Azerbaijan Journal of Mathematics
V. 10, No 1, 2020, January
ISSN 2218-6816

Contraction Quasi Semigroups and Their Applications in Decomposing Hilbert Spaces

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Abstract. This paper addresses the problem of implementations of a strongly continuous quasi semigroup in analyzing non-autonomous Cauchy problems induced by dissipative operators. The implementations are closely related to contraction quasi semigroups. Lumer-Phillips Theorem for the contraction quasi semigroups is proved. Relationships between the contraction quasi semigroups and their cogenerator are also explored. Furthermore, we show that the contraction quasi semigroups are applicable in decomposing Hilbert spaces.

Key Words and Phrases: contraction quasi semigroup, decomposing, Lumer-Phillips Theorem, non-autonomous Cauchy problems.

2010 Mathematics Subject Classifications: 35R20

1. Introduction

In this paper we focus on non-autonomous Cauchy problems (NCP) on a Banach space X:

$$\begin{aligned} \dot{x}(t) &= A(t)x(t), \qquad t \geq 0 \\ x(0) &= x_0, \qquad\quad x_0 \in X. \end{aligned} \tag{1}$$

Here x is an unknown function from the real interval $[0, \infty)$ into X and $A(t)$ is a densely defined closed linear operator in X with domain $\mathcal{D}(A(t)) = \mathcal{D}$, independent of t. The NCP (1) often appears in description of transport-reaction phenomena arising in physical and biological systems, e.g. a heat conduction of a material undergoing decay or radioactive damage [1], and Schrödinger operators for particles in external electric fields [2]. In general, NCP (1) is divided into two types, namely parabolic and hyperbolic types, see [3, 4, 5].

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The well-posedness is a main problem of NCP (1). This is related to the existence and uniqueness of the solution. For $t \in [0, \tau]$, where $0 < \tau < \infty$, the well-posedness of NCP (1) can be characterized by evolution operators $\{U(t,s)\}_{t,s\geq 0}$, a family of linear operators on X depending on t and s, with $t \geq s$, see [3, 4, 5, 6, 7]. The assumptions for $A(t)$ in order to NCP (1) admits a unique solution are given by Kato [7] and Kato and Tanabe [8, 9]. These sufficient conditions are known as Sobolevski-Tanabe and Kato's theory. Unfortunately, this method requires that both parabolic and hyperbolic types have a very strong sufficient condition to be well-posed on $[0, \tau]$. Based on the evolution operators, evolution semigroups have been constructed to investigate the well-posedness of NCP (1), see [10, 11, 12]. Recently, Schmid and Griesemer [13] have developed a quasi contractive strongly continuous group from the evolution operators to investigate the well-posedness of NCP (1) in uniformly convex spaces.

In autonomous case i.e. $A(t) = A$ is independent of t, $U(t,s) = T(t-s)$, where $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup of bounded linear operators on X with an infinitesimal generator A. In many applications, operator A may be dissipative. In this context, A is the infinitesimal generator of a C_0-semigroup of contraction. This operator plays an important role within both the abstract operator theory and its more specialized applications in other fields, see [14, 15]. Actually, if A is the infinitesimal generator A of contraction semigroup $\{T(t)\}_{t\geq 0}$ on a Banach space X, then A is maximal dissipative (m-dissipative). Conversely, if A is m-dissipative, then A is the infinitesimal generator of a contraction semigroup on X. This statement is known as Lumer-Phillips Theorem, see [16, 17, 18].

In the autonomous case, if A is the infinitesimal generator of a C_0-semigroup on a Banach space X, then NCP (1) admits a unique solution

$$x(t) = T(t)x_0.$$

This means that the time evolution of $\{T(t)\}_{t\geq 0}$ determines the qualitative properties of the solution $x(t)$. This implies that the stability of C_0-semigroup directly affects the stability of the solution. Concerning this stability, Eisner [19] has developed some types of stabilities of C_0-semigroups including a weak stability. In particular, if X is a Hilbert space, the weak stability can be used to identify a classical theorem on decomposition of the space in contraction semigroup term. In fact, Foguel [20] has decomposed a Hilbert space into weakly stable and weakly unstable parts. Basically, the decomposition can use other ways, as done by Nagy and Foiaş [21]. They have decomposed a Hilbert space into unitary and completely non-unitary parts. The similar works have been done by Eisner [19].

The results in the autonomous cases urge a generalization to the non-autonomous ones. It is natural to take over some assumptions on the infinitesimal generator A into $A(t)$ for each t. Leiva and Barcenas [22] have established a quasi semigroup theory, an alternative family of two-parameter operators which is different from $\{U(t,s)\}_{t,s\geq 0}$. By this approach, the sufficient condition in order to NCP (1) admits a unique solution is the family $\{A(t)\}_{t\geq 0}$ is the infinitesimal generator of a C_0-quasi semigroup, regardless of whether parabolic or hyperbolic. The other advantage is that the domain of $A(t)$ must not be a bounded interval $[0,\tau]$. The properties of C_0-quasi semigroups and their applications were discussed comprehensively in [22, 23, 24]. In general, C_0-semigroups and C_0-quasi semigroups have different properties. For example, in the C_0-quasi semigroups, the infinitesimal generator must not be densely defined and closed. Nevertheless, Sutrima *et al.* [25] have generalized the various concept of stabilities of C_0-semigroups into C_0-quasi semigroups including the weak stability.

In this paper, we deal with Lumer-Phillips Theorem for the contraction quasi semigroups on Banach spaces and the decomposition theorem on Hilbert spaces. This paper is structured as follows. In Section 2, we prove the Lummer-Phillips Theorem for the contraction quasi semigroups and explore relationships between the quasi semigroups and their cogenerator. Decomposition theorem for the contraction quasi semigroups is presented in Section 3. In the last section, we illustrate our results by two examples.

2. Contraction quasi semigroups

Before discussing the main results, we recall the definition of a strongly continuous quasi semigroup initiated by Leiva and Barcenas [22]. In this paper, we rewrite the weaker definition, which follows the definition of C_0-semigroup.

Definition 1. *Let $\mathcal{L}(X)$ be the set of all bounded linear operators on a Banach space X. A two-parameter commutative family $\{R(t,s)\}_{s,t\geq 0}$ in $\mathcal{L}(X)$ is called a strongly continuous quasi semigroup (C_0-quasi semigroup, in short) on X, if:*

(a) $R(t,0)=I$, *the identity operator on* X,

(b) $R(t,s+r)=R(t+r,s)R(t,r)$,

(c) $\lim_{s\to 0^+}\|R(t,s)x-x\|=0$,

(d) there is a continuous increasing function $M:[0,\infty)\to[0,\infty)$ *such that*

$$\|R(t,s)\|\leq M(t+s),$$

for all $r,s,t\geq 0$ *and* $x\in X$.

Let $\mathcal{D}$ be the set of all $x \in X$ such that the following limits exist:

$$\lim_{s\to 0^+} \frac{R(0,s)x - x}{s} \quad \text{and} \quad \lim_{s\to 0^+} \frac{R(t,s)x - x}{s} = \lim_{s\to 0^+} \frac{R(t-s,s)x - x}{s}, \quad t > 0.$$

For $t \geq 0$, we define an operator $A(t)$ on $\mathcal{D}$ as

$$A(t)x = \lim_{s\to 0^+} \frac{R(t,s)x - x}{s}.$$

Next, the family $\{A(t)\}_{t\geq 0}$ is called an infinitesimal generator of the C_0-quasi semigroup $\{R(t,s)\}_{s,t\geq 0}$.

Condition (c) of Definition 1 means that for a fixed $t \geq 0$, the mapping $s \mapsto R(t,s)$ is strongly continuous at 0. This implies that the mapping is strongly continuous in s. Conversely, for a fixed $s \geq 0$, the mapping $t \mapsto R(t,s)$ is strongly continuous in t. In fact, for any $t_0 \geq 0$, $0 \leq t \leq s$, M is increasing, and for $x \in X$, we obtain

$$\begin{aligned} \|R(t,s)x - R(t_0,s)x\| &= \|R(t,s-t)[R(s,t)x - R(t_0,t)x]\| \\ &\leq M(s)\left[\|R(s,t)x - x\| + \|x - R(t_0,t)x\|\right]. \end{aligned}$$

Condition (c) of Definition 1 implies the assertion.

Definition 2. *A C_0-quasi semigroup $\{R(t,s)\}_{s,t\geq 0}$ is said to be a contraction on a Banach space X if condition (d) of Definition 1 is satisfied by the constant function $M = 1$.*

Following theory of C_0-semigroups for the autonomous case, we can define a cogenerator of a C_0-quasi semigroup $R(t,s)$ to investigate its characterizations. Let $A(t)$ be the infinitesimal generator of a C_0-quasi semigroup $R(t,s)$ on a Banach space X such that $1 \in \rho(A(t))$. We define a cogenerator of $R(t,s)$ to be a family $\{V(t)\}_{t\geq 0}$ given by

$$V(t) := (A(t) + I)(A(t) - I)^{-1}, \tag{2}$$

where I is the identity operator on X. We obtain

$$V(t) = I - 2\mathcal{R}(1, A(t)) \quad \text{and} \quad A(t) = I + 2(V(t) - I)^{-1},$$

where $\mathcal{R}(\lambda, A(t)) := (\lambda I - A(t))^{-1}$ is the resolvent operator of $A(t)$ with resolvent set $\rho(A(t))$.

In the sequel, $R(t,s)$, $A(t)$, and $V(t)$ denote the quasi semigroup $\{R(t,s)\}_{s,t\geq 0}$, the infinitesimal generator $\{A(t)\}_{t\geq 0}$, and the cogenerator $\{V(t)\}_{t\geq 0}$, respectively.

We also denote by $\mathcal{D}$ the domain of $A(t)$ for $t \geq 0$. In this section, we deal with the sufficient conditions for $A(t)$ to be an infinitesimal generator of a contraction quasi semigroup.

First, we generalize the Yosida approximation for the non-autonomous case. For $\lambda \in \rho(A(t))$, $t \geq 0$, the Yosida approximation is defined as

$$A_\lambda(t) = \lambda A(t)\mathcal{R}(\lambda, A(t)) = \lambda^2 \mathcal{R}(\lambda, A(t)) - \lambda I \in \mathcal{L}(X).$$

We note that

$$A(t)\mathcal{R}(\lambda, A(t)) = \lambda \mathcal{R}(\lambda, A(t)) - I \quad \text{and} \quad A(t)\mathcal{R}(\lambda, A(t))x = \mathcal{R}(\lambda, A(t))A(t)x,$$

for all $x \in \mathcal{D}$.

Theorem 1. *Let for $t \geq 0$, $A(t)$ be a closed and densely defined in a Banach space X with domain $\mathcal{D}$, $[0, \infty) \subset \rho(A(t))$, and the mapping $t \mapsto A(t)y$ be continuous from $\mathbb{R}^+$ to X for all $y \in \mathcal{D}$. If $\mathcal{R}(\lambda, A(\cdot))$ is locally integrable and*

$$\|\mathcal{R}(\lambda, A(t))\| \leq \frac{1}{\operatorname{Re}\lambda},$$

for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$, then $A(t)$ generates a contraction quasi semigroup.

Proof. For $t \geq 0$, let $A_n(t)$ be Yosida approximation of $A(t)$,

$$A_n(t) := nA(t)\mathcal{R}(n, A(t)) = n^2\mathcal{R}(n, A(t)) - nI,$$

for all $n > \omega$, $n \in \mathbb{N}$ for some $\omega > 0$. Lemma 3.6 of [26] gives

$$\lim_{n\to\infty} A_n(t)y = A(t)y \tag{3}$$

for all $y \in \mathcal{D}$. From hypothesis, we can define

$$G_n(t) := \int_0^t A_n(\tau)d\tau,$$

for all $n \in \mathbb{N}$ and $t \geq 0$. Since $G_n(t) \in \mathcal{L}(X)$, we can construct a C_0-quasi semigroup

$$R_n(t, s)x := e^{G_n(t+s) - G_n(t)}x,$$

for all $t, s \geq 0$ and $x \in X$. The infinitesimal generator of $R_n(t, s)$ is $A_n(t)$, and

$$\|R_n(t, s)\| \leq e^{-ns} e^{\int_t^{t+s} n^2 \|\mathcal{R}(n, A(\tau))\| d\tau}$$

$$= e^{-ns}.e^{ns} = 1, \tag{4}$$

for all $t, s \geq 0$ and $n \in \mathbb{N}$.

Next, we have $A_m(t)A_n(t) = A_n(t)A_m(t)$ and $A_n(t)R_m(t,s) = R_m(t,s)A_n(t)$, for all $m, n \in \mathbb{N}$ and $t, s \geq 0$. Hence, for $x \in X$,

$$\begin{aligned} R_m(t,s)x - R_n(t,s)x &= \int_0^s \frac{\partial}{\partial r}\left(R_m(t,r)R_n(t+r,s-r)x\right) dr \\ &= \int_0^s R_m(t,r)R_n(t+r,s-r)(A_m(t+r)x - A_n(t+r)x)dr. \end{aligned}$$

Since the mapping $r \mapsto A(r)y$ is uniformly continuous on the compact interval $[0,s]$ for $y \in \mathcal{D}$, by (3) for a fixed $t \geq 0$ we obtain

$$\lim_{m,n\to\infty} \sup_{r\in[0,s]} \|A_m(t+r)y - A_n(t+r)y\| = 0. \tag{5}$$

From (4) it follows that

$$\|R_m(t,s)y - R_n(t,s)y\| \leq s \sup_{r\in[0,s]} \|A_m(t+r)y - A_n(t+r)y\|. \tag{6}$$

for all $y \in \mathcal{D}$. Hence, by (5) the right-hand side converges to 0 as $m, n \to \infty$. Therefore, $(R_n(t,s)y)$ is a Cauchy sequence in X for all $t, s \geq 0$, and so it converges in X. By (4) for each $x \in X$ the set $\{R_n(t,s)x\}$ is bounded. Since $\mathcal{D}$ is dense in X, Theorem 18 of Chapter II of [27] guarantees that the convergence can be extended for each $x \in X$. Therefore, we can define

$$R(t,s)x := \lim_{n\to\infty} R_n(t,s)x, \tag{7}$$

for all $s, t \geq 0$ and $x \in X$. We easily verify that $R(t,s)$ satisfies the definition of quasi semigroup. Moreover, we have

$$\|R(t,s)x\| \leq \liminf_{n\to\infty} \|R_n(t,s)x\| \leq \|x\|,$$

for all $x \in X$, i.e. $\|R(t,s)\| \leq 1$ for all $t, s \geq 0$. Taking limits in (6) as $m \to \infty$, we get that for $y \in \mathcal{D}$

$$\|R(t,s)y - R_n(t,s)y\| \leq s \sup_{r\in[0,s]} \|A(t+r)y - A_n(t+r)y\|, \tag{8}$$

for all $t \geq 0$, $s > 0$, and $n \in \mathbb{N}$.

Next, we show that the for a fixed $t \geq 0$, limit in (7) is uniform on compact interval for every $x \in X$. For $x \in X$ and $y \in \mathcal{D}$, we have

$$\|R(t,s)x - R_n(t,s)x\| \leq \|R(t,s)(x-y)\| + \|R(t,s)y - R_n(t,s)y\| + \|R_n(t,s)(y-x)\|.$$

For arbitrary $\epsilon > 0$ and a fixed x, by choosing $y \in \mathcal{D}$ such that $\|x - y\| < \frac{\epsilon}{3}$, the first and the last term on the right-hand side above become less than $\frac{\epsilon}{3}$. For every compact interval $J \subset [0, \infty)$, according to (8) there is $N \in \mathbb{N}$ such that for $s \in J$ and $n \geq N$, the middle term on the right-hand side above become less than $\frac{\epsilon}{3}$. Thus, for any compact interval $J \subset [0, \infty)$, there is $N \in \mathbb{N}$ such that $\|R(t, s)y - R_n(t, s)y\| < \frac{\epsilon}{3}$ for all $s \in J$ and $n \geq N$. Therefore, limit in (7) is uniform on compact interval for every $x \in X$. This implies that for each $x \in X$ the mapping $s \to R(t, s)x$ is continuous. In other words, $R(t, s)$ is strongly continuous.

Let $B(t)$ be the infinitesimal generator of $R(t, s)$. For $s > 0$ and $y \in \mathcal{D}$, by Theorem 2.1 (c) of [22] and Theorem 3.2 (d) of [25], we obtain

$$\frac{1}{s}(R(t, s)y - y) = \lim_{n\to\infty} \frac{1}{s}(R_n(t, s)y - y) = \frac{1}{s}\int_0^s R(t, r)A(t + r)ydr.$$

Taking $s \to 0^+$, we obtain $y \in \mathcal{D}(B(t))$ and $B(t)y = A(t)y$. Thus, $\mathcal{D} \subseteq \mathcal{D}(B(t))$.

Next, let $y \in \mathcal{D}(B(t))$ and $\lambda > 0$. If $x = (\lambda I - A(t))^{-1}(\lambda y - B(t)y)$, then $x \in \mathcal{D} \subseteq \mathcal{D}(B(t))$ and $\lambda x - B(t)x = \lambda x - A(t)x = \lambda y - B(t)y$. Since $\lambda I - B(t) : \mathcal{D}(B(t)) \to X$ is injective, we have $y = x \in \mathcal{D}$. This shows that $\mathcal{D}(B(t)) \subseteq \mathcal{D}$. Thus, $\mathcal{D} = \mathcal{D}(B(t))$ and $B(t) = A(t)$. ◄

Theorem 1 gives the sufficiency for the infinitesimal generator of a contraction quasi semigroup although the proof is quite crucial. Even in the case of relatively simple problem, it is not easy to check all of the desired properties. For example, estimating the norm of the resolvent operator is very difficult. The dissipativity of $A(t)$ gives the sufficient and necessary conditions of an infinitesimal generator of a contraction quasi semigroup which is simpler condition.

We recall that a linear operator A on a Banach space X is said to be dissipative if for every $x \in \mathcal{D}(A)$ there exists $x' \in J(x)$ such that

$$\operatorname{Re} x'(Ax) \leq 0,$$

where X' is a dual space of X and $J(x)$ is the duality set defined by

$$J(x) = \{x' \in X' : x'(x) = \|x\|^2 = \|x'\|^2\}.$$

In case X is a Hilbert space, the operator A is dissipative if and only if

$$\operatorname{Re}\langle Ax, x\rangle \leq 0,$$

for all $x \in X$, where $\langle\cdot, \cdot\rangle$ denotes an inner product in X. A dissipative operator A is said to be maximal dissipative (m-dissipative) if A does not have a proper dissipative extension.

Theorem 2. *Let $A(t)$, $t \geq 0$, be a linear, closed and densely defined operator on $\mathcal{D}$ in a Banach space X such that for $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$ $\mathcal{R}(\lambda, A(\cdot))$ is locally integrable. Family $A(t)$ is an infinitesimal generator of a contraction quasi semigroup if and only if each $A(t)$ is m-dissipative.*

Proof. Let $A(t)$ be the infinitesimal generator of a contraction quasi semigroup $R(t,s)$. The contractiveness of $R(t,s)$ implies

$$\begin{aligned}\operatorname{Re} x'(A(t)x) &= \lim_{s\to 0^+} \operatorname{Re} x'\left(\frac{R(t,s)-I}{s}x\right)\\ &= \lim_{s\to 0^+} \frac{1}{s}\left(\operatorname{Re} x'(R(t,s)x) - \|x\|^2\right)\\ &\leq \limsup_{s\to 0^+} \frac{1}{s}\left(\|x'\|.\|x\| - \|x\|^2\right) = 0,\end{aligned}$$

for all $x \in \mathcal{D}$ and $x' \in J(x)$. This means $A(t)$ is dissipative.

Next, for every $x \in \mathcal{D}$ and $\lambda > 0$ we have

$$\begin{aligned}\|x\|\|\lambda x - A(t)x\| &= \|x'\|\|\lambda x - A(t)x\|\\ &\geq \operatorname{Re} x'(\lambda x - A(t)x)\\ &= \operatorname{Re} \lambda x'(x) - \operatorname{Re} x'(A(t)x) \geq \operatorname{Re}\lambda\|x\|^2.\end{aligned}$$

It follows that

$$\|\lambda x - A(t)x\| \geq \operatorname{Re}\lambda\|x\|, \tag{9}$$

for all $x \in \mathcal{D}$ and $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. Consequently, the operator $\lambda I - A(t) : \mathcal{D} \to X$ is injective for all $\lambda \in \mathbb{C}$ with $\operatorname{Re}\lambda > 0$. Moreover, by the closedness of $A(t)$, Lemma 3.1.4 of [18] gives that $\operatorname{ran}(\lambda I - A(t))$ is closed in X. Therefore, if we set

$$\Delta := \{\lambda \in \mathbb{C} : \operatorname{Re}\lambda > 0, \quad \overline{\operatorname{ran}(\lambda I - A(t))} = X\},$$

then $\Delta \subset \rho(A(t))$. Thus, $\operatorname{ran}(\lambda I - A(t)) = X$ for some $\lambda \in \Delta$, i.e. $A(t)$ is m-dissipative.

Conversely, by the dissipativity of $A(t)$ we have (9). Therefore, we obtain

$$\|\mathcal{R}(\lambda, A(t))\| \leq \frac{1}{\operatorname{Re}\lambda},$$

for all $\lambda \in \mathbb{C}$ such that $\operatorname{Re}\lambda > 0$. Theorem 1 implies that $A(t)$ is the infinitesimal generator of a contraction quasi semigroup. ◄

We call Theorem 2 as a version of the Lummer-Phillips Theorem for contraction quasi semigroups. This is parallel with a version of the Lummer-Phillips Theorem (Theorem 7.2.11) for contraction semigroups given by Buhler and Salamon [28].

Corollary 1. *If $A(t)$ is the infinitesimal generator of a contraction quasi semigroup on a Banach space X, then for each $x_0 \in \mathcal{D}$ the initial value problem*

$$\dot{x}(t) = A(t)x(t), \;\; x(0) = x_0 \tag{10}$$

admits a unique solution.

Proof. Let $A(t)$ be the infinitesimal generator of a contraction quasi semigroup $R(t,s)$. Proof follows from the proof of Theorem 2.2 of [22] with $r = 0$. ◄

Remark 1. *The Lummer-Phillips theorem is an alternative sufficiency of the Sobolevski-Tanabe and Kato's theory for the non-autonomous abstract Cauchy problems of dissipation, see [4]. The theorem also has the potential to be developed to investigate the well-posedness of other non-autonomous linear evolution equations, see [2, 13].*

In the rest of this section, we focus on the characterization of the contraction quasi semigropus on Hilbert spaces using the cogenerator. This is also helpful in decomposing the Hilbert spaces in the contraction quasi semigroup term.

Theorem 3. *If $R(t,s)$ is a contraction quasi semigroup on a Hilbert space X, then its cogenerator is a family of contraction operators on X.*

Proof. Let $A(t)$ be the infinitesimal generator of $R(t,s)$ and $V(t)$ be its cogenerator. Theorem 2 implies that each $A(t)$ is a dissipative operator, and by the polarization identity, we have

$$\|(A(t)+I)x\|^2 - \|(A(t)-I)x\|^2 = 4\mathrm{Re}\,\langle A(t)x, x\rangle \le 0,$$

for all $x \in X$. This shows that

$$\|(A(t)+I)x\| \le \|(A(t)-I)x\|,$$

for all $x \in X$. Therefore,

$$\|V(t)x\| = \|(A(t)+I)(A(t)-I)^{-1}x\| \le \|(A(t)-I)(A(t)-I)^{-1}x\| = \|x\|,$$

for all $x \in X$. This means that $V(t)$ is a contraction on X. ◄

There is a simple characterization of an operator to become the cogenerator of a contraction quasi semigroup on a Hilbert space. This characterization follows Theorem III.8.1 of [29].

Theorem 4. *For each $t \geq 0$, $V(t)$ is a contraction operator on a Hilbert space X. The $V(t)$ is the cogenerator of a contraction quasi semigroup on X if and only if $1 \notin \sigma_p(V(t))$ for every $t \geq 0$, where $\sigma_p(V(t))$ denotes the point spectrum of $V(t)$.*

Proof. By assumptions and Lemma 2.2.6 of [18], for $t \geq 0$ the operator $I - V(t)$ is invertible. We may define

$$A(t) := -(I + V(t))(I - V(t))^{-1},$$

on domain $\mathcal{D} := \operatorname{ran}(I - V(t))$. We just prove that $A(t)$ is the infinitesimal generator of a contraction quasi semigroup on X if and only if $1 \in \rho(A(t))$ for every $t \geq 0$.

First, assume that $A(t)$ is the infinitesimal generator of a contraction quasi semigroup on X. From the definition of $A(t)$ and Yosida approximation, we have $A(t) = I - 2(I - V(t))^{-1}$. Hence, $(I - A(t))^{-1} = \frac{1}{2}(I - V(t))$ exists which shows $1 \in \rho(A(t))$.

Conversely, assume that for each $t \geq 0$, $V(t)$ is a contraction operator on a Hilbert space X and $1 \in \rho(A(t))$. For $x \in \mathcal{D}$ and $y = (V(t) - I)^{-1}x$, we have

$$\begin{aligned}\langle A(t)x, x\rangle &= \big\langle (I + V(t))(V(t) - I)^{-1}x, x\big\rangle \\ &= \langle (I + V(t))y, (V(t) - I)y\rangle \\ &= \|V(t)y\|^2 - \|y\|^2 + 2i.\, Im\langle y, V(t)y\rangle,\end{aligned}$$

where i and $\operatorname{Im}\alpha$ denote the imaginary unit with $i^2 = -1$ and the imaginary part of a complex number α, respectively. Contractiveness of $V(t)$ gives $\operatorname{Re}\langle A(t)x, x\rangle \leq 0$. This means that $A(t)$ is dissipative.

Next, for each $t \geq 0$, $\|V(t)^n\| \leq 1$ for all $n \in \mathbb{N}$, so $V(t)$ is mean ergodic. But, since $I - V(t)$ is injective (or $\ker(I - V(t)) = \{0\}$) and X is reflexive, Theorem 3.6.9 of [28] implies that $\overline{\mathcal{D}} = \overline{\operatorname{ran}(I - V(t))} = X$. This shows that $A(t)$ is densely defined in X. Analogously, we prove $\overline{\operatorname{ran}(I - A(t))} = X$. On the other hand, since $\rho(A(t))$ is not empty, by Remark 2.2.4 of [18], $A(t)$ is closed. Consequently, Lemma 3.1.4 of [18] guarantees that $\operatorname{ran}(I - A(t)) = X$. Therefore, each $A(t)$ is m-dissipative. Thus, Theorem 2 implies that $A(t)$ is the infinitesimal generator of a contraction quasi semigroup on X. ◀

3. Decomposition on Hilbert Spaces

In this section we prove the classical theorem on decomposition of the contraction quasi semigroups on a Hilbert space with respect to different qualitative behaviors. Technique of decomposing follows the approach developed by Eisner [19], Foguel [20], and Nagy and Foiaş [21]. We begin with a weakly stable concept of C_0-quasi semigroups introduced by Sutrima *et al.* [25].

Definition 3. *A C_0-quasi semigroup $R(t,s)$ on a Hilbert space X is said to be weakly stable if* $\lim_{s\to\infty}\langle R(t,s)x,y\rangle = 0$ *for every $x, y \in X$ and $t \geq 0$.*

We recall that an operator T is unitary on a Hilbert space X if it satisfies $T^*T = TT^* = I$ on X where T^* is the adjoint operator of T on X.

Theorem 5. *Let $R(t,s)$ be a contraction quasi semigroup on a Hilbert space X. For every $t \geq 0$ there exists two $R(t,\cdot)-$invariant and $R^*(t,\cdot)-$invariant subspaces X_1 and X_2 such that X is an orthogonal sum of X_1 and X_2. Moreover,*

(a) X_1 is the maximal subspace on which the restriction of $R(t,\cdot)$ is unitary;

(b) the restriction of $R(t,\cdot)$ and $R^*(t,\cdot)$ to X_2 are weakly stable.

Proof. (a) For $t \geq 0$ fixed, we define

$$X_1 := \{x \in X : \|R(t,s)x\| = \|R^*(t,s)x\| = \|x\| \quad \text{for all } \ s \geq 0\}.$$

By this definition, $R(t,s)$ and $R^*(t,s)$ are operators from X_1 onto X_1. Moreover, for any $x \in X_1$ and $s \geq 0$, we have

$$\langle x,x\rangle = \langle R(t,s)x, R(t,s)x\rangle = \langle R^*(t,s)R(t,s)x, x\rangle.$$

This means that $R^*(t,s)R(t,s)x = x$ for every $x \in X_1$. Analogously, we can show that $R(t,s)R^*(t,s)x = x$ for every $x \in X_1$. Hence, we can represent X_1 as

$$X_1 = \{x \in X : R^*(t,s)R(t,s)x = R(t,s)R^*(t,s)x = x \quad \text{for all } \ s \geq 0\}.$$

This means that X_1 is the maximal closed subspace on which $R(t,\cdot)$ is unitary. The invariance of X_1 under $R(t,s)$ and $R^*(t,s)$ follows from the definition of X_1 and the equality $R^*(t,s)R(t,s) = R(t,s)R^*(t,s)$ on X_1.

(b) We take $X_2 := X_1^{\perp}$. Since X_1 is invariant under $R(t,s)$ and $R^*(t,s)$, so is X_2. For $x \in X_2$ and $t \geq 0$ fixed, suppose that $R(t,s)x$ does not converge weakly to zero as $s \to \infty$. Equivalently, there exist $y \in X$, $\epsilon > 0$, and a sequence (s_n) such that $|\langle R(t,s_n)x,y\rangle| \geq \epsilon$ for every $n \in \mathbb{N}$.

Since every bounded set in a reflexive Banach space is weakly compact (Corollary 8 V.4 of [27]) and weak compactness on Banach spaces coincides with weak sequential compactness (Eberlein-Smulian Theorem V.6 of [27]), there is a weakly converging subsequence of $(R(t,s_n)x)$. For simplicity, we assume that the subsequence is $(R(t,s_n)x)$ itself and its limit is x_0. The closedness and $R(t,s)$-invariance of X_2 imply that $x_0 \in X_2$.

For $s_0 \geq 0$ fixed and a fact $R(t+s, s_0)R(t,s) = R(t, s+s_0)$, we obtain

$$\begin{aligned}
&\|R^*(t+s,s_0)R(t+s,s_0)R(t,s)x - R(t,s)x\|^2 \\
&\leq \|R^*(t+s,s_0)R(t,s+s_0)x\|^2 - 2\langle R^*(t+s,s_0)R(t,s+s_0)x, R(t,s)x\rangle \\
&\quad + \|R(t,s)x\|^2 \\
&\leq \|R(t,s+s_0)x\|^2 - 2\langle R(t,s+s_0)x, R(t+s,s_0)R(t,s)x\rangle + \|R(t,s)x\|^2 \\
&= \|R(t,s+s_0)x\|^2 - 2\|R(t,s+s_0)x\|^2 + \|R(t,s)x\|^2 \\
&= \|R(t,s)x\|^2 - \|R(t,s+s_0)x\|^2.
\end{aligned}$$

Since the function $t \mapsto \|R(t,s)x\|$ is monotone decreasing on $\mathbb{R}^+$, the right-hand side converges to zero as $s \to \infty$. Therefore, we obtain

$$\|R^*(t+s,s_0)R(t+s,s_0)R(t,s)x - R(t,s)x\| \to 0 \quad \text{as } s \to \infty.$$

By assumption, $R(t,s_n)x \to x_0$ weakly as $n \to \infty$. This implies that

$$R^*(t+s,s_0)R(t+s,s_0)R(t,s_n)x \to R^*(t+s,s_0)R(t+s,s_0)x_0 \quad \text{as } n \to \infty.$$

On the other hand, we have

$$R^*(t+s,s_0)R(t+s,s_0)R(t,s_n)x \to x_0 \quad \text{as } n \to \infty.$$

By the uniqueness of limit we have $R^*(t+s,s_0)R(t+s,s_0)x_0 = x_0$. Analogously, we obtain $R(t+s,s_0)R^*(t+s,s_0)x_0 = x_0$. These imply that $x_0 \in X_1$. Since $X_1 \cap X_2 = \{0\}$, it follows that $x_0 = 0$. This contradicts the assumption that $R(t,s)x$ does not converge weakly to zero.

Analogously we can show that the restriction of $R^*(t,s)x$ to X_2 converges weakly to zero as well. ◀

Indeed, due to Foguel [20], we can decompose a Hilbert space into weakly stable and weakly unstable parts. The following theorem is a generalization of the Foguel's result for the contraction quasi semigroups.

Theorem 6. *Let $R(t,s)$ be a contraction quasi semigroup on a Hilbert space X. If for any $t \geq 0$ we define*

$$W = \left\{x \in X : \lim_{s\to\infty} \langle R(t,s)x, x\rangle = 0\right\},$$

then

$$\begin{aligned}
W &= \left\{x \in X : \lim_{s\to\infty} R(t,s)x = 0 \ \textit{weakly}\right\} \\
&= \left\{x \in X : \lim_{s\to\infty} R^*(t,s)x = 0 \ \textit{weakly}\right\},
\end{aligned}$$

W is a closed $R(t,s)-$invariant and $R^*(t,s)-$invariant subspace of X and also the restriction of $R(t,s)$ to $W^\perp$ is unitary.

Proof. For any $t \geq 0$ fixed, we have to show that $R(t,s)x \to 0$ weakly as $s \to \infty$ for all $x \in W$. For $x \in W$, by Theorem 5 we may assume $x \in X_1$. If we take $Z := \overline{\text{span}\,\{R(t,s)x : s \geq 0\}}$, then by the decomposition $X = Z \oplus Z^{\perp}$, it is enough to show that $\langle R(t,s)x, y\rangle \to 0$ as $s \to \infty$ for all $y \in Z$. We set $y := R(t,s_0)x$ for some $s_0 \geq 0$. Since the restriction of $R(t,s)$ to X_1 is unitary, for $s \geq s_0$ we have

$$\begin{aligned}\langle R(t,s)x, y\rangle &= \langle R^*(t,s_0)R(t,s_0)R(t+s_0, s-s_0)x, x\rangle \\ &= \langle R(t+s_0, s-s_0)x, x\rangle \to 0 \quad \text{as } s \to \infty.\end{aligned}$$

By the density of span $\{R(t,s)x : s \geq 0\}$ in Z we have that $\langle R(t,s)x, y\rangle \to 0$ as $s \to \infty$ for all $y \in Z$. Hence $R(t,s)x \to 0$ weakly as $s \to \infty$. Analogously, $R^*(t,s)x \to 0$ weakly as $s \to \infty$. The converse of implication, closedness and invariance of W are clear. Theorem 5 implies directly that the restriction of $R(t,s)$ to $W^{\perp}$ is unitary. ◄

Remark 2. *It is easily understood that one of the components in Theorem 5 or Theorem 6 may be absent. In this context, the corresponding subspace may reduce to* $\{0\}$*. Moreover, the set* W *in Theorem 6 is a closed subspace of* X *containing* X_2 *which is defined in Theorem 5.*

4. Applications

In this section, we consider some examples of application of the contraction quasi semigroups in solving the non-autonomous Cauchy problems and in decomposing a Hilbert space.

Example 1. *Consider the boundary value problem of the heat conduction of a material undergoing decay or radioactive damage*

$$\begin{aligned}&\frac{\partial x}{\partial t}(t,\xi) = a(t)\frac{\partial^2 x}{\partial \xi^2}(t,\xi), \quad 0 < \xi < 1, \quad t \geq 0 \\ &x(t,0) = x(t,1) = 0,\end{aligned} \tag{11}$$

where a *is the time-dependent thermal diffusivity, a continuous function with* $a(t) > 0$ *for* $t \geq 0$.

The problem (1) is taken from [1]. Therein the author focuses on determining unknown time-dependent thermal diffusivity a from overspecified data. Let X be a Hilbert space $L_2(0,1)$ and the operator $A : \mathcal{D}(A) \to X$ given by

$$Ax := \frac{d^2 x}{d\xi^2}$$

and

$$\mathcal{D}(A) = \left\{ x \in X : x, \tfrac{dx}{d\xi} \text{ are absolutely continuous}, \tfrac{d^2x}{d\xi^2} \in X, x(0) = x(1) = 0 \right\}.$$

The equation (11) can be rewritten as

$$\dot{x}(t) = a(t)Ax(t) \quad t \geq 0. \tag{12}$$

Obviously, A is self-adjoint and $-A \geq 0$ on $\mathcal{D}(A)$. Proposition 3.3.5 of [18] implies that A is m-dissipative. Therefore, $A(t) = a(t)A$ is the infinitesimal generator of a contraction quasi semigroup $R(t,s)$ given by

$$R(t,s)x = \sum_{n=1}^{\infty} e^{-\lambda_n (g(t+s)-g(t))} \langle \phi_n, x \rangle \phi_n,$$

where $\lambda_n = n^2\pi^2$, $\phi_n(\xi) = \sin(n\pi\xi)$, and $g(t) = \int_0^t a(s)ds$. In this case, $\mathcal{D} = \mathcal{D}(A)$ is dense in X. According to Corollary 1, for each $x_0 \in \mathcal{D}$ the problem (12) admits a unique solution

$$x(t) = R(0,t)x_0, \quad x(0) = x_0.$$

Hence, the problem (11) has a unique solution

$$x(t,\xi) = R(0,t)x_0(\xi).$$

Example 2. *Let X be a Hilbert space $L_2(\mathbb{R}^+, Y)$ where Y is a Hilbert space. Define a contraction quasi semigroup $R(t,s)$ on X by*

$$R(t,s)x(\xi) = \begin{cases} x(\xi - (s^2 + 2st)), & \xi \geq s^2 + 2st \\ 0, & 0 \leq \xi < s^2 + 2st, \end{cases} \tag{13}$$

for all $t, s \geq 0$. We shall examine the decomposition of X due to $R(t,s)$.

It is obvious that $R(t,s)$ is isometric on X. Moreover, the adjoint of $R(t,s)$ is given by

$$R^*(t,s)x(\xi) = x(s^2 + 2st + \xi),$$

for all $t, s \geq 0$, $x \in X$, and $\xi \geq 0$. Hence, we have

$$\|R(t,s)x\|^2 = \int_{s^2+2st}^{\infty} \|x(\tau)\|_Y^2 d\tau \to 0 \quad \text{as } s \to \infty.$$

This shows that $R(t,s)$ is weakly stable on X. Analogously, $R^*(t,s)$ is also weakly stable on X. Direct calculation shows that the infinitesimal generator $A(t)$ of $R(t,s)$ is

$$A(t)x(\xi) = -2t\frac{dx}{d\xi}(\xi),$$

with domain $\mathcal{D} = \{x \in X : x \text{ is absolutely continuous, } \frac{dx}{d\xi} \in X\}$.

Let $V(t) := (A(t)+I)(A(t)-I)^{-1}$ be a cogenerator of $R(t,s)$ defined by (2) with adjoint $V^*(t)$. A simple calculation gives

$$V(t)x(\xi) = x(\xi) - \frac{e^{-\xi/2t}}{t}\int_0^\xi e^{\tau/2t}x(\tau)d\tau$$

and

$$V^*(t)x(\xi) = x(\xi) - \frac{e^{\xi/2t}}{t}\int_\xi^\infty e^{-\tau/2t}x(\tau)d\tau,$$

respectively. If we define

$$\begin{aligned} X_1 &= \{x \in X : x = V(t)z,\ z \in X\} \\ X_2 &= \{x \in X : V^*(t)x = 0\}, \end{aligned}$$

then $X = X_1 \oplus X_2$. By solving the equation $V^*(t)x = 0$ we have $X_2 = \{x \in X : x(\xi) = e^{-\xi/2t}\}$. Moreover, for every $x \in X$ we verify that $V^*(t)V(t)x = x$. For any $x \in X_1$, $x = V(t)y$ for some $y \in X$, we have

$$\begin{aligned} V(t)V^*(t)x(\xi) &= V^*(t)x(\xi) - \frac{e^{-\xi/2t}}{t}\int_0^\xi e^{\tau/2t}V^*(t)x(\tau)d\tau \\ &= V^*(t)V(t)y(\xi) - \frac{e^{-\xi/2t}}{t}\int_0^\xi e^{\tau/2t}V^*(t)V(t)y(\tau)d\tau \\ &= y(\xi) - \frac{e^{-\xi/2t}}{t}\int_0^\xi e^{\tau/2t}y(\tau)d\tau \\ &= V(t)y(\xi) = x(\xi). \end{aligned}$$

This shows that the operator $V(t)$ is unitary on X_1. Stone's Theorem (Theorem 13.38 of [30]) implies that the quasi semigroup $R(t,s)$ is also unitary on X_1. Actually, we can directly verify that $R(t,s)$ is also unitary on X_1.

For $t = 0$ we have $X_1 = X$ and $X_2 = \{0\}$ in which $R(0,s)$ and $R^*(0,s)$ satisfy the conclusion of Theorem 5. Thus, we have confirmed Theorem 5 by the quasi semigroup $R(t,s)$.

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Received 05 March 2019
Accepted 14 May 2019