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Invertibility of Multipliers in Hilbert C^*-modules

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Abstract. Multipliers are operators which have important applications for signal processing and acoustics. In this paper, we investigate invertibility of multipliers and Riesz multipliers in Hilbert C^*-modules. We show that unlike Riesz multipliers in Hilbert spaces, Riesz multipliers in Hilbert C^*-modules may not be invertible. In addition, using the modular Riesz bases and uniqueness of dual, we prove that the Riesz multipliers of those Riesz bases in Hilbert C^*-modules are invertible. Also, we obtain some necessary conditions for invertibility of multipliers in Hilbert C^*-modules. Furthermore, we show that the inverse of any invertible multiplier operator in Hilbert C^*-module is a multiplier operator.

Key Words and Phrases: frame, Bessel sequence, Riesz basis, Hilbert C^*-module, multiplier operator.

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1. Introduction

Frames for Hilbert spaces were first introduced in 1952 by Duffin and Schaeffer [11] for study of nonharmonic Fourier series. They were reintroduced and further developed in 1986 by Daubechies, Grossmann and Meyer [10], and popularized from then on. For more complete treatment of frame theory we recommend the excellent book of Christensen, see [9].

Hilbert C^*-modules form a wide category between Hilbert spaces and Banach spaces. Their structure was first used by Kaplansky [18] in 1952. They are an often used tool in operator theory and in operator algebra theory. They serve as a major class of examples in operator C^*-module theory.

The notions of frames in Hilbert C^*-modules were introduced and investigated in [12]. Frank and Larson [12, 13] defined the standard frames in Hilbert C^*-modules in 1998 and got a series of results for standard frames in finitely or

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countably generated Hilbert C^*-modules over unital C^*-algebras. Extending the results to this more general framework is not a routine generalization, as there are essential differences between Hilbert C^*-modules and Hilbert spaces. For example, any closed subspace in a Hilbert space has an orthogonal complement, but this fails in Hilbert C^*-module. Also there is no explicit analogue of the Riesz representation theorem of continuous functionals in Hilbert C^*-modules. We refer the readers to [17] and [21] for more details on Hilbert C^*-modules and to [13] and [27, 28, 29] for a discussion on basic properties of frame in Hilbert C^*-modules and their generalizations.

Multipliers are operators which are defined based on two sequences with elements from a Hilbert space and one scalar sequence. Several basic properties of these operators were investigated in [3]. Multipliers are not only interesting from a theoretical point of view, they are also used in applications, in particular in the fields of audio and acoustics. In signal processing applications like wireless communication or medical imaging, "time-invariant filters" , i.e. convolution operators are used very often. Such operators are called "Fourier multipliers" [8]. In these applications, they are used for Gabor frames under the name "Gabor filters" [22].

Recently, the concept of multipliers has been extended and introduced for continuous frames [4], fusion frames [2], p-Bessel sequences [24], generalized frames [23], controlled frames [25], Banach frames [6, 7], Hilbert C^*-modules [20] and etc.

From a theoretical point of view, it is very natural to investigate the invertibility of multiplier and to represent the inverse of a Bessel, frame and Riesz multiplier. In the series of papers [30, 31, 32] and [5], P. Balazs and D.T. Stoeva have given a full and detailed characterization on the invertibility of multipliers based on analysis, synthesis and symbol sequences.

In this paper, we investigate invertibility of multipliers and Riesz multipliers in Hilbert C^*-modules. In Hilbert C^*-modules, a Riesz multiplier may not be invertible unlike Riesz multipliers in Hilbert spaces. By using modular Riesz bases and uniqueness of dual, we prove that the Riesz multipliers of those Riesz bases in Hilbert C^*-modules are invertible. Also, we obtain some necessary conditions for invertibility of multipliers in Hilbert C^*-modules and we show that the inverse of any invertible multiplier operator in Hilbert C^*-module is a multiplier operator.

The paper is organized as follows. In Section 2, we review the concepts Hilbert C^*-modules, frames and Riesz bases in Hilbert C^*-modules. Also the analysis, synthesis, frame operators and dual frames be reviewed. Section 3 deals with invertibility of Riesz multipliers in Hilbert C^*-modules. In Section 4, we focus on invertible operators and obtain necessary condition for invertibility of multipliers in Hilbert C^*-modules.

2. Frames, Riesz bases and multipliers in Hilbert C^*-modules

In this section, we collect the basic notations and some preliminary results on frames in Hilbert C^*-modules.

Hilbert C^*-modules form a wide category between Hilbert spaces and Banach spaces. Their structure was first used by Kaplansky [18] in 1952. They are an often used tool in operator theory and in operator algebra theory. They serve as a major class of examples in operator C^*-module theory.

Let A be a C^*-algebra with involution $*$. An inner product A-module (or pre Hilbert A-module) is a complex linear space $\mathcal{H}$ which is a left A-module with an inner product map $\langle\cdot,\cdot\rangle : \mathcal{H}\times\mathcal{H}\to A$ which satisfies the following properties:

1. $\langle \alpha f+\beta g, h\rangle = \alpha\langle f,h\rangle + \beta\langle g,h\rangle$ for all $f,g,h\in\mathcal{H}$ and $\alpha,\beta\in\mathbb{C}$;
2. $\langle af,g\rangle = a\langle f,g\rangle$ for all $f,g\in\mathcal{H}$ and $a\in A$;
3. $\langle f,g\rangle = \langle g,f\rangle^*$ for all $f,g\in\mathcal{H}$;
4. $\langle f,f\rangle\geq 0$ for all $f\in\mathcal{H}$ and $\langle f,f\rangle = 0$ iff $f=0$.

For $f\in\mathcal{H}$, we define a norm on $\mathcal{H}$ by $\|f\|_{\mathcal{H}} = \|\langle f,f\rangle\|_A^{1/2}$. If $\mathcal{H}$ is complete with this norm, it is called a (left) Hilbert C^*-module over A or a (left) Hilbert A-module.

An element a of a C^*-algebra A is positive if $a^*=a$ and its spectrum is a subset of positive real numbers. In this case, we write $a\geq 0$. By condition (4) in the definition $\langle f,f\rangle\geq 0$ for every $f\in\mathcal{H}$, hence we define $|f| = \langle f,f\rangle^{1/2}$. We call $Z(A)=\{a\in A : ab=ba, \forall b\in A\}$ the center of A. If $a\in Z(A)$, then $a^*\in Z(A)$, and if a is an invertible element of $Z(A)$, then $a^{-1}\in Z(A)$, also if a is a positive element of $Z(A)$, then $a^{\frac{1}{2}}\in Z(A)$. Let $Hom_A(M,N)$ denote the set of all A-linear operators from M to N.

We are focusing on finitely and countably generated Hilbert C^*-modules over unital C^*-algebra A. A Hilbert A-module $\mathcal{H}$ is finitely generated if there exists a finite set $\{x_1,x_2,...,x_n\}\subseteq\mathcal{H}$ such that every $x\in\mathcal{H}$ can be expressed as $x=\sum_{i=1}^n a_ix_i$, $a_i\in A$. A Hilbert A-module $\mathcal{H}$ is countably generated if there exits a countable set of generators.

In Hilbert C^*-module, in contrast with Hilbert space, every bounded operator is not adjointable. We denote the set of all adjointable maps from $\mathcal{H}$ to $\mathcal{K}$ by $End_A^*(\mathcal{H},\mathcal{K})$. Let

$$\ell^2(A) = \left\{ \{a_j\}\subseteq A : \sum_{j\in J} a_j^*a_j \ converges \ in \|.\| \right\}$$

with inner product $\langle\{a_j\},\{b_j\}\rangle=\sum_{j\in J}a_j^*b_j, \quad \{a_j\},\{b_j\}\in\ell^2(A)$ and $\|\{a_j\}\|:=\sqrt{\|\sum a_j^*a_j\|}$. It was shown that [33], $\ell^2(A)$ is a Hilbert A-module.

The notion of (standard) frames in Hilbert C^*-modules is first defined by Frank and Larson [13]. Basic properties of frames in Hilbert C^*-modules are discussed in [14, 15].

If $\mathcal{H}$ is a Hilbert C^*-module, and J a set which is finite or countable, a system $\{f_j\}_{j\in J}\subseteq\mathcal{H}$ is called a frame for $\mathcal{H}$ if there exist constants $C,D>0$ such that

$$C\langle f,f\rangle\leq\sum_{j\in J}\langle f,f_j\rangle\langle f_j,f\rangle\leq D\langle f,f\rangle \tag{1}$$

for all $f\in\mathcal{H}$. The constants C and D are called the frame bounds. If $C=D$, it is called a tight frame and in the case $C=D=1$ it is called a Parseval frame. It is called a Bessel sequence if the upper inequality in (1) holds.

Let $\{f_j : j\in J\}$ be a frame in Hilbert A-module $\mathcal{H}$ over a unital C^*-algebra A and $\{g_j : j\in J\}$ be a sequence of $\mathcal{H}$. Then $\{g_j\}_{j\in J}$ is called a dual sequence of $\{f_j\}_{j\in J}$ if

$$f=\sum_{j\in J}\langle f,g_j\rangle f_j$$

for all $f\in\mathcal{H}$. The sequences $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ are called a dual frame pair when $\{g_j\}_{j\in J}$ is also a frame..

For the frame $\{f_j : j\in J\}$ in Hilbert A-module $\mathcal{H}$ over a unital C^*-algebra A, the operator S defined by

$$Sf=\sum_{j\in J}\langle f,f_j\rangle f_j,\ \ f\in\mathcal{H}$$

is called the frame operator. It is proved that [13], S is invertible, positive, adjointable and self-adjoint.

Let $\tilde{f}_j=S^{-1}f_j$. Then

$$f=\sum_{j\in J}\langle f,\tilde{f}_j\rangle f_j=\sum_{j\in J}\langle f,f_j\rangle\tilde{f}_j,$$

for any $f\in\mathcal{H}$. The sequence $\{\tilde{f}_j : j\in J\}$ is also a frame for $\mathcal{H}$ which is called the canonical dual frame of $\{f_j : j\in J\}$.

Like ordinary frames in Hilbert spaces, the notions of analysis and synthesis operators can be defined as follows:

Let $\{f_j\}_{j\in J}$ be a frame in Hilbert A-module $\mathcal{H}$ over a unital C^*-algebra A. Then the related analysis operator $U:\mathcal{H}\to\ell^2(A)$ is defined by

$$Uf=\{\langle f,f_j\rangle : j\in J\},$$

for all $f \in \mathcal{H}$. We define the synthesis operator $T : \ell^2(A) \to \mathcal{H}$ by

$$T(\{a_j\}) = \sum_{j \in J} a_j f_j,$$

for all $\{a_j\}_{j \in J} \in \ell^2(A)$.

Next theorem is a corollary from Theorem 3.4 in [1] on characterization of dual frames in Hilbert C^*-modules.

Theorem 1. *Let $\Phi = (\phi_j)$ be a frame in Hilbert C^*-module $\mathcal{H}$ over a unital C^*-algebra A with the canonical dual frame $\widetilde{\Phi} = (\tilde{\phi}_j)$. Then any dual frames of $\Phi = (\phi_j)$ are the sequences*

$$\left(\tilde{\phi}_j + h_j - \sum_{n=1}^{\infty} \langle \tilde{\phi}_j, \phi_n \rangle h_n \right)_{j=1}^{\infty},$$

where $(h_j)_{j=1}^{\infty}$ is a Bessel sequence in Hilbert A-module $\mathcal{H}$.

In Hilbert spaces every Riesz basis has a unique dual which is also a Riesz basis. But in Hilbert C^*-modules, a Riesz basis may have many dual modular frames and it may even admit two different dual modular frames both of which are Riesz bases.

Definition 1. *[13] A frame $\{f_j\}_{j \in J}$ in Hilbert A-module $\mathcal{H}$ over a unital C^*-algebra A is called a Riesz basis if it satisfies:*

1. *$f_j \neq 0$ for any $j \in J$;*
2. *if an A-linear combination $\sum_{j \in K} a_j f_j$ is equal to zero, then every summand $a_j f_j$ is equal to zero, where $\{a_j\}_{j \in K} \subseteq A$ and $K \subseteq J$.*

A. Khosravi and B. Khosravi introduced modular Riesz bases in Hilbert C^*-modules [19], which share many properties with Riesz bases in Hilbert spaces.

Definition 2. *Let A be a unital C^*-algebra with identity 1_A. A sequence $\{f_j\}_{j \in J}$ in Hilbert A-module $\mathcal{H}$ is called a modular Riesz basis for $\mathcal{H}$ if there exists an invertible operator $T \in End_A^*(\ell^2(A), \mathcal{H})$ such that $Te_j = f_j$ for each $j \in J$, where $\{e_j\}_{j \in J}$ is the standard orthonormal basis of $\ell^2(A)$, i.e. $e_j = (\delta_{ij} 1_A)_{j \in J}$.*

In Hilbert C^*-module setting, every modular Riesz basis is a Riesz basis, but every Riesz basis is not a modular Riesz basis [26].

3. Invertibility of Riesz multipliers in Hilbert C^*-modules

In this section, we study the concept of multiplier for Riesz bases and modular Riesz bases in Hilbert C^*-modules and we show some of its properties.

Definition 3. *Let A be a unital C^*-algebra, J be a finite or countable index set and $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$ be Hilbert C^*-modules Bessel sequences for $\mathcal{H}$. For $m = \{m_j\}_{j\in J} \in \ell^\infty(A)$, where $m_j \in Z(A)$, for each $j \in J$, the operator $M_{m,\{f_j\},\{g_j\}} : \mathcal{H} \to \mathcal{H}$ defined by*

$$M_{m,\{f_j\},\{g_j\}} f := \sum_{j\in J} m_j \langle f, f_j\rangle g_j, \quad f \in \mathcal{H}$$

is called the multiplier operator of $\{f_j\}_{j\in J}$ and $\{g_j\}_{j\in J}$. The sequence $m = \{m_j\}$ is called the symbol of $M_{m,\{f_j\},\{g_j\}}$.

The symbol of m has an important role in the study of multiplier operators. In this paper m is always a sequence $m = \{m_j\}_{j\in J} \in \ell^\infty(A)$ with $m_j \in Z(A)$, for each $j \in J$.

One of the major questions in the study of multiplier is the invertibility of multiplier operator. In the sequel, we list results of invertibility of Riesz multipliers in Hilbert spaces.

Lemma 1. *[3] Let $(\psi_j) \subset \mathcal{H}_1$ be a Bessel sequence with no zero elements, and $(\phi_j) \subset \mathcal{H}_2$ a Riesz sequence. Then the mapping $m \mapsto M_{m,\phi_j,\psi_j}$ is injective from $\ell^\infty(A)$ into $End^*_A(\mathcal{H}_1, \mathcal{H}_2)$.*

Proposition 1. *[3] Let (ψ_j) be a Riesz basis with bounds C, D and (ϕ_j) be a frame with bounds C', D'. Then*

$$\sqrt{CC'}\|m\|_\infty \leq \|M_{m,(\phi_j),(\psi_j)}\|_{Op} \leq \sqrt{DD'}\|m\|_\infty.$$

The following theorem gives sufficient and necessary conditions for invertibility of multipliers for Riesz bases.

Theorem 2. *[30] Let Φ be a Riesz basis for Hilbert $\mathcal{H}$. Then the following hold.*

1. *If Ψ is a Riesz basis for $\mathcal{H}$, then $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) is invertible on $\mathcal{H}$ if and only if m is semi-normalized, i.e. $0 < \inf |m_i| \leq \sup\inf |m_i| < \infty$.*
2. *If m is semi-normalized, then $M_{m,\Phi,\Psi}$ (resp. $M_{m,\Psi,\Phi}$) is invertible on $\mathcal{H}$ if and only if Ψ is a Riesz basis for $\mathcal{H}$.*

But the above results don't hold in Hilbert C^*-module. We show this by the following example.

Example 1. *Let $A = M_{2\times 2}(\mathbb{C})$ be the C^*-algebra of all 2×2 complex matrices. Let $\mathcal{H} = A$ and for any $A, B \in \mathcal{H}$ define $\langle A, B\rangle = AB^*$. Then $\mathcal{H}$ is a Hilbert A-module. Let $E_{i,j}$ be the matrix with 1 in the (i, j)'th entry and 0 elsewhere, where $1 \leq i, j \leq 2$. Then $\Phi = \{E_{1,1}, E_{2,2}\}$ is a Riesz basis of $\mathcal{H}$. It is easy to see $\Psi = \{E_{2,1}, E_{1,2}\}$ is a tight frame and therefore a Bessel sequence in $\mathcal{H}$. Let $M \in \mathcal{H}$ be arbitrary. Then for any $m = \{m_1, m_2\} \subset \mathbb{C}$ we have*

$$M_{m,\Phi,\Psi} = m_1\langle M, E_{2,1}\rangle E_{1,1} + m_2\langle M, E_{1,2}\rangle E_{2,2} = 0.$$

Then $M_{m,\Phi,\Psi}$ is not invertible.

In the following, we characterize modular Riesz bases and obtain sufficient conditions for invertibility of Riesz multipliers in Hilbert C^*-modules.

Theorem 3. *Let $\Phi = \{\phi_j\}_{j\in J}$ be a Riesz basis in Hilbert C^*-module $\mathcal{H}$. Then the following statements are equivalent:*

1. *$\Phi = \{\phi_j\}_{j\in J}$ is a modular Riesz basis;*
2. *$\Phi = \{\phi_j\}_{j\in J}$ has a unique dual frame which is a modular Riesz basis;*
3. *the synthesis operator T_Φ is invertible;*
4. *for the analysis operator U_Φ, $Rang(U) = \ell^2(A)$;*
5. *if $\sum_{j\in J} a_j\phi_j = 0$ for some sequence $\{a_j\}_{j\in J} \in \ell^2(A)$, then $a_j = 0$ for each $j \in J$.*

In case the equivalent conditions are satisfied, Riesz multiplier $M_{m,\Phi,\Phi}$ is invertible, where the symbol $m = (m_j)$ is invertible and $|m_j|$ has a lower positive bound for each $j \in J$.

Proof. (1) $\Rightarrow$ (2) By definition of modular Riesz basis (Definition 2) there exists an invertible operator $T \in End^*_A(\ell^2(A), \mathcal{H})$ such that $Te_j = f_j$ for each $j \in J$. For every $f \in \mathcal{H}$, $T^{-1}(f) \in \ell^2(A)$, so

$$T^{-1}(f) = \sum_{j\in J}\langle T^{-1}f, e_j\rangle e_j = \sum_{j\in J}\langle f, (T^{-1})^* e_j\rangle e_j.$$

Therefore

$$\begin{aligned} f = T(T^{-1}f) &= U(\sum_{j\in J}\langle f, (T^{-1})^* e_j\rangle e_j) \\ &= \sum_{j\in J}\langle f, (T^{-1})^* e_j\rangle T(e_j) = \sum_{j\in J}\langle f, (T^{-1})^* e_j\rangle f_j. \end{aligned}$$

Now $(T^{-1})^* : \ell^2(A) \to \mathcal{H}$ is adjointable and invertible. The sequence $\{g_j = (T^{-1})^*(e_j)\}_{j\in J}$ is a modular Riesz basis for $\mathcal{H}$ and for every $f \in \mathcal{H}$, so we have

$$f = \sum_{j\in J}\langle f, g_j\rangle f_j.$$

Therefore $\{g_j\}_{j\in J}$ is a dual frame of $\{f_j\}_{j\in J}$.

Let $\{h_j\}_{j\in J}$ be any dual frame of $\{f_j\}_{j\in J}$. Then $f = \sum_{j\in J}\langle f, h_j\rangle f_j$ for all $f \in \mathcal{H}$. Hence

$$\sum_{j\in J}\langle f, g_j\rangle f_j - \sum_{j\in J}\langle f, h_j\rangle f_j = 0,$$

i.e. $\sum_{j\in J}\langle f, g_j - h_j\rangle f_j = 0$ and so, $g_j = h_j$ for all $j \in J$.

(2) ⇒ (3) Since Φ is a Riesz basis, its synthesis operator T_Φ is well defined and surjective [16]. It is enough to show that the synthesis operator T_Φ is injective or equivalently, $KerT_\Phi = \{0\}$. Assume on the contrary that $KerT_\Phi \neq \{0\}$. Let P_Φ be the orthogonal projection from $\ell^2(A)$ onto $KerT_\Phi$ and U an adjointable operator from $KerT_\Phi$ to $\mathcal{H}$. Now, let $\{\tilde{\phi_j}\}_{j\in J}$ be the canonical dual of $\{\tilde{\phi_j}\}_{j\in J}$ and set $\pi_j = \tilde{\phi_j} + UP_\Phi e_j$.

For any $f \in \mathcal{H}$,

$$\sum_{j\in J}\langle f, UP_\Phi e_j\rangle \phi_j = T_\Phi \sum_{j\in J}\langle P_\Phi U^* f, e_j\rangle e_j = T_\Phi P_\Phi U^* f = 0.$$

This yields $f = \sum_{j\in J}\langle f, \pi_j\rangle \phi_j$ for all $f \in \mathcal{H}$. Therefore $\{\phi_j\}_{j\in J}$ is a dual frame of $\{\phi_j\}_{j\in J}$ and is different from $\{\tilde{\phi_j}\}_{j\in J}$, which contradicts the uniqueness of the dual frame of $\{\phi_j\}_{j\in J}$.

(3) ⇔ (4) By Theorem 15.3.8 in [33], we have

$$\ell^2(A) = RangU_\Phi \oplus KerT_\Phi.$$

(3) ⇔ (5) It is obvious.

(3) ⇒ (1) As a corollary of Theorem 4.3 in [19].

In case the equivalent conditions are satisfied, with regard to $M_{m,\Phi,\Phi} = T_\Phi D_m U_\Phi$ $U_\Phi^{-1} = T_{\widetilde{\Phi}}$ and $T_\Phi^{-1} = U_{\widetilde{\Phi}}$, therefore

$$M_{m,\Phi,\Phi}^{-1} = U_\Phi^{-1} D_m T_\Phi^{-1} = T_{\widetilde{\Phi}} D_{m^{-1}} U_{\widetilde{\Phi}} = M_{m^{-1},\widetilde{\Phi},\widetilde{\Phi}}.$$

◀

Remark 1. *Suppose Φ and Ψ are Riesz bases in Hilbert C^*-module $\mathcal{H}$. In case the equivalent conditions in Theorem 3 are satisfied for Φ and Ψ, then by the proof of the previous theorem Riesz multiplier $M_{m,\Phi,\Psi}$ is invertible and $M_{m,\Phi,\Psi}^{-1} = M_{m^{-1},\widetilde{\Psi},\widetilde{\Phi}}$.*

4. Necessary conditions for invertibility of multipliers in Hilbert C^*-module

In this section, we obtain some necessary conditions for invertibility of multipliers in Hilbert C^*-modules and show that the inverse of any invertible multiplier operator in Hilbert C^*-module is a multiplier operator.

In the next theorem, we show that in the Hilbert A-module $\mathcal{H}$ for a given frame, the union of all coefficients of its dual frame is dense in $\ell^2(A)$. Also a frame is uniquely determined by the set of its dual frames. This is a generalization of Theorem 1.2 in [5].

Theorem 4. *Let Φ be a frame for Hilbert A-module $\mathcal{H}$. Then the following statements hold.*

1. *The closure of the union of all sets $Rang(U_{\Phi^d})$, where Φ^d runs through all dual frames of Φ is $\ell^2(A)$, i.e.*

$$\overline{\bigcup_{\Phi^d} Rang(U_{\Phi^d})} = \ell^2(A).$$

2. *Let Ψ be a frame for Hilbert A-module $\mathcal{H}$. If every dual frame Φ^d of Φ is a dual frame of Ψ, then $\Psi = \Phi$.*

Proof.

1. Let the sequence $a = (a_j) \in \ell^2(A)$ and $a \perp Rang(U_{\Phi^d})$ for every dual frame Φ^d of Φ. Then

$$T_{\Phi^d} a = \sum a_j \phi_j^d = 0, \tag{2}$$

for every dual frame Φ^d of Φ. Then by Theorem 1, the dual frames of Φ are precisely the sequences

$$\left(\tilde{\phi}_j + h_j - \sum_{n=1}^{\infty} \langle \tilde{\phi}_j, \phi_n \rangle h_n \right)_{j=1}^{\infty},$$

where $(h_j)_{j=1}^{\infty}$ is a Bessel sequence in Hilbert A-module $\mathcal{H}$. Therefore

$$\sum_{j=1}^{\infty} a_j (\tilde{\phi}_j + h_j - \sum_{n=1}^{\infty} \langle \tilde{\phi}_j, \phi_n \rangle h_n) = 0$$

for every Bessel sequence $(h_j)_{j=1}^{\infty}$ in Hilbert A-module $\mathcal{H}$. By (2), we have $T_{\tilde{\phi}}a = 0$, which implies that

$$\sum_{j=1}^{\infty} a_j(h_j - \sum_{n=1}^{\infty} \langle \tilde{\phi}_j, \phi_n \rangle h_n) = 0 \tag{3}$$

for every Bessel sequence $(h_j)_{j=1}^{\infty}$ in Hilbert A-module $\mathcal{H}$. Using (3) with the Bessel sequence $(h_j)_{j=1}^{\infty} = (e_1, 0, 0, ...)$, we obtain

$$a_1 e_1 - \sum_{j=1}^{\infty} a_j \langle \tilde{\phi}_j, \phi_1 \rangle e_1 = 0.$$

By $(2)\sum_{j=1}^{\infty} a_j \tilde{\phi}_j = 0$, it follows that $a_1 = 0$. Similarly, using (3) for the Bessel sequence $(h_n)_{n=1}^{\infty} = (0, ..., 0, e_n, 0, ...)$, where e^n is at the n'th position, we obtain $a_n = 0$ for every $n \geq 2$. Therefore, $a = (0)$, which completes the proof. ◀

2. Assume that all dual frames Φ^d of Φ are dual frame of Ψ. Then $T_\Phi U_{\Phi^d} = Id_{\mathcal{H}} = T_\Psi U_{\Phi^d}$, which by (1) implies that $T_\Phi = T_\Psi$ and hence $\Phi = \Psi$.

Next theorem says that in Hilbert C^*-module any invertible frame multiplier with an inverse symbol can always be represented as a multiplier with the inverse symbol and dual frames of the given ones, where one of these dual frames is uniquely determined and the other one can be arbitrarily chosen. This is a generalization of Theorem 1.1 of [5].

Theorem 5. *Let Φ and Ψ be frames for Hilbert A-module $\mathcal{H}$, the symbol $m = (m_j) \in \ell^\infty(Z(A))$ be invertible and $|m_j|$ have a lower positive bound for each $j \in J$. Assume that $M_{m,\Phi,\Psi}$ is invertible. Then there exists a unique dual frame $\Phi^+(\Psi^+)$ of $\Phi(\Psi)$ such that for any dual frame $\Psi^d(\Phi^d)$ of $\Psi(\Phi)$*

$$M_{m,\Phi,\Psi}^{-1} = M_{m^{-1},\Psi^d,\Phi^+} \quad \textit{and} \quad (M_{m,\Phi,\Psi}^{-1} = M_{m^{-1},\Psi^+,\Phi^d}).$$

Proof. Denote $M := M_{m,\Phi,\Psi}$ and $\Psi^+ = (M^{-1}(m_j\phi_j))_{j=1}^{\infty}$. First observe that Ψ^+ is a dual frame of Ψ. Therefore, $M^{-1}T_\Phi\delta_j = T_{\Psi^+}D_{m^{-1}}\delta_j$, $j \in \mathbb{N}$. Now the boundedness of this operator implies that $M^{-1}T_\Phi = T_{\Psi^+}D_{m^{-1}}$ on $\ell^2(A)$. Using any dual frame Φ^d of Φ we get $M^{-1} = T_{\Psi^+}D_{m^{-1}}U_{\Phi^d}$ on Hilbert A-module $\mathcal{H}$. Similarly, it follows that $\Phi^+ = ((M^{-1})^*(m_j^*\psi_j))_{j=1}^{\infty}$ is a dual frame of Φ and hence $(M^{-1})^*T_\psi = T_{\Phi^+}D_{m^{*-1}}$ on $\ell^2(A)$. Therefore

$$M^{-1} = T_{\Psi^d}D_{m^{-1}}U_{\Psi^+} = M_{m^{-1},\Psi^d,\Phi^+}.$$

◀

Corollary 1. *With the assumptions of Theorem 5, we have the additional properties:*

1. *If $F = (f_n)$ is a Bessel sequence in Hilbert C^*-module $\mathcal{H}$ such that $M^{-1}_{m,\Phi,\Psi} = M_{m^{-1},F,\Phi^+}$ (resp. $M^{-1}_{m,\Phi,\Psi} = M_{m^{-1},\Psi^+,F}$), then F must be a dual frame of Ψ (resp.Φ).*

2. *Ψ^+ is the only Bessel sequence in Hilbert C^*-module $\mathcal{H}$ which satisfies $M^{-1}_{m,\Phi,\Psi} = M_{m^{-1},\Psi^+,\Phi^d}$ for all dual frames Φ^d of Φ.*

3. *Φ^+ is the only Bessel sequence in Hilbert C^*-module $\mathcal{H}$ which satisfies $M^{-1}_{m,\Phi,\Psi} = M_{m^{-1},\Psi^d,\Phi^+}$ for all dual frames Ψ^d of Ψ.*

Proof.

1. Let $F = (f_n)$ be a Bessel sequence in Hilbert C^*-module $\mathcal{H}$ which satisfies $M^{-1}_{m,\Phi,\Psi} = M_{m^{-1},F,\Phi^+}$. Regarding the proof of Theorem 5, it follows that

$$T_\Psi U_F = M^* T_{\Phi^+} D_{m^{*-1}} U_F = M^*(M^{-1})^* = Id_{\mathcal{H}},$$

which implies that F is a dual frame of Ψ.
In a similar way, every Bessel sequence F in Hilbert C^*-module $\mathcal{H}$ which satisfies $M^{-1}_{m,\Phi,\Psi} = M_{m^{-1},\Psi^+,F}$ must be a dual frame of Φ.

2. Let $F = (f_n)$ be a Bessel sequence in Hilbert C^*-module $\mathcal{H}$ which satisfies $M_{m^{-1},F,\Phi^d} = M_{m^{-1},\Psi^+,\Phi^d}$ for all dual frames Φ^d of Φ, which by Theorem 4 (1) implies that $T_F D_{m^{-1}} = T_{\Psi^+} D_{m^{-1}}$. Since m is invertible (so, $D_{m^{-1}}$ is invertible on $\ell^2(A)$) it follows that $T_F = T_{\Psi^+}$ and hence $F = \Psi^+$.

3. The proof follows in a similar way as (2). ◀

The next proposition determines some classes of multipliers in Hilbert C^*-modules which are invertible and it is a generalization of Proposition 4.3 in [5].

Proposition 2. *Let Φ and Ψ be frames for Hilbert C^*-module $\mathcal{H}$ and $(m_j) = (a)$, where a is an invertible constant in A. Then the following assertions hold.*

1. *If $Rang(U_\Phi) \subseteq Rang(U_\Psi)$, then $M_{a^{-1},\widetilde{\Psi},\widetilde{\Phi}}$ is a bounded right inverse of $M_{(a),\Phi,\Psi}$.*

2. *If $Rang(U_\Psi) \subseteq Rang(U_\Phi)$, then $M_{a^{-1},\widetilde{\Psi},\widetilde{\Phi}}$ is a bounded left inverse of $M_{(a),\Phi,\Psi}$.*

3. *If* $Rang(U_\Phi) = Rang(U_\Psi)$*, then* $M_{(a),\Phi,\Psi}$ *is invertible and* $M^{-1}_{(a),\Phi,\Psi} = M_{a^{-1},\widetilde{\Psi},\widetilde{\Phi}}$.

4. *If* $Rang(U_\Phi) \subsetneq Rang(U_\Psi)$*, then* $M_{(a),\Phi,\Psi}$ *is not invertible.*

5. *If* $Rang(U_\Psi) \subsetneq Rang(U_\Phi)$*, then* $M_{(a),\Phi,\Psi}$ *is not invertible.*

Proof.

1. Assume that $Rang(U_\Phi) \subseteq Rang(U_\Psi)$. For every $h \in \mathcal{H}$, the element $U_\Phi S_\Phi^{-1} h$ can be written as $U_\Psi g^h$ for some $g^h \in \mathcal{H}$ and

$$M_{(a),\Phi,\Psi} M_{a^{-1},\widetilde{\Psi},\widetilde{\Phi}} h = T_\Phi U_\Psi S_\Psi^{-1} T_\Psi U_\Phi S_\Phi^{-1} h = T_\Phi U_\Psi g^h = h.$$

2. Can be proved in a similar way as (1).

3. Follows from (1) and (2).

4. Assume that $Rang(U_\Phi) \subseteq Rang(U_\Psi)$ with $Rang(U_\Phi) \neq Rang(U_\Psi)$. By (1), the operator $M_{a^{-1},\widetilde{\Psi},\widetilde{\Phi}}$ is a bounded right inverse of $M_{(a),\Phi,\Psi}$. We will prove that $M_{a^{-1},\widetilde{\Psi},\widetilde{\Phi}}$ is not a left inverse of $M_{(a),\Phi,\Psi}$, which will imply that $M_{(a),\Phi,\Psi}$ cannot be invertible. Consider an arbitrary element $g \in Rang(U_\Psi)$, $g \notin Rang(U_\Phi)$. Write $g = U_\Psi h$ for some $h \in \mathcal{H}$. Since $\ell^2(A) = Rang(U_\Phi) \bigoplus Ker(T_\Phi)$, we can also write $g = U_\Phi f + d$ for some $f \in \mathcal{H}$ and some $d \in Ker(T_\Phi)$, $d \neq 0$. Since $d = g - U_\Phi f \in Rang(U_\Psi)$, it follows that $d \notin Ker T_\Psi$, which implies that $S_\Psi^{-1} T_\Psi d \neq 0$. Then

$$\begin{aligned} M_{a^{-1},\widetilde{\Psi},\widetilde{\Phi}} M_{(a),\Phi,\Psi} h =& S_\Psi^{-1} T_\Psi U_\Phi S_\Phi^{-1} T_\Phi U_\Psi h \\ =& S_\Psi^{-1} T_\Psi U_\Phi S_\Phi^{-1} T_\Phi (U_\Phi f + d) \\ =& S_\Psi^{-1} T_\Psi (U_\Psi h - d) \\ =& h - S_\Psi^{-1} T_\Psi d \neq h, \end{aligned}$$

which implies that $M_{a^{-1},\widetilde{\Psi},\widetilde{\Phi}}$ is not a left inverse of $M_{(a),\Phi,\Psi}$.

5. In a similar way as in (4). ◄

The following theorem is a generalization of Theorem 4.6 in [5].

Theorem 6. *Let* Φ *and* Ψ *be frames for* $\mathcal{H}$*. The following statements are equivalent.*

1. *$M_{(a),\Phi,\Psi}$ is invertible and $M^{-1}_{(a),\Phi,\Psi} = M_{a^{-1},\widetilde{\Psi},\widetilde{\Phi}}$.*

2. *Φ and Ψ are equivalent frames.*

3. *$M_{(a),\Phi,\Psi}$ is invertible and the unique frame Ψ^+ in Theorem 5 is $\widetilde{\Psi}$.*

4. *$M_{(a),\Phi,\Psi}$ is invertible and the unique frame Φ^+ in Theorem 5 is $\widetilde{\Phi}$.*

5. *$M_{(a),\Phi,\Psi}$ is invertible and $M^{-1}_{(a),\Phi,\Psi} = M_{a^{-1},\widetilde{\Psi},\Phi^d}$ for all dual frames Φ^d of Φ.*

6. *$M_{(a),\Phi,\Psi}$ is invertible and $M^{-1}_{(a),\Phi,\Psi} = M_{a^{-1},\Psi^d,\widetilde{\Phi}}$ for all dual frames Ψ^d of Ψ.*

Proof. Without loss of generality, we may consider $a = 1$. For a closed subspace U of $\ell^2(A)$, we denote the orthogonal projection on U by P_U.
(1)⇒ (2) By (1), we have $T_{\widetilde{\Psi}}U_{\widetilde{\Phi}}T_\Phi U_\Psi = Id_{\mathcal{H}}$ and hence, $U_\Psi T_{\widetilde{\Psi}}U_{\widetilde{\Phi}}T_\Phi U_\Psi T_{\widetilde{\Psi}} = U_\Psi T_{\widetilde{\Psi}}$. Then $P_{Rang(U_\Psi)}P_{Rang(U_{\widetilde{\Phi}})}P_{Rang(U_\Psi)} = P_{Rang(U_\Psi)}$ which implies that $Rang(U_\Psi) \subseteq Rang(U_{\widetilde{\Phi}})$.
Similarly it follows that $Rang(U_{\widetilde{\Phi}}) \subseteq Rang(U_\Psi)$. Therefore $Rang(U_{\widetilde{\Phi}}) = Rang(U_\Psi)$. This implies that Φ and Ψ are equivalent.
(2)⇒ (3) and (4) Since $\Psi^+_n = M^{-1}(\Phi_n)$, $n \in \mathbb{N}$, it follows that Ψ^+ is equivalent to Ψ. Therefore, $\Psi^+ = \widetilde{\Psi}$. The validity of (4) follows in a similar way.
(3)⇒(5) and (4)⇒ (6) Use Theorem 5.
(5)⇒(1) and (6)⇒ (1) Clear. ◄

Proposition 3. *Let Φ and Ψ be frames for Hilbert C^*-module $\mathcal{H}$ and let the symbol $m = (m_j)$ be invertible and $|m_j|$ have a lower positive bound for each $j \in J$. Assume that $M_{m,\Phi,\Psi}$ is invertible. If Ψ is equivalent to $(m_j\Phi_j)$ or Φ is equivalent to $(m_j^*\Psi_j)$, then*

$$M^{-1}_{m,\Phi,\Psi} = M_{m^{-1},\widetilde{\Psi},\widetilde{\Phi}}.$$

The next proposition which generalizes Proposition 3.1 in [30] shows that if one of the sequences is Bessel, the invertibility of $M_{(1),\Phi,\Psi}$ implies that the other one must satisfy the lower frame condition.

Proposition 4. *Let $M_{m,\Phi,\Psi}$ be invertible on Hilbert C^*-module $\mathcal{H}$.*

1. *If Ψ (resp. Φ) is a Bessel sequence for $\mathcal{H}$ with bound B, then $m\Phi$ (resp. $m\Psi$) satisfies the lower frame condition for $\mathcal{H}$ with bound $\frac{1}{B\|M_{m,\Phi,\Psi}^{-1}\|^2}$.*

2. *If Ψ (resp. Φ) and $m\Phi$ (resp. $m\Psi$) are Bessel sequences for $\mathcal{H}$, then they are frames for $\mathcal{H}$.*

3. *If Ψ (resp. Φ) is a Bessel sequence for $\mathcal{H}$ and $m \in \ell^\infty(A)$, then Φ (resp. Ψ) satisfies the lower frame condition for $\mathcal{H}$.*

4. *If Φ and Ψ are Bessel sequences for $\mathcal{H}$ and $m \in \ell^\infty(A)$, then Ψ, Φ, $m\Phi$ and $m\Psi$ are frames for $\mathcal{H}$.*

Proof. Denote $M := M_{m,\Phi,\Psi}$.
First step: For $m = 1$. Assume that Ψ is a Bessel sequence for $\mathcal{H}$ with bound B_Ψ. For those $g \in \mathcal{H}$, for which $\sum |\langle g, \Phi_j\rangle|^2 = \infty$ or $g = 0$, clearly the lower frame condition holds. Now let $g \in \mathcal{H}$ be such that $\sum |\langle g, \Phi_j\rangle|^2 < \infty$ or $g \neq 0$. For every $f \in \mathcal{H}$

$$|\langle Mf, g\rangle| \leq \sqrt{B_\Psi}\|f\| \left(\sum |\langle \Phi_j, g\rangle|^2\right)^{\frac{1}{2}}.$$

For $f = M^{-1}g$, it follows that

$$\|g\| \leq \sqrt{B_\Psi}\|f\| \left(\sum |\langle \Phi_j, g\rangle|^2\right)^{\frac{1}{2}}.$$

Therefore, Φ satisfies the lower frame condition with bound $\frac{1}{B\|M^{-1}\|^2}$.
The case where Φ is a Bessel sequence can be proved in a similar way.
Second Step: For general m. Apply the first step to the multiplier $M_{(1),m\Phi,\Psi}$ (resp. $M_{(1),\Phi,m^*\Psi}$).
(2)-(4) follow now easily. ◄

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