}(1-x^n) + \frac{2}{q[n]_q}\left[1-(1-x)_q^{n+1} - x^{n+1}\right]\right\} \\
&= x + \frac{[n+1]_q \lambda x(1-x^n)}{[n]_q([n]_q+1)} + \frac{2[n+1]_q \lambda x^2(1-x^{n-1})(1-q)}{[n]_q^2-1} \\
&+ \frac{1-(1-x)_q^{n+1} - x^{n+1} - [n+1]_q x(1-x^n)}{q[n]_q([n]_q-1)}\lambda \\
(iii) B_{n,\lambda}^q(e_2;x) &= x^2 + \frac{x(x-1)}{[n]_q} + \frac{[n+1]_q x\lambda}{[n]_q([n]_q-1)}\left[\frac{1-x^n}{[n]_q} + qx(1-x^{n-1})\right] \\
&- \frac{2[n+1]_q\lambda}{[n]_q([n]_q^2-1)}\left[\frac{x(1-x^n)}{[n]_q} + q(2+q)x^2(1-x^{n-1})\right. \\
&\left. + q^3[n-1]_q x^3(1-x^{n-2})\right]
\end{aligned}$$

$$-\frac{\lambda}{q[n]_q([n]_q+1)}\Big\{[n+1]_q x^2(1-x^{n-1})$$
$$-\frac{[n+1]_q x(1-x^n)}{q[n]_q}+\frac{1-(1-x)_q^{n+1}-x^{n+1}}{q[n]_q}\Big\}$$
$$+\frac{2\lambda}{[n]_q([n]_q^2-1)}\Big\{q[n-1]_q[n+1]_q x^3(1-x^{n-2})$$
$$-\frac{(1-q)[n+1]_q x^2(1-x^{n-1})}{q}+\frac{[n+1]_q x(1-x^n)}{q^2[n]_q}$$
$$-\frac{1-(1-x)_q^{n+1}-x^{n+1}}{q^2[n]_q}\Big\}.$$

2. Construction of operators

Considering the revised form of (λ, q)-Bernstein operators [5], we construct the Chlodowsky of (λ, q)-Bernstein-Stancu operators as

$$B_{n,\lambda,q}^{(\alpha,\beta)}(f;x)=\sum_{k=0}^{n}\widehat{b}_{n,k}(x;q)f\left(\frac{[k]_q+\alpha}{[n]_q+\beta}b_n\right). \tag{7}$$

Lemma 2. *Let $e_r(t)=t^r$, $r\in\mathbb{N}\cup\{0\}$, and $n>1$. For the Stancu-Chlodowsky type (λ, q)-Bernstein operators $B_{n,\lambda,q}^{(\alpha,\beta)}(f;x)$, we have*

$$(i)B_{n,\lambda,q}^{(\alpha,\beta)}(e_0;x) = 1$$
$$(ii)B_{n,\lambda,q}^{(\alpha,\beta)}(e_1;x) = \frac{b_n}{[n]_q+\beta}\Big\{\frac{[n]_q x}{b_n}+\lambda\Big[\frac{((q-1)[n]_q-q-1)[n+1]_q x}{q([n]_q^2-1)b_n}$$
$$-\frac{2(1-q)[n]_q[n+1]_q x^2}{([n]_q^2-1)b_n^2}$$
$$+\frac{((q-1)(2q-1)[n]_q+q)[n+1]_q+(1-q)[n]_q}{q([n]_q^2-1)}\left(\frac{x}{b_n}\right)^{n+1}$$
$$-\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right)\Big]+\alpha\Big\}$$

$$(iii)B_{n,\lambda,q}^{(\alpha,\beta)}(e_2;x) = \frac{b_n^2}{([n]_q+\beta)^2}\Big\{\alpha^2+(2\alpha+1)\frac{[n]_q x}{b_n}+\lambda\Big[\Big(2q\alpha((q-1)[n]_q$$
$$-q-1)+(q^2+1)[n]_q-q^2+1\Big)\frac{[n+1]_q x}{q^2([n]_q^2-1)b_n}$$

$$+\Bigg(\Bigg[([n]_q(q^2-1)-q(2q^2+3q-2)-1)$$

$$-4q\alpha(1-q)\Bigg]\frac{[n]_q[n+1]_q}{q([n]_q^2-1)}+2\alpha[n]_q([n]_q-1)\Bigg)\frac{x^2}{b_n^2}$$

$$+\frac{2q(1-q^2)[n]_q[n+1]_q[n-1]_q x^3}{([n]_q^2-1)b_n^3}+\Bigg\{\Bigg[((q^2-1)(2q-1)[n]_q$$

$$+3q^2+1)[n]_q[n+1]_q+2\alpha\Bigg(((q-1)(2q-1)[n]_q+q)[n+1]_q$$

$$+(1-q)[n]_q\Bigg)\Bigg]\frac{1}{q([n]_q^2-1)}-\frac{[n+1]_q}{[n]_q+1}$$

$$-\frac{[n]_q}{q([n]_q-1)}\Bigg\}\left(\frac{x}{b_n}\right)^{n+1}+\frac{2q\alpha-1}{q^2([n]_q-1)}\Bigg(1-(1-\frac{x}{b_n})_q^{n+1}\Bigg)\Bigg]\Bigg\}.$$

Proof. We can obtain (i) easily by the fact that $\sum_{k=0}^n \widehat{b}_{n,k}(x;q)=1$. Next, by (7), we have

(ii) Using $[k]_q=\frac{[k+1]_q}{q}-\frac{1}{q}$ and $[k]_q^2=\frac{[k]_q[k+1]_q}{q}-\frac{[k]_q}{q}$, we have

$$\begin{aligned}
B_{n,\lambda,q}^{(\alpha,\beta)}(e_1;x) &= \sum_{k=0}^{n}\widehat{b}_{n,k}(x;q)\left(\frac{[k]_q+\alpha}{[n]_q+\beta}b_n\right)\\
&= \frac{b_n}{[n]_q+\beta}\Bigg(\sum_{k=0}^{n}[k]_q\widehat{b}_{n,k}(x;q)+\alpha\Bigg)
\end{aligned}$$

$$= \frac{b_n}{[n]_q+\beta}\Bigg(\sum_{k=0}^{n}[k]_q b_{n,k}(x;q)$$

$$+\lambda\Bigg(\sum_{k=0}^{n}[k]_q\frac{[n]_q-2[k]_q+1}{[n]_q^2-1}b_{n+1,k}(x;q)$$

$$-\sum_{k=1}^{n-1}[k]_q\frac{[n]_q-2q[k]_q-1}{[n]^2-1}b_{n+1,k+1}(x;q)\Bigg)+\alpha\Bigg)$$

$$= \frac{b_n}{[n]_q+\beta}\Bigg(\frac{[n]_q x}{b_n}+\lambda\Big(\Delta_1(n;x)-\Delta_2(n;x)\Big)+\alpha\Bigg).$$

Now, let us calculate $\Delta_1(n;x)$ and $\Delta_2(n;x)$:

$$\Delta_1(n;x) = \sum_{k=0}^{n}[k]_q\frac{[n]_q-2[k]_q+1}{[n]_q^2-1}b_{n+1,k}(x;q)$$

$$
\begin{aligned}
&= \frac{1}{[n]_q-1}\sum_{k=0}^{n}[k]_q b_{n+1,k}(x;q) - \frac{2}{[n]_q^2-1}\sum_{k=0}^{n}[k]_q^2 b_{n+1,k}(x;q)\\
&= \frac{[n+1]_q x}{([n]_q-1)b_n}\sum_{k=0}^{n-1} b_{n,k}(x;q) - \frac{2q[n]_q[n+1]_q x^2}{([n]_q^2-1)b_n^2}\sum_{k=0}^{n-2} b_{n-1,k}(x;q)\\
&\quad -\frac{2[n+1]_q x}{([n]_q^2-1)b_n}\sum_{k=0}^{n-1} b_{n,k}(x;q)\\
&= \frac{[n+1]_q x}{([n]_q+1)b_n} - \frac{2q[n]_q[n+1]_q x^2}{([n]_q^2-1)b_n^2} + \frac{((2q-1)[n]_q+1)[n+1]_q}{[n]_q^2-1}\left(\frac{x}{b_n}\right)^{n+1}
\end{aligned} \tag{8}
$$

$$
\begin{aligned}
\Delta_2(n;x) &= \sum_{k=1}^{n-1}[k_q]\frac{[n]_q-2q[k]_q-1}{[n]_q^2-1} b_{n+1,k+1}(x;q)\\
&= \frac{1}{[n]_q+1}\sum_{k=1}^{n-1}[k]_q b_{n+1,k+1}(x;q) - \frac{2q}{[n]_q^2-1}\sum_{k=1}^{n-1}[k]_q^2 b_{n+1,k+1}(x;q)\\
&= \frac{[n+1]_q x}{q([n]_q+1)b_n}\sum_{k=1}^{n-1} b_{n,k}(x;q) - \frac{1}{q([n]_q+1)}\sum_{k=1}^{n-1} b_{n+1,k+1}(x;q)\\
&\quad -\frac{2[n]_q[n+1]_q x^2}{([n]_q^2-1)b_n^2}\sum_{k=0}^{n-2} b_{n-1,k}(x;q) + \frac{2[n+1]_q x}{q([n]_q^2-1)b_n}\sum_{k=1}^{n-1} b_{n,k}(x;q)\\
&\quad -\frac{2}{q([n]_q^2-1)}\sum_{k=1}^{n-1} b_{n+1,k+1}(x;q)\\
&= \frac{[n+1]_q x}{q([n]_q-1)b_n} - \frac{2[n]_q[n+1]_q x^2}{([n]_q^2-1)b_n^2}\\
&\quad +\frac{((2q-1)[n+1]_q-q+1)[n]_q}{q([n]_q^2-1)}\left(\frac{x}{b_n}\right)^{n+1}\\
&\quad +\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right).
\end{aligned} \tag{9}
$$

Combining (8) and (9), we have

$$
B_{n,\lambda,q}^{(\alpha,\beta)}(e_1;x) = \frac{b_n}{[n]_q+\beta}\left\{\frac{[n]_q x}{b_n} + \lambda\left[\frac{((q-1)[n]_q-q-1)[n+1]_q x}{q([n]_q^2-1)b_n}\right.\right.
$$

$$-\frac{2(1-q)[n]_q[n+1]_qx^2}{([n]_q^2-1)b_n^2}$$
$$+\frac{((q-1)(2q-1)[n]_q+q)[n+1]_q+(1-q)[n]_q}{q([n]_q^2-1)}\left(\frac{x}{b_n}\right)^{n+1}$$
$$-\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right)\Bigg]+\alpha\Bigg\}.$$

$$\begin{aligned}
(iii)B_{n,\lambda,q}^{(\alpha,\beta)}(e_2;x) &= \sum_{k=0}^{n}\widehat{b}_{n,k}(x;q)\left(\frac{[k]_q+\alpha}{[n]_q+\beta}b_n\right)^2\\
&= \frac{b_n^2}{([n]_q+\beta)^2}\Bigg(\alpha^2\sum_{k=0}^{n}\widehat{b}_{n,k}(x;q)+2\alpha\sum_{k=0}^{n}[k]_q\widehat{b}_{n,k}(x;q)\\
&\quad+\sum_{k=0}^{n}[k]_q^2\widehat{b}_{n,k}(x;q)\Bigg)\\
&= \frac{b_n^2}{([n]_q+\beta)^2}\Bigg\{\alpha^2+\frac{2[n]_q\alpha x}{b_n}+2\alpha\lambda\Bigg[\\
&\quad\times\frac{((q-1)[n]_q-q-1)[n+1]_qx}{q([n]_q^2-1)b_n}-\frac{2(1-q)[n]_q[n+1]_qx^2}{([n]_q^2-1)b_n^2}\\
&\quad+\frac{((q-1)(2q-1)[n]_q+q)[n+1]_q+(1-q)[n]_q}{q([n]_q^2-1)}\left(\frac{x}{b_n}\right)^{n+1}\\
&\quad-\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right)\Bigg]+\frac{[n]_qx}{b_n}\\
&\quad+\frac{[n]_q([n]_q-1)x^2}{b_n^2}+\lambda\Bigg(\sum_{k=0}^{n}[k]_q^2\frac{[n]_q-2[k]_q+1}{[n]_q^2-1}b_{n+1,k}(x;q)
\end{aligned}$$

$$-\sum_{k=1}^{n-1}[k]_q^2\frac{[n]_q-2q[k]_q-1}{[n]_q^2-1}b_{n+1,k+1}(x;q)\Bigg)\Bigg\}$$
$$=\frac{b_n^2}{([n]_q+\beta)^2}\Bigg\{\alpha^2+(2\alpha+1)\frac{[n]_qx}{b_n}+2\alpha\lambda\Bigg[$$
$$\times\frac{((q-1)[n]_q-q-1)[n+1]_qx}{q([n]_q^2-1)b_n}$$
$$+\left([n]_q([n]_q-1)-\frac{2(1-q)[n]_q[n+1]_q}{([n]_q^2-1)}\right)\frac{x^2}{b_n^2}$$

$$+\frac{((q-1)(2q-1)[n]_q+q)[n+1]_q+(1-q)[n]_q}{q([n]_q^2-1)}\left(\frac{x}{b_n}\right)^{n+1}$$
$$-\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right)\Bigg]+\lambda\Big(\Delta_3(n;x)-\Delta_4(n;x)\Big)\Bigg\}.$$

Calculate $\Delta_3(n;x)$ and $\Delta_4(n;x)$. Using $[k]_q^3=[k]_q+q(2+q)[k]_q[k-1]_q+q^3[k]_q[k-1]_q[k-2]_q$, we have

$$\begin{aligned}
\Delta_3(n;x) &= \sum_{k=0}^{n}[k]_q^2\frac{[n]_q-2[k]_q+1}{[n]_q^2-1}b_{n+1,k}(x;q)\\
&= \frac{1}{[n]_q-1}\sum_{k=0}^{n}[k]_q^2 b_{n+1,k}(x;q)-\frac{2}{[n]_q^2-1}\sum_{k=0}^{n}[k]_q^3 b_{n+1,k}(x;q)\\
&= \frac{q[n]_q[n+1]_q x^2}{([n]_q-1)b_n^2}\sum_{k=0}^{n-2}b_{n-1,k}(x;q)+\frac{[n+1]_q x}{([n]_q-1)b_n}\sum_{k=0}^{n-1}b_{n,k}(x;q)\\
&\quad -\frac{2[n+1]_q x}{([n]_q^2-1)b_n}\sum_{k=0}^{n-1}b_{n,k}(x;q)-\frac{2q(2+q)[n]_q[n+1]_q x^2}{([n]_q^2-1)b_n^2}\sum_{k=0}^{n-2}b_{n-1,k}(x;q)\\
&\quad -\frac{2q^3[n]_q[n+1]_q[n-1]_q x^3}{([n]_q^2-1)b_n^3}\sum_{k=0}^{n-3}b_{n-2,k}(x;q)\\
&= \frac{[n+1]_q x}{([n]_q+1)b_n}+\frac{q([n]_q-2q-3)[n]_q[n+1]_q x^2}{([n]_q^2-1)b_n^2}\\
&\quad -\frac{2q^3[n]_q[n+1]_q[n-1]_q x^3}{([n]_q^2-1)b_n^3}
\end{aligned}$$

$$+\left[\frac{q((2q-1)[n]_q+3)[n]_q[n+1]_q}{[n]_q^2-1}-\frac{[n+1]_q}{[n]_q+1}\right]\left(\frac{x}{b_n}\right)^{n+1}. \tag{10}$$

Using $[k]_q^2=\frac{[k+1]_q[k]_q}{q}-\frac{[k+1]_q}{q^2}+\frac{1}{q^2}$ and $[k]_q^3=[k+1]_q[k]_q[k-1]_q$ $-\frac{[k+1]_q[k]_q(1-q)}{q^2}+\frac{[k+1]_q}{q^3}-\frac{1}{q^3}$, we have

$$\Delta_4(n;x) = \sum_{k=1}^{n-1}[k]_q^2\frac{[n]_q-2q[k]_q-1}{[n]_q^2-1}b_{n+1,k+1}(x;q)$$

$$
\begin{aligned}
&= \frac{1}{[n]_q+1}\sum_{k=1}^{n-1}[k]_q^2 b_{n+1,k+1}(x;q) - \frac{2q}{[n]_q^2-1}\sum_{k=1}^{n-1}[k]_q^3 b_{n+1,k+1}(x;q) \\
&= \frac{[n]_q[n+1]_q x^2}{q([n]_q+1)b_n^2}\sum_{k=0}^{n-2} b_{n-1,k}(x;q) - \frac{[n+1]_q x}{q^2([n]_q+1)b_n}\sum_{k=1}^{n-1} b_{n,k}(x;q) \\
&\quad + \frac{1}{q^2([n]_q+1)}\sum_{k=1}^{n-1} b_{n+1,k+1}(x;q) \\
&\quad - \frac{2q[n]_q[n+1]_q[n-1]_q x^3}{([n]_q^2-1)b_n^3}\sum_{k=0}^{n-3} b_{n-2,k}(x;q) \\
&\quad + \frac{2(1-q)[n]_q[n+1]_q x^2}{q([n]_q^2-1)b_n^2}\sum_{k=0}^{n-2} b_{n-1,k}(x;q) \\
&\quad - \frac{2[n+1]_q x}{q^2([n]_q^2-1)b_n}\sum_{k=1}^{n-1} b_{n,k}(x;q) + \frac{2}{q^2([n]_q^2-1)}\sum_{k=1}^{n-1} b_{n+1,k+1}(x;q) \\
&= -\frac{[n+1]_q x}{q^2([n]_q-1)b_n} + \frac{([n]_q-2q+1)[n]_q[n+1]_q x^2}{q([n]_q^2-1)b_n^2} \\
&\quad - \frac{2q[n]_q[n+1]_q[n-1]_q x^3}{([n]_q^2-1)b_n^3} \\
&\quad + \left[\frac{((2q-1)[n]_q-1)[n]_q[n+1]_q}{q([n]_q^2-1)} + \frac{[n]_q}{q([n]_q-1)}\right]\left(\frac{x}{b_n}\right)^{n+1} \\
&\quad + \frac{1}{q^2([n]_q-1)}\left(1-(1-\frac{x}{b_n})_q^{n+1}\right). \qquad (11)
\end{aligned}
$$

Combining (10) and (11), we have

$$
\begin{aligned}
B_{n,\lambda,q}^{(\alpha,\beta)}(e_2;x) \quad = \quad & \frac{b_n^2}{([n]_q+\beta)^2}\Bigg\{\alpha^2 + (2\alpha+1)\frac{[n]_q x}{b_n} \\
& +2\alpha\lambda\Bigg[\frac{((q-1)[n]_q-q-1)[n+1]_q x}{q([n]_q^2-1)b_n} \\
& +\left([n]_q([n]_q-1) - \frac{2(1-q)[n]_q[n+1]_q}{([n]_q^2-1)}\right)\frac{x^2}{b_n^2}
\end{aligned}
$$

$$
+\frac{((q-1)(2q-1)[n]_q+q)[n+1]_q+(1-q)[n]_q}{q([n]_q^2-1)}\left(\frac{x}{b_n}\right)^{n+1}
$$

$$
\begin{aligned}
&-\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right)\Big]\\
&+\lambda\Big(\frac{((q^2+1)[n]_q-q^2+1)[n+1]_qx}{q^2([n]_q^2-1)b_n}+\\
&+\frac{([n]_q(q^2-1)-q(2q^2+3q-2)-1)[n]_q[n+1]_qx^2}{q([n]_q^2-1)b_n^2}\\
&+\frac{2q(1-q^2)[n]_q[n+1]_q[n-1]_qx^3}{([n]_q^2-1)b_n^3}\\
&+\Big[\frac{((q^2-1)(2q-1)[n]_q+3q^2+1)[n]_q[n+1]_q}{q([n]_q^2-1)}-\frac{[n+1]_q}{[n]_q+1}\\
&-\frac{[n]_q}{q([n]_q-1)}\Big]\left(\frac{x}{b_n}\right)^{n+1}-\frac{1}{q^2([n]_q-1)}\left(1-(1-\frac{x}{b_n})_q^{n+1}\right)\Big)\Big\}.
\end{aligned}
$$

◄

Lemma 3. *Taking Lemma 2 into the account, we get the following central moments:*

$$
\begin{aligned}
(i)B_{n,\lambda,q}^{(\alpha,\beta)}(e_1-x;x) &= \left(\frac{[n]_q}{[n]_q+\beta}-1\right)x+\frac{b_n}{[n]_q+\beta}\Big\{\lambda\Big[\\
&\times\frac{((q-1)[n]_q-q-1)[n+1]_qx}{q([n]_q^2-1)b_n}-\frac{2(1-q)[n]_q[n+1]_qx^2}{([n]_q^2-1)b_n^2}\\
&+\frac{((q-1)(2q-1)[n]_q+q)[n+1]_q+(1-q)[n]_q}{q([n]_q^2-1)}\left(\frac{x}{b_n}\right)^{n+1}\\
&-\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right)\Big]+\alpha\Big\}
\end{aligned}
$$

$$
\begin{aligned}
(ii)B_{n,\lambda,q}^{(\alpha,\beta)}((e_1-x)^2;x) &= \left(1-\frac{2[n]_q}{[n]_q+\beta}\right)x^2-\frac{2b_nx}{[n]_q+\beta}\Big\{\lambda\Big[\\
&\times\frac{((q-1)[n]_q-q-1)[n+1]_qx}{q([n]_q^2-1)b_n}\\
&-\frac{2(1-q)[n]_q[n+1]_qx^2}{([n]_q^2-1)b_n^2}\\
&+\frac{((q-1)(2q-1)[n]_q+q)[n+1]_q+(1-q)[n]_q}{q([n]_q^2-1)}\\
&\times\left(\frac{x}{b_n}\right)^{n+1}-\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right)\Big]+\alpha\Big\}
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{b_n^2}{([n]_q+\beta)^2}\Bigg\{\alpha^2+(2\alpha+1)\frac{[n]_q x}{b_n}\\
&+\lambda\Bigg[\Bigg(2q\alpha((q-1)[n]_q-q-1)\\
&+(q^2+1)[n]_q-q^2+1\Bigg)\frac{[n+1]_q x}{q^2([n]_q^2-1)b_n}\\
&+\Bigg(\Bigg[([n]_q(q^2-1)-q(2q^2+3q-2)-1)-4q\alpha(1\\
&-q)\Bigg]\frac{[n]_q[n+1]_q}{q([n]_q^2-1)}+2\alpha[n]_q([n]_q-1)\Bigg)\frac{x^2}{b_n^2}\\
&+\frac{2q(1-q^2)[n]_q[n+1]_q[n-1]_q x^3}{([n]_q^2-1)b_n^3}+\Bigg\{\Bigg[((q^2-1)(2q\\
&-1)[n]_q+3q^2+1)[n]_q[n+1]_q+2\alpha\Bigg(((q-1)(2q\\
&-1)[n]_q+q)[n+1]_q+(1-q)[n]_q\Bigg)\Bigg]\frac{1}{q([n]_q^2-1)}\\
&-\frac{[n+1]_q}{[n]_q+1}-\frac{[n]_q}{q([n]_q-1)}\Bigg\}\left(\frac{x}{b_n}\right)^{n+1}\\
&+\frac{2q\alpha-1}{q^2([n]_q-1)}\left(1-(1-\frac{x}{b_n})_q^{n+1}\right)\Bigg]\Bigg\}.
\end{aligned}
$$

Lemma 4. *If $0 \le x \le b_n$ and b_n is a sequence of positive numbers such that $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \frac{b_n}{[n]_q} = 0$, $\lambda \in [-1,1]$, then*

$$
\begin{aligned}
(i)\ \lim_{n\to\infty}\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}(e_1-x;x) &= \alpha\\
(ii)\ \lim_{n\to\infty}\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}((e_1-x)^2;x) &= (4\alpha+1)x.
\end{aligned}
$$

3. Korovkin-type approximation theorem

We now give the Gadzhiev's results in weighted spaces. Therefore we need to introduce the notations of [7]. Let $\rho(x) = 1 + x^2$, $-\infty < x < \infty$ and $\mathbf{B}_\rho$ be the set of all functions f defined on the real axis satisfying the condition $| f(x) | \le \mathbf{M}_f \rho(x)$, where $\mathbf{M}_f$ is a constant depending only on f. $\mathbf{B}_\rho$ is a normed space with the norm $\| f \|_\rho = \sup_{x\geq 0} \frac{f(x)}{\rho(x)}$, $f \in \mathbf{B}_\rho$. $\mathbf{C}_\rho$ denotes the subspace of all continuous functions belonging to $\mathbf{B}_\rho$ and $\mathbf{C}_\rho^k$ denotes the subspace of all functions $f \in \mathbf{C}_\rho$ with $\lim_{|x|\to\infty} \frac{f(x)}{\rho(x)} = k$, where k is a constant depending on f.

Theorem 1. *Let* (B_n) *be the sequence of positive linear operators which act from* $\mathbf{C}_\rho$ *to* $\mathbf{B}_\rho$ *such that*

$$\lim_{n\to\infty} \|B_n(t^i;x) - x^i\|_\rho = 0, \quad i \in \{0,1,2\}.$$

Then for any function $f \in C_\rho^k$

$$\lim_{n\to\infty} \|B_n f - f\|_\rho = 0,$$

and there exists a function $f^* \in C_\rho \backslash C_\rho^k$ *such that*

$$\lim_{n\to\infty} \|B_n f^* - f^*\|_\rho \geq 1.$$

Theorem 2. *Let* $B_{n,\lambda,q}^{(\alpha,\beta)}$ *be the sequence of positive linear operators defined by* (7) *and* $\rho(x) = 1 + x^2$, *with* $0 \leq x \leq b_n$, $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \dfrac{b_n}{[n]_q} = 0$, $\lambda \in [-1,1]$. *Then for each* $f \in C_\rho^k$

$$\lim_{n\to\infty} \|B_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x)\|_\rho = 0.$$

Proof. It is enough to prove that the conditions of the weighted Korovkin type theorem given by Theorem 1 are satisfied. From Lemma2(i), it is immediate that

$$\lim_{n\to\infty} \|B_{n,\lambda,q}^{(\alpha,\beta)}(e_0;x) - 1\|_\rho = 0. \tag{12}$$

Using Lemma2(ii), we have

$$\begin{aligned}
\|B_{n,\lambda,q}^{(\alpha,\beta)}(e_1;x) - x\|_\rho \quad = \quad & \left(\frac{[n]_q}{[n]_q+\beta} - 1\right) \sup_{x\in R_0} \frac{x}{1+x^2} \\
& + \frac{b_n}{[n]_q+\beta} \Bigg\{ \lambda \Bigg[\frac{((q-1)[n]_q - q - 1)[n+1]_q x}{q([n]_q^2 - 1) b_n} \\
& - \frac{2(1-q)[n]_q[n+1]_q x^2}{([n]_q^2 - 1) b_n^2} \\
& + \frac{((q-1)(2q-1)[n]_q + q)[n+1]_q + (1-q)[n]_q}{q([n]_q^2-1)} \left(\frac{x}{b_n}\right)^{n+1}
\end{aligned}$$

$$-\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right)\Bigg]+\alpha\Bigg\}\sup_{x\in R_0}\frac{1}{1+x^2}.$$

Hence we obtain

$$\lim_{n\to\infty}\|B_{n,\lambda,q}^{(\alpha,\beta)}(e_1;x)-x\|_\rho=0. \tag{13}$$

By means of Lemma 2 (iii), we get

$$\begin{aligned}
\|B_{n,\lambda,q}^{(\alpha,\beta)}(e_2;x)-x^2\|_\rho &= \left(1-\frac{2[n]_q}{[n]_q+\beta}\right)\sup_{x\in R_0}\frac{x^2}{1+x^2}\\
&-\frac{2b_n}{[n]_q+\beta}\Bigg\{\lambda\Bigg[\frac{((q-1)[n]_q-q-1)[n+1]_q x}{q([n]_q^2-1)b_n}\\
&-\frac{2(1-q)[n]_q[n+1]_q x^2}{([n]_q^2-1)b_n^2}\\
&+\frac{((q-1)(2q-1)[n]_q+q)[n+1]_q+(1-q)[n]_q}{q([n]_q^2-1)}\left(\frac{x}{b_n}\right)^{n+1}\\
&-\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right)\Bigg]+\alpha\Bigg\}\sup_{x\in R_0}\frac{x}{1+x^2}\\
&+\frac{b_n^2}{([n]_q+\beta)^2}\Bigg\{\alpha^2+(2\alpha+1)\frac{[n]_q x}{b_n}+\lambda\Bigg[\Bigg(2q\alpha((q-1)[n]_q\\
&-q-1)+(q^2+1)[n]_q-q^2+1\Bigg)\frac{[n+1]_q x}{q^2([n]_q^2-1)b_n}\\
&+\Bigg(\Bigg[([n]_q(q^2-1)-q(2q^2+3q-2)-1)-4q\alpha(1\\
&-q)\Bigg]\frac{[n]_q[n+1]_q}{q([n]_q^2-1)}+2\alpha[n]_q([n]_q-1)\Bigg)\frac{x^2}{b_n^2}\\
&+\frac{2q(1-q^2)[n]_q[n+1]_q[n-1]_q x^3}{([n]_q^2-1)b_n^3}+\Bigg\{\Bigg[((q^2-1)(2q\\
&-1)[n]_q+3q^2+1)[n]_q[n+1]_q+2\alpha\Bigg(((q-1)(2q-1)[n]_q\\
&+q)[n+1]_q+(1-q)[n]_q\Bigg)\Bigg]\frac{1}{q([n]_q^2-1)}-\frac{[n+1]_q}{[n]_q+1}\\
&-\frac{[n]_q}{q([n]_q-1)}\Bigg\}\left(\frac{x}{b_n}\right)^{n+1}\\
&+\frac{2q\alpha-1}{q^2([n]_q-1)}\left(1-(1-\frac{x}{b_n})_q^{n+1}\right)\Bigg]\Bigg\}\sup_{x\in R_0}\frac{1}{1+x^2}
\end{aligned}$$

$$\lim_{n\to\infty} \|B^{(\alpha,\beta)}_{n,\lambda,q}(e_2;x) - x^2\|_\rho = 0. \tag{14}$$

From (12), (13) and (14), for $i \in \{0,1,2\}$, we have

$$\lim_{n\to\infty} \|B^{(\alpha,\beta)}_{n,\lambda,q}(t^i;x) - x^i\|_\rho = 0.$$

Applying Theorem 1, we obtain the desired result. ◄

Now, we present a weighted approximation theorem for functions in C^k_ρ.

Theorem 3. *Let* $0 \le x \le b_n$, $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty} \dfrac{b_n}{[n]_q} = 0$, $\lambda \in [-1,1]$, $f \in C^k_\rho$ *and* $a > 0$. *Then we have*

$$\lim_{n\to\infty} \sup_{x\in[0,\infty)} \frac{| B^{(\alpha,\beta)}_{n,\lambda,q}(f;x) - f(x) |}{(1+x^2)^{1+a}} = 0.$$

Proof. For any fixed $x_0 > 0$,

$$\begin{aligned}
\sup_{x\in[0,\infty)} \frac{| B^{(\alpha,\beta)}_{n,\lambda,q}(f;x) - f(x) |}{(1+x^2)^{1+a}} &\le \sup_{x\le x_0} \frac{| B^{(\alpha,\beta)}_{n,\lambda,q}(f;x) - f(x) |}{(1+x^2)^{1+a}} \\
&\quad + \sup_{x>x_0} \frac{| B^{(\alpha,\beta)}_{n,\lambda,q}(f;x) - f(x) |}{(1+x^2)^{1+a}} \\
&\le \|B^{(\alpha,\beta)}_{n,\lambda,q}(f;x) - f(x)\|_{C[0,x_0]} \\
&\quad + \|f\|_\rho \sup_{x>x_0} \frac{| B^{(\alpha,\beta)}_{n,\lambda,q}(1+t^2;x) |}{(1+x^2)^{1+a}} \\
&\quad + \sup_{x>x_0} \frac{| f(x) |}{(1+x_0^2)^{1+a}} \\
&= I_1 + I_2 + I_3.
\end{aligned} \tag{15}$$

Since $| f(x) | \le \|f\|_\rho (1+x^2)$, we have

$$I_3 = \sup_{x>x_0} \frac{| f(x) |}{(1+x^2)^{1+a}} \le \sup_{x>x_0} \frac{\|f\|_\rho}{(1+x^2)^a} \le \frac{\|f\|_\rho}{(1+x_0^2)^a}.$$

Let $\epsilon > 0$ be arbitrary.
There exists $n_1 \in \mathbb{N}$ such that

$$\|f\|_\rho \frac{\mid B_{n,\lambda,q}^{(\alpha,\beta)}(1+t^2;x)\mid}{(1+x^2)^{1+a}} < \frac{1}{(1+x^2)^{1+a}}\|f\|_\rho\left((1+x^2)+\frac{\epsilon}{3\|f\|_\rho}\right), \qquad \forall n \geq n_1$$

$$< \frac{\|f\|_\rho}{(1+x^2)^a}+\frac{\epsilon}{3} \qquad \forall n \geq n_1. \tag{16}$$

Hence

$$\|f\|_\rho \sup_{x\geq x_0} \frac{\mid B_{n,\lambda,q}^{(\alpha,\beta)}(1+t^2;x)\mid}{(1+x^2)^{1+a}} < \frac{\|f\|_\rho}{(1+x_0^2)^a}+\frac{\epsilon}{3}, \qquad \forall n \geq n_1.$$

Thus

$$I_2+I_3 < \frac{2\|f\|_\rho}{(1+x_0^2)^a}+\frac{\epsilon}{3}, \qquad \forall n \geq n_1.$$

Now, let us choose x_0 to be so large that $\frac{\|f\|_\rho}{(1+x^2)^a} < \frac{\epsilon}{6}$.

Then,

$$I_2+I_3 < \frac{2\epsilon}{3}, \qquad \forall n \geq n_1. \tag{17}$$

$$I_1 = \|B_{n,\lambda,q}^{(\alpha,\beta)}(f)-f\|_{C[0,x_0]} < \frac{\epsilon}{3}, \qquad \forall n \geq n_2. \tag{18}$$

Let $n_0 = \max(n_1, n_2)$. Then, combining (15)-(18), we obtain

$$\sup_{x\in[0,\infty)} \frac{\mid B_{n,\lambda,q}^{(\alpha,\beta)}(f;x)-f(x)\mid}{(1+x^2)^{1+a}} < \epsilon, \qquad \forall n \geq n_0.$$

This completes the proof.◀

4. Order of convergence

Now we give the rate of convergence of the operators $B_{n,\lambda,q}^{(\alpha,\beta)}(f;x)$ in terms of the elements of the usual Lipschitz class $Lip_M(\gamma)$.

Let $f \in C_B[0,\infty)$, $M > 0$ and $0 < \gamma \leq 1$. We recall that f belongs to the class $Lip_M(\gamma)$ if the inequality

$$\mid f(t)-f(x)\mid \leq M \mid t-x\mid^\gamma \quad t,x \in [0,\infty)$$

is satisfied.

Theorem 4. *Let $0 \leq x \leq b_n$, $0 \leq \alpha \leq \beta$ and $\lambda \in [-1,1]$. Then for each $f \in Lip_M(\gamma)$, we have*

$$| B_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x) | \leq M(\delta_n(x))^{\frac{\gamma}{2}},$$

where

$$\delta_n(x) = B_{n,\lambda,q}^{(\alpha,\beta)}((t-x)^2;x).$$

Proof. For $f \in Lip_M(\gamma)$, we obtain
$| B_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x) |$

$$\begin{aligned} &= \left| \sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \left(f\left(\frac{[k]_q + \alpha}{[n]_n + \beta} b_n \right) - f(x) \right) \right| \\ &\leq \sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \left| f\left(\frac{[k]_q + \alpha}{[n]_q + \beta} b_n \right) - f(x) \right| \\ &\leq M \sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \left| \frac{[k]_q + \alpha}{[n]_q + \beta} b_n - x \right|^{\gamma}. \end{aligned}$$

Applying Hölder's inequality with $p = \frac{2}{\gamma}$ and $q = \frac{2}{2-\gamma}$, we get following inequality:

$$\begin{aligned} | B_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x) | \leq\ & M \left(\sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \left(\frac{[k]_q + \alpha}{[n]_q + \beta} b_n - x \right)^2 \right)^{\frac{\gamma}{2}} \\ & \times \left(\sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \right)^{\frac{2-\gamma}{2}}. \end{aligned}$$

From Lemma 2 we get

$$\begin{aligned} &= M \left(B_{n,\lambda,q}^{(\alpha,\beta)} \left((t-x)^2;x \right) \right)^{\frac{\gamma}{2}} \left(B_{n,\lambda,q}^{(\alpha,\beta)} (1;x) \right)^{\frac{2-\gamma}{2}} \\ &= M \left(B_{n,\lambda,q}^{(\alpha,\beta)} \left((t-x)^2;x \right) \right)^{\frac{\gamma}{2}}. \end{aligned}$$

Choosing $\delta : \delta_n(x) = B_{n,\lambda}^{(\alpha,\beta)} \left((t-x)^2;x \right)$
we obtain

$$| B_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x) | \leq M(\delta_n(x))^{\frac{\gamma}{2}}.$$

Hence, the desired result is obtained. ◄

We will estimate the rate of convergence in terms of modulus of continuity. Let $f \in C_B[0,\infty)$, and suppose the modulus of continuity of f denoted by $\omega(f,\delta)$ gives the maximum oscillation of f in any interval of length not exceeding $\delta > 0$ and it is given by the relation

$$\omega(f,\delta) = \max_{|y-x|\leq\delta} \mid f(y) - f(x) \mid, \quad x, y \in [0,\infty).$$

It is known that $\lim_{\delta\to 0+} \omega(f,\delta) = 0$ for $f \in C_B[0,\infty)$ and for any $\delta > 0$ one has

$$\mid f(y) - f(x) \mid \leq \left(\frac{\mid y - x \mid}{\delta} + 1 \right) \omega(f,\delta). \tag{19}$$

Theorem 5. *If $f \in C_B[0,\infty)$, then*

$$\mid B^{(\alpha,\beta)}_{n,\lambda,q}(f;x) - f(x) \mid \leq 2\omega(f;(\sqrt{\delta_n(x)}),$$

where $\omega(f;\cdot)$ is modulus of continuity of f and $\delta_n(x)$ is the same as in Theorem 4.

Proof. Using triangular inequality, we get

$$\begin{aligned}
\mid B^{(\alpha,\beta)}_{n,\lambda,q}(f;x) - f(x) \mid &= \left| \sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \left(f\left(\frac{[k]_q + \alpha}{[n]_q + \beta} b_n \right) - f(x) \right) \right| \\
&\leq \sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \left| f\left(\frac{[k]_q + \alpha}{[n]_q + \beta} b_n \right) - f(x) \right|.
\end{aligned}$$

Now using inequality (19), Hölder's inequality and Lemma 2, we get

$$\begin{aligned}
\mid B^{(\alpha,\beta)}_{n,\lambda,q}(f;x) - f(x) \mid &= \sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \left(\frac{\mid \frac{[k]_q+\alpha}{[n]_q+\beta} b_n - x \mid}{\delta} + 1 \right) \omega(f,\delta) \\
&\leq \omega(f,\delta) \sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \\
&\quad + \frac{\omega(f,\delta)}{\delta} \sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \left| \frac{[k]_q + \alpha}{[n]_q + \beta} b_n - x \right| \\
&= \omega(f,\delta) + \frac{\omega(f,\delta)}{\delta} \left(\sum_{k=0}^{n} \widehat{b}_{n,k}(x;q) \left(\frac{[k]_q + \alpha}{[n]_q + \beta} b_n - x \right)^2 \right)^{\frac{1}{2}}
\end{aligned}$$

$$= \omega(f,\delta) + \frac{\omega(f,\delta)}{\delta}\left(B_{n,\lambda,q}^{(\alpha,\beta)}((t-x)^2;x)\right)^{\frac{1}{2}}.$$

Choosing $\delta = \delta_n(x)$ as in Theorem 4, we have

$$\mid B_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x) \mid \leq 2\omega(f;(\sqrt{\delta_n(x)}).$$

This completes the proof. ◄

Theorem 6. *If f is a differentiable function on $[0,\infty)$ and $f \in C_B[0,\infty)$, then for any $x \in [0,\infty)$ and $\delta > 0$, it follows*

$$\begin{aligned}|B_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x)| &= \Bigg|\left(\frac{[n]_q}{[n]_q+\beta} - 1\right)x + \frac{b_n}{[n]_q+\beta}\Bigg\{\lambda\Bigg[\\ &\times\frac{((q-1)[n]_q - q - 1)[n+1]_q x}{q([n]_q^2-1)b_n} - \frac{2(1-q)[n]_q[n+1]_q x^2}{([n]_q^2-1)b_n^2}\\ &+\frac{((q-1)(2q-1)[n]_q + q)[n+1]_q + (1-q)[n]_q}{q([n]_q^2-1)}\left(\frac{x}{b_n}\right)^{n+1}\\ &-\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1} - 1\right)\Bigg] + \alpha\Bigg\}\Bigg|f'(x) + 2\delta\omega(f',\delta),\end{aligned}$$

with $\delta = \left(B_{n,\lambda,q}^{(\alpha,\beta)}((e_1-x)^2;x)\right)^{\frac{1}{2}}$.

Proof. Starting with the identity

$$f(t) - f(x) = f'(x)(t-x) + f(t) - f(x) - f'(x)(t-x), \tag{20}$$

we get for ξ between t and x

$$|f(t) - f(x) - f'(x)(t-x)| = |f'(\xi) - f'(x)||t-x|,$$

using the Lagrange mean value theorem ($f(t)-f(x) = f'(\xi)(t-x)$, with ξ between t and x). As $|\xi - x| \leq |t-x|$, it follows

$$|f'(\xi) - f'(x)| \leq \omega(f',|\,t-x\,|) \leq \left(1 + \frac{1}{\delta}\mid t-x\mid\right)\omega(f',\delta),$$

and

$$|f(t)-f(x)-f'(x)(t-x)| \;\leq\; \left(\mid t-x \mid + \frac{1}{\delta}(t-x)^2 \right)\omega(f',\delta).$$

Applying the linear positive Stancu-Chlodowsky type (λ, q)-Bernstein operators (7) to the inequality

$$|f(t)-f(x)| \;\leq\; |f'(x)(t-x)| + \left(\mid t-x \mid + \frac{1}{\delta}(t-x)^2 \right)\omega(f',\delta),$$

obtained from (20) and the above relations, we get

$$|B^{(\alpha,\beta)}_{n,\lambda,q}(f;x)-f(x)| \leq f'(x)\left|B^{(\alpha,\beta)}_{n,\lambda,q}(e_1-x;x)\right|$$

$$+\left(B^{(\alpha,\beta)}_{n,\lambda,q}(e_1-x;x)+\frac{1}{\delta}B^{(\alpha,\beta)}_{n,\lambda,q}((e_1-x)^2;x)\right)\omega(f',\delta). \tag{21}$$

The Cauchy-Schwarz inequality for linear positive operators leads to

$$B^{(\alpha,\beta)}_{n,\lambda,q}(\mid e_1-x\mid;x) \leq \left(B^{(\alpha,\beta)}_{n,\lambda,q}(e_0;x)\right)^{\frac{1}{2}} \cdot \left(B^{(\alpha,\beta)}_{n,\lambda,q}((e_1-x)^2;x)\right)^{\frac{1}{2}}. \tag{22}$$

Due to the relation (22) and the result presented by Lemma 4,the inequality (21) becomes

$$\begin{aligned}
|B^{(\alpha,\beta)}_{n,\lambda,q}(f;x)-f(x)| \;=\; & \Bigg|\left(\frac{[n]_q}{[n]_q+\beta}-1\right)x+\frac{b_n}{[n]_q+\beta}\Bigg\{\lambda\Bigg[\\
& \times\frac{((q-1)[n]_q-q-1)[n+1]_q x}{q([n]_q^2-1)b_n}-\frac{2(1-q)[n]_q[n+1]_q x^2}{([n]_q^2-1)b_n^2} \\
& +\frac{((q-1)(2q-1)[n]_q+q)[n+1]_q+(1-q)[n]_q}{q([n]_q^2-1)}\left(\frac{x}{b_n}\right)^{n+1} \\
& -\frac{1}{q([n]_q-1)}\left((1-\frac{x}{b_n})_q^{n+1}-1\right)\Bigg]+\alpha\Bigg\}\Bigg|f'(x)+2\delta\omega(f',\delta)
\end{aligned}$$

with $\delta=\left(B^{(\alpha,\beta)}_{n,\lambda,q}((e_1-x)^2;x)\right)^{\frac{1}{2}}$. ◀

5. Direct results

In this section we discuss the direct result and Voronovskaja type asymptotic formula for the operators $B^{(\alpha,\beta)}_{n,\lambda,q}$. By $C_B[0,\infty)$, we denote the space of real-valued continuous and bounded functions f defined on the interval $[0,\infty)$. The norm $\|\cdot\|$ on the space $C_B[0,\infty)$ is given by

$$\| f \| = \sup_{0\le x<\infty} | f(x) | .$$

Further let us consider the following K-functional:

$$K_2(f,\delta) = \inf_{g\in W^2}\{\| f-g \| + \delta \| g'' \|\},$$

where $\delta > 0$ and $W^2 = \{g \in C_B[0,\infty) : g', g'' \in C_B[0,\infty)\}$. By Theorem (2.4) of [6], there exists an absolute constant $C > 0$ such that

$$K_2(f,\delta) \le C\omega_2(f,\sqrt{\delta}), \tag{23}$$

where

$$\omega_2(f,\sqrt{\delta}) = \sup_{0<h\le\sqrt{\delta}} \sup_{x\in[0,\infty)} | f(x+2h) - 2f(x+h) + f(x) |$$

is the second order modulus of smoothness of $f \in C_B[0,\infty)$. The usual modulus of continuity of $f \in C_B[0,\infty)$ is defined by

$$w(f,\delta) = \sup_{0<h\le\delta} \sup_{x\in[0,\infty)} | f(x+h) - f(x) | .$$

Theorem 7. *Let $x \in [0,b_n]$, $f \in C_B[0,\infty)$, $0 \le x \le b_n$, $0 \le \alpha \le \beta$ and $\lambda \in [-1,1]$. Then for all $n \in \mathbb{N}$, there exists a positive constant $C > 0$ such that*

$$| B^{(\alpha,\beta)}_{n,\lambda,q}(f;x) - f(x) | \le C\omega_2(f,\delta_n(x)) + \omega(f,\alpha_n(x)),$$

where

$$\delta_n(x) = \sqrt{B^{(\alpha,\beta)}_{n,\lambda,q}((e_1-x)^2;x) + (\alpha_n(x))^2},$$

$$\alpha_n(x) = b_n\alpha + (a_n + d_n x + \lambda b_n c_n - 1)x + \lambda b_n\Bigg(r_n(\frac{x}{b_n})^{n+1} - u_n((1-\frac{x}{b_n})^{n+1} - 1)\Bigg),$$

$$a_n = \frac{[n]_q}{[n]_q+\beta}, b_n = \frac{b_n}{[n]_q+\beta}, c_n = \frac{((q-1)[n]_q-q-1)[n+1]_q}{q([n]_q^2-1)b_n},$$

$$d_n = \frac{2(1-q)[n]_q[n+1]_q}{([n]_q^2-1)b_n^2},\ r_n = \frac{((q-1)(2q-1)[n]_q+q)[n+1]_q+(1-q)[n]_q}{q([n]_q^2-1)},$$

$$u_n = \frac{1}{q([n]_q-1)}.$$

Proof. For $x \in [0,\infty)$, we consider the auxiliary operators $\bar{B}_n^*$ defined by

$$\begin{aligned}\bar{B}_{n,q}^*(f;x) &= B_{n,\lambda,q}^{(\alpha,\beta)}(f;x)+f(x)-f\Bigg(b_n\alpha+(a_n+d_nx+\lambda b_nc_n)x \\ &\quad +\lambda b_n\Bigg(r_n(\frac{x}{b_n})^{n+1}-u_n((1-\frac{x}{b_n})^{n+1}-1)\Bigg).\end{aligned}$$

From Lemma 2(i)(ii) and Lemma 3(i), we observe that the operators $\bar{B}_n^*(f;x)$ are linear and reproduce the linear functions. Hence

$$\begin{aligned}\bar{B}_{n,q}^*(e_0;x) &= B_{n,\lambda,q}^{(\alpha,\beta)}(e_0;x)+1-1=1 \\ \bar{B}_{n,q}^*(e_1;x) &= B_{n,\lambda,q}^{(\alpha,\beta)}(e_1;x)+x-\Bigg(b_n\alpha+(a_n+d_nx+\lambda b_nc_n)x \\ &\quad +\lambda b_n\Bigg(r_n(\frac{x}{b_n})^{n+1}-u_n((1-\frac{x}{b_n})^{n+1}-1)\Bigg)=x \\ \bar{B}_{n,q}^*((e_1-x);x) &= \bar{B}_{n,q}^*(e_1;x)-x\bar{B}_{n,q}^*(e_0;x)=0.\end{aligned}$$

Let $x \in [0,\infty)$ and $g \in C_B^2[0,\infty)$. Using the Taylor's formula, we get

$$g(t) = g(x)+g'(x)(t-x)+\int_x^t (t-u)g''(u)\mathrm{d}u.$$

Applying $\bar{B}_{n,q}^*$ to both sides of the above equation, we have

$$\begin{aligned}\bar{B}_{n,q}^*(g;x)-g(x) &= g'(x)\bar{B}_{n,q}^*((t-x);x)+\bar{B}_{n,q}^*\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right) \\ &= B_{n,\lambda,q}^{(\alpha,\beta)}\left(\int_x^t (t-u)g''(u)\mathrm{d}u;x\right) \\ &\quad -\int_x^{b_n\alpha+(a_n+d_nx+\lambda b_nc_n)x+\lambda b_n(r_n(\frac{x}{b_n})^{n+1}-u_n((1-\frac{x}{b_n})^{n+1}-1))}\Bigg(b_n\alpha \\ &\quad +(a_n+d_nx+\lambda b_nc_n)x+\lambda b_n\Bigg(r_n(\frac{x}{b_n})^{n+1}-u_n((1-\frac{x}{b_n})^{n+1}\end{aligned}$$

$$-1)\Big)-u\Big)g''(u)\mathrm{d}u.$$

On the other hand, since

$$\begin{aligned}\left|\int_x^t (t-u)g''(u)\mathrm{d}u\right| &\leq \int_x^t \mid t-u \mid\mid g''(u)\mid \mathrm{d}u\\ &\leq \parallel g''\parallel \left|\int_x^t \mid t-u\mid \mathrm{d}u\right| \leq (t-x)^2 \parallel g''\parallel\end{aligned}$$

and

$$\Big|\int_x^{b_n\alpha+(a_n+d_nx+\lambda b_nc_n)x+\lambda b_n(r_n(\frac{x}{b_n})^{n+1}-u_n((1-\frac{x}{b_n})^{n+1}-1))}\Big(b_n\alpha+(a_n+d_nx+\lambda b_nc_n)x$$
$$+\lambda b_n\Big(r_n(\frac{x}{b_n})^{n+1}-u_n((1-\frac{x}{b_n})^{n+1}-1)\Big)-u\Big)g''(u)\mathrm{d}u\Big|$$

$$\begin{aligned}\leq\ &\Big(b_n\alpha+(a_n+d_nx+\lambda b_nc_n)x\\ &+\lambda b_n\Big(r_n(\frac{x}{b_n})^{n+1}-u_n((1-\frac{x}{b_n})^{n+1}-1)\Big)-x\Big)^2 \parallel g''\parallel,\end{aligned}$$

we conclude that

$$\Big|\bar{B}^*_{n,q}(g;x)-g(x)\Big| \leq \Big\|B^{(\alpha,\beta)}_{n,\lambda,q}\Big(\int_x^t (t-u)g''(u)\mathrm{d}u;x\Big)$$
$$-\int_x^{b_n\alpha+(a_n+d_nx+\lambda b_nc_n)x+\lambda b_n(r_n(\frac{x}{b_n})^{n+1}-u_n((1-\frac{x}{b_n})^{n+1}-1))}\Big(b_n\alpha$$
$$+(a_n+d_nx+\lambda b_nc_n)x+\lambda b_n\Big(r_n(\frac{x}{b_n})^{n+1}-u_n((1-\frac{x}{b_n})^{n+1}$$
$$-1)\Big)-u\Big)g''(u)\mathrm{d}u\Big\|$$
$$\leq \parallel g''\parallel B^{(\alpha,\beta)}_{n,\lambda,q}((t-x)^2;x)+\parallel g''\parallel \Big(b_n\alpha+(a_n+d_nx$$
$$+\lambda b_nc_n)x+\lambda b_n\Big(r_n(\frac{x}{b_n})^{n+1}$$

$$-u_n((1-\frac{x}{b_n})^{n+1}-1)\bigg)-x\bigg)^2$$
$$=\| g'' \| \delta_n^2(x).$$

Now, taking into account Lemma 2(i), we have

$$| \bar{B}^*_{n,q}(f;x) | \leq | B^{(\alpha,\beta)}_{n,\lambda,q}(f;x) | +2 \| f \| \leq 3 \| f \| .$$

Therefore

$$\begin{aligned} | B^{(\alpha,\beta)}_{n,\lambda,q}(f;x)-f(x) | \leq \; & | \bar{B}^*_{n,q}(f-g;x)-(f-g)(x) | \\ & +\bigg| f\bigg(b_n\alpha+(a_n+d_nx \\ & +\lambda b_nc_n)x+\lambda b_n\bigg(r_n(\frac{x}{b_n})^{n+1} \\ & -u_n((1-\frac{x}{b_n})^{n+1}-1)\bigg)\bigg)-f(x)\bigg| \\ & +| \bar{B}^*_{n,q}(g;x)-g(x) | \\ \leq \; & 4 \| f-g \| +\omega(f,\alpha_n(x))+\delta_n^2(x) \| g'' \| . \end{aligned}$$

Hence, taking the infimum on the right-hand side over all $g \in C_B^2[0,\infty)$, we have the following result:

$$| B^{(\alpha,\beta)}_{n,\lambda,q}(f;x)-f(x) | \leq 4K_2(f,\delta_n^2(x))+\omega(f,\alpha_n(x)).$$

In view of the property of K-functional, we get

$$| B^{(\alpha,\beta)}_{n,\lambda,q}(f;x)-f(x) | \leq C\omega_2(f,\delta_n(x))+\omega(f,\alpha_n(x)).$$

This completes the proof of the theorem. ◄

Let $B_{x^2}[0,\infty)=\{f:$ for every $x\in[0,\infty),\ | f(x) | \ \leq M_f(1+x^2),\ M_f$ being a constant depending on $f\}$.

We denote by $C_{x^2}[0,\infty)$, the space of all continuous functions on $[0,\infty)$ belonging to $B_{x^2}[0,\infty)$.

Our next result in this section is the Voronovskaja type asymptotic formula:

Theorem 8. *For any function $f\in C_{x^2}[0,\infty)$ such that $f', f''\in C_{x^2}[0,\infty)$, $x\in[0,b_n]$, $0\leq x\leq b_n$, $0\leq\alpha\leq\beta$, $\lim_{n\to\infty} b_n=\infty$, $\lim_{n\to\infty}\dfrac{b_n}{[n]_q}=0$, and $\lambda\in[-1,1]$, we have*

$$\lim_{n\to\infty}\frac{[n]_q}{b_n}[B^{(\alpha,\beta)}_{n,\lambda,q}(f;x)-f(x)] \;=\; \alpha f'(x)+(4\alpha+1)xf''(x).$$

Proof. Let $f, f', f'' \in C_2^*[0,\infty)$ and $x \in [0,b_n)$ be fixed. By Taylor expansion we can write

$$f(t) = f(x) + (t-x)f'(x) + \frac{(t-x)^2}{2!}f''(x) + r(t,x)(t-x)^2,$$

where $r(t,x)$ is the Peano form of the remainder, $r(t,x) \in C_B[0,\infty)$ and $\lim_{t\to x} r(t,x) = 0$. Applying $B_{n,\lambda,q}^{(\alpha,\beta)}$, we get

$$\frac{[n]_q}{b_n}[B_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x)] = f'(x)\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}(t-x;x) + \frac{f''(x)}{2!}\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}((t-x)^2;x) + \frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}(r(t,x)(t-x)^2;x).$$

Therefore,

$$\begin{aligned}\lim_{n\to\infty}\frac{[n]_q}{b_n}[B_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x)] &= f'(x)\lim_{n\to\infty}\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}(t-x;x) \\ &\quad + \frac{f''(x)}{2!}\lim_{n\to\infty}\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}((t-x)^2;x) \\ &\quad + \lim_{n\to\infty}\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}(r(t,x)(t-x)^2;x) \\ &= \alpha f'(x) + (4\alpha+1)xf''(x) \\ &\quad + \lim_{n\to\infty}\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}(r(t,x)(t-x)^2;x) \\ &= \alpha f'(x) + (4\alpha+1)xf''(x) + E.\end{aligned}$$

By Cauchy-Schwarz inequality, we have

$$|E| \le \lim_{n\to\infty}\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}(r^2(t,x);x)^{\frac{1}{2}}B_{n,\lambda,q}^{(\alpha,\beta)}((t-x)^4;x)^{\frac{1}{2}}. \tag{24}$$

Observe that $r^2(x,x) = 0$ and $r^2(\cdot,x) \in C_2^*[0,\infty)$. Then, it follows that

$$\lim_{n\to\infty}\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}(r^2(t,x);x) = (r^2(x,x);x) = 0 \tag{25}$$

uniformly with respect to $x \in [0,b_n]$. Now from (24) and (25) we obtain

$$\lim_{n\to\infty}\frac{[n]_q}{b_n}B_{n,\lambda,q}^{(\alpha,\beta)}(r(t,x)(t-x)^2;x) = 0.$$

Hence, $E = 0$, and using Lemma 4, we have

$$\lim_{n\to\infty}\frac{[n]_q}{b_n}[B_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x)] = \alpha f'(x) + (4\alpha+1)xf''(x),$$

which completes the proof. ◄

6. Generalization of the Stancu type (λ, q)-Bernstein-Chlodowsky operators

We now give a generalization of Chlodowsky-type (λ, q)-Bernstein-Stancu operators. For $x \geq 0$, consider any continuous function $\omega(x) \geq 1$ and define

$$G_f(t) = f(t)\frac{1+t^2}{\omega(x)}.$$

Let us consider the generalization of $B_{n,\lambda,q}^{(\alpha,\beta)}$ as follows:

$$\widehat{B}_{n,\lambda,q}^{(\alpha,\beta)}(f;x) = \frac{\omega(x)}{1+t^2}\sum_{k=0}^{n}\widehat{b}_{n,k}(x;q)G_f\left(\frac{[k]_q}{[n]_q+\beta}b_n + \frac{\alpha}{[n]_q+\beta}b_n\right)$$

where $0 \leq x \leq b_n$ and (b_n) has the properties of Chlodowsky variant of (λ, q)-Bernstein-Stancu operators. Notice that this kind of generalization was considered earlier for the Bernstein-Chlodowsky polynomials [7], q-Bernstein-Chlodowsky polynomials [3] and Chlodowsky variant of q-Bernstein-Schurer-Stancu operators [18]. Now we have the following approximation theorem.

Theorem 9. *For the continuous functions satisfying*

$$\lim_{x\to\infty}\frac{f(x)}{\omega(x)} = K_f < \infty,$$

we have

$$\lim_{x\to\infty}\sup_{0\leq x\leq b_n}\frac{\mid \widehat{B}_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x)\mid}{\omega(x)} = 0,$$

provided that $x \in [0, b_n]$, $0 \leq x \leq b_n$, $0 \leq \alpha \leq \beta$, $\lim_{n\to\infty} b_n = \infty$, $\lim_{n\to\infty}\frac{b_n}{[n]_q} = 0$, *and* $\lambda \in [-1, 1]$.

Proof.

$$\widehat{B}_{n,\lambda,q}^{(\alpha,\beta)}(f;x) - f(x) = \frac{\omega(x)}{1+t^2}\left(\sum_{k=0}^{n}\widehat{b}_{n,k}(x;q)G_f\left(\frac{[k]_q}{[n]_q+\beta}b_n + \frac{\alpha}{[n]_q+\beta}b_n\right) - G_f(x)\right)$$

and therefore

$$\sup_{0\leq x\leq b_n} \frac{\mid \widehat{B}_{n,\lambda,q}^{(\alpha,\beta)}(f;x)-f(x)\mid}{\omega(x)} = \sup_{0\leq x\leq b_n} \frac{\mid \widehat{B}_{n,\lambda,q}^{(\alpha,\beta)}(G_f;x)-G_f(x)\mid}{1+x^2}.$$

From $\mid f(x)\mid\leq M_f\omega(x)$ and the continuity of the function f, we have $\mid G_f(x)\mid\leq M_f(1+x^2)$, for $x\geq 0$ and $G_f(x)$ is a continuous function on $[0,\infty)$. Using Theorem 2, we get the desired result. ◀

Finally note that in case $\omega(x)=1+x^2$, the operator $\widehat{B}_{n,\lambda,q}^{(\alpha,\beta)}$ reduces to $B_{n,\lambda,q}^{(\alpha,\beta)}$.

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