Azerbaijan Journal of Mathematics
V. 10, No 1, 2020, January
ISSN 2218-6816

The A-integral and Restricted Complex Riesz Transform

R.A. Aliev*, Kh.I. Nebiyeva

Abstract. In this paper, we prove that the restricted complex Riesz transform of a Lebesgue integrable function is A-integrable and we obtain an analogue of Riesz's equality.

Key Words and Phrases: complex Riesz transform, A-integral, Riesz's equality, covering theorem.

2010 Mathematics Subject Classifications: 44A15, 26A39, 30C62, 42B20

---

1. Introduction

For every $k \in Z$, $k \neq 0$, the complex Riesz transform of a function $f \in L_p(C)$, $1 \leq p < \infty$, is defined as the following singular integral (see [7]):

$$(R^k f)(z) = \frac{|k|}{2\pi i^{|k|}} \lim_{\varepsilon \to 0} \int_{\{w \in C : |z-w| > \varepsilon\}} \frac{(\overline{z} - \overline{w})^k}{|z-w|^{k+2}} f(w) dm(w),$$

whose multiplier is

$$\mathbf{m}_k(\tau) = \left(\frac{\overline{\tau}}{|\tau|} \right)^k, \tau \neq 0.$$

Of course, for $k = 0$ we set R^0 as the identity operator, $(R^0 f)(z) = f(z)$. Note that in the case $k = 2$ we get the Ahlfors–Beurling transform.

---

*Corresponding author.

Let Ω be a bounded domain in the complex plane and $f \in L_1(\Omega)$. In the present paper we consider some modification of R^k. Namely, the restricted complex Riesz transform R^k_Ω is defined as

$$(R^k_\Omega f)(z) = R^k(\chi_\Omega f)(z) = \frac{|k|}{2\pi i^{|k|}} \lim_{\varepsilon\to 0} \int_{\{w\in\Omega:\,|z-w|>\varepsilon\}} \frac{(\overline{z}-\overline{w})^k}{|z-w|^{k+2}} f(w)dm(w), z \in \Omega.$$

The complex Riesz transform is one of the important operators in complex analysis. It has been shown in [7, 16] that this transform plays an essential role in applications to the theory of quasiconformal mappings and to the elliptic partial differential equations with discontinuous coefficients.

It is known from the theory of singular integrals (see [14]) that the complex Riesz transform is a bounded operator in the space $L_p(\Omega)$, $1 < p < \infty$, that is, if $f \in L_p(\Omega)$, then $R^k_\Omega(f) \in L_p(\Omega)$ and

$$\|R^k_\Omega f\|_{L_p} \le C_p \|f\|_{L_p}. \tag{1}$$

In the case $f \in L_1(\Omega)$ only the weak inequality holds,

$$m\{z \in \Omega:\ |(R^k_\Omega f)(z)| > \lambda\} \le \frac{C_1}{\lambda}\|f\|_{L_1}, \tag{2}$$

where m stands for the Lebesgue measure, C_p, C_1 are constants independent of f. From inequalities (1), (2) it follows that the complex Riesz transform of the function $f \in L_1(\Omega)$ satisfies the condition

$$m\{z \in \Omega:\ |(R^k_\Omega f)(z)| > \lambda\} = o(\frac{1}{\lambda}), \lambda \to +\infty. \tag{3}$$

Indeed, if $f \in L_1(\Omega)$, then for every $\varepsilon > 0$ there exists $n \in N$ such that $\|f - [f]^n\|_{L_1} \le \frac{\varepsilon}{4C_1}$, where $[f(z)]^n = f(z)$ for $|f(z)| \le n$ and $[f(z)]^n = 0$ for $|f(z)| > n$. It follows from (2) that

$$m\{z \in \Omega:\ |R^k_\Omega(f - [f]^n)(z)| > \frac{\lambda}{2}\} \le \frac{2C_1}{\lambda} \cdot \|f - [f]^n\|_{L_1} \le \frac{\varepsilon}{2\lambda}. \tag{4}$$

Since the function $[f(z)]^n$ is bounded, $[f]^n \in L_p(\Omega)$ for every $p \ge 1$; whence $R^k_\Omega([f]^n) \in L_p(\Omega)$ for every $p > 1$. Therefore $R^k_\Omega([f]^n) \in L_1(\Omega)$. It follows that for sufficiently large values of $\lambda > 0$

$$\frac{\lambda}{2} m\{z \in \Omega:\ |(R^k_\Omega [f]^n)(z)| > \frac{\lambda}{2}\}$$

$$\le \int_{\{z\in\Omega:\ |(R^k_\Omega [f]^n)(z)|>\frac{\lambda}{2}} |(R^k_\Omega [f]^n)(z)| dm(z) < \frac{\varepsilon}{4}. \tag{5}$$

From (4) and (5) for sufficiently large values of $\lambda > 0$ we obtain

$$m\{z \in \Omega: \ |(R^k_\Omega f)(z)| > \lambda\}$$

$$\leq m\{z \in \Omega: \ |R^k_\Omega(f - [f]^n)(z)| > \frac{\lambda}{2}\} + m\{z \in \Omega: \ |(R^k_\Omega [f]^n)(z)| > \frac{\lambda}{2}\} < \frac{\varepsilon}{\lambda}.$$

This means that the condition (3) holds.

Note that the complex Riesz transform of a function $f \in L_1(\Omega)$ is not Lebesgue integrable. In the present paper, we prove that the complex Riesz transform of a function $f \in L_1(\Omega)$ is A-integrable on Ω and we obtain an analogue of Riesz's equality.

2. A-integral

For a measurable complex function $f(z)$ on domain Ω we set

$$[f(z)]_n = [f(z)]^n = f(z) \text{ for } |f(z)| \leq n$$

$$[f(z)]_n = n \cdot \operatorname{sgn} f(z), \ [f(z)]^n = 0 \text{ for } |f(z)| > n, \ n \in N,$$

where $\operatorname{sgn} w = \frac{w}{|w|}$ for $w \neq 0$ and $\operatorname{sgn} 0 = 0$.

In 1928, Titchmarsh [15] introduced the notions of Q- and Q'-integrals of a function measurable on Ω.

Definition 1. *If the finite limit* $\lim\limits_{n\to\infty} \int_\Omega [f(z)]_n dm(z)$ ($\lim\limits_{n\to\infty} \int_\Omega [f(z)]^n dm(z)$, *respectively) exists, then* f *is said to be* Q*-integrable (*Q'*-integrable, respectively) on* Ω*; that is,* $f \in Q(\Omega)$ ($f \in Q'(\Omega)$)*. The value of this limit is referred to as the* Q*-integral (*Q'*-integral) of this function and is denoted by* $(Q)\int_\Omega f(z)dm(z)$ ($(Q')\int_\Omega f(z)dm(z)$).

In the same paper, when studying properties of trigonometric series conjugate to Fourier series of Lebesgue integrable functions, Titchmarsh established that the Q-integrability leads to a series of natural results. A very uncomfortable fact impeding the application of Q-integrals and Q'-integrals when dealing with diverse problems of function theory is the absence of the additivity property; that is, the Q-integrability (Q'-integrability) of two functions does not imply the Q-integrability (Q'-integrability) of their sum. If one adds the condition

$$m\{\, z \in \Omega: \ |f(z)| > \lambda \,\} = o(\frac{1}{\lambda}), \lambda \to +\infty \tag{6}$$

to the definition of Q-integrability (Q'-integrability) of a function f, then the Q-integral and Q'-integral coincide ($Q(\Omega) = Q'(\Omega)$), and these integrals become additive.

Definition 2. *If $f \in Q'(\Omega)$ (or $f \in Q(\Omega)$) and condition* (6) *holds, then f is said to be A-integrable on Ω, $f \in A(\Omega)$, and the limit $\lim_{n\to\infty} \int_\Omega [f(z)]_n dm(z)$ (or the limit $\lim_{n\to\infty} \int_\Omega [f(z)]^n dm(z)$) is denoted in this case by $(A)\int_\Omega f(z)dm(z)$.*

The properties of Q- and Q'-integrals were investigated in [2, 9, 10, 15, 17]; for the applications of A-, Q- and Q'-integrals in the theory of functions of real and complex variables we refer the reader to [1–6, 12, 13, 17, 18].

3. A-integrability and Riesz's equality for the complex Riesz transform of Lebesgue integrable functions

From the properties of singular integrals it follows that (see [14]) if $f \in L_p(\Omega)$, $p > 1$ and $g \in L_q(\Omega)$, $q > 1$, $\frac{1}{p} + \frac{1}{q} = 1$, then

$$\int_\Omega g(z)(R_\Omega^k f)(z)dm(z)$$

$$= \frac{|k|}{2\pi i^{|k|}} \lim_{\varepsilon\to 0} \iint_{\{w,z\in\Omega\,:\,|z-w|>\varepsilon\}} \frac{(\overline{z}-\overline{w})^k}{|z-w|^{k+2}} f(w)g(z)dm(w)dm(z)$$

$$= (-1)^k \int_\Omega f(z)(R_\Omega^k g)(z)dm(z). \tag{7}$$

In this section we prove that the complex Riesz transform of a function $f \in L_1(\Omega)$ is A-integrable on Ω and put forward an analogue of (7).

Theorem 1. *Let $f \in L_1(\Omega)$ and $g(z)$ be a bounded function on Ω with bounded $(R_\Omega^k g)(z)$ on Ω. Then the function $g(z) \cdot (R_\Omega^k f)(z)$ is A-integrable on Ω and*

$$(A)\int_\Omega g(z)(R_\Omega^k f)(z)dm(z) = (-1)^k \int_\Omega f(z)(R_\Omega^k g)(z)dm(z). \tag{8}$$

Note that in the case $k = 2$ (that is, in the case of Ahlfors–Beurling transform) theorem 3.1 was proved in [4].

Proof. Since the A-integral satisfies the additivity property, it can be assumed that the function f is real, $f(z) \geq 0$ for any $z \in \Omega$, and

$$\max_{z\in\Omega}\{|g(z)|,\, |(R_\Omega^k g)(z)|\} \leq 1.$$

For $z \notin \Omega$ we assume that $f(z) = 0$.

Our proof will depend on a certain refinement of Besicovitch's method [8] for a direct proof of the existence of conjugate function (this method employs only the machinery of the theory of sets of points). This method was improved by Titchmarsh [15] and Ul'yanov [17] for the study of properties of the conjugate function. It is worth noting that Besicovitch–Titchmarsh–Ul'yanov's method is applicable only to functions of one real variable (because this method relies on some facts that are valid only in one-dimensional case). For example, it depends on the fact that any open set is a union of at most a countable number of intervals (to overcome this difficulty, we used Vitali's covering lemma). To make this method to work in the setting of functions of complex variable, we have slightly improved the construction, which, for simplicity of presentation, is divided into three steps.

Step 1. In this part we construct and study properties of the sets G_p, L_n, L_n', T_n and the functions $\Phi_n(z)$, $\Phi_n^*(z)$, which we shall use later.

Denote $\Phi_n(z) = f(z) - [f(z)]^n$. Then $\alpha_n = \int_\Omega \Phi_n(z)dm(z) \to 0$ as $n \to \infty$. Take $n \in N$ such that $\alpha_n < 1$. Let $E_n = \{z \in \Omega : \ f(z) > n\}$. For any $z \in E_n$ we set

$$r_z = \sup \left\{ r > 0 : \int_{B(z;r)} \Phi_n\left(w\right) dm\left(w\right) = \frac{1}{2}\pi r^2 \cdot n \right\}$$

if $\left\{ r > 0 : \int_{B(z;r)} \Phi_n\left(w\right) dm\left(w\right) = \frac{1}{2}\pi r^2 \cdot n \right\} \neq \emptyset$, and define $r_z = 0$ otherwise, where $B\left(z;\, r\right)$ is an open ball with center z and radius r. Note that if $z \in E_n$ is a Lebesgue point of the function $\Phi_n\left(z\right)$, then $r_z > 0$ and, therefore, the set $E_n \backslash E_n'$ has a zero measure, where $E_n' = \{z \in E_n : \ \ r_z > 0\}$.

Consider the system of sets $\{B(z;\, r_z)\}_{z \in E_n'}$. It follows from the Vitali's covering lemma (see [11]) that there exists at most a countable set of points $z_j \in E_n'$, $j \in I \subset N$ such that the balls $B(z_j;\, r_{z_j})$, $j \in I$ are pairwise disjoint and

$$\bigcup_{z \in E_n'} B(z; r_z) \subset \bigcup_{j \in I} B(z_j;\, 5r_{z_j}).$$

Denote (see [6])

$$G_1 = B(z_1; 5r_{z_1}) \backslash \bigcup_{j>1} B(z_j; r_{z_j}),$$

$$G_p = B(z_p; 5r_{z_p}) \backslash [\bigcup_{j=1}^{p-1} G_j \bigcup (\bigcup_{j>p} B(z_j; r_{z_j}))], \qquad p \geq 2, \quad p \in I.$$

Then the measurable sets G_p, $p \in I$ are pairwise disjoint, and moreover,

$$B(z_p; r_{z_p}) \subset G_p \subset B(z_p; 5r_{z_p}), p \in I,$$

$$E'_n \subset \bigcup_{z\in E'_n} B(z; r_z) \subset \bigcup_{p\in I} G_p = \bigcup_{p\in I} B(z_p; 5r_{z_p}).$$

Denote $\Phi_n^*(z) = \frac{1}{m(G_p)} \int_{G_p} \Phi_n(w)dm(w)$ for $z \in G_p$, $p \in I$ and $\Phi_n^*(z) = 0$ for $z \in C\backslash \bigcup_{p\in I} G_p$. Then for any $p \in I$ we have

$$\int_{G_p} \Phi_n(z)dm(z) = \int_{G_p} \Phi_n^*(z)dm(z). \tag{9}$$

Note that (see [4]), for any $z \in G_p$, $p \in I$,

$$0 \le \Phi_n^*(z) \le \frac{25n}{2}.$$

Setting $L_n = \bigcup_{p\in I} G_p$, $L'_n = \bigcup_{p\in I} B\left(z_p;\, 10r_{z_p}\right)$, we have (see [6])

$$m(L_n) \le \frac{50\alpha_n}{n}, m(L'_n) \le \frac{200\alpha_n}{n}. \tag{10}$$

Let $T_n = \Omega\backslash L'_n$. We first prove the inequality

$$\int_{T_n} |R_\Omega^k(\Phi_n - \Phi_n^*)(z)|dm(z) < d_k \cdot \alpha_n, \tag{11}$$

where $d_k = 4000 \cdot 3^{|k|} \cdot |k|$. Denote $h_n(z) = R_\Omega^k(\Phi_n - \Phi_n^*)(z)$. For any $z \in T_n$ we have

$$|h_n(z)| = \frac{|k|}{2\pi} \left| \int_\Omega \frac{(\overline{z} - \overline{w})^k}{|z-w|^{k+2}} \cdot [\Phi_n(w) - \Phi_n^*(w)]dm(w) \right|$$

$$= \frac{|k|}{2\pi} \left| \sum_{p\in I} \int_{G_p} \frac{(\overline{z} - \overline{w})^k}{|z-w|^{k+2}} \cdot [\Phi_n(w) - \Phi_n^*(w)]dm(w) \right|$$

$$\le \frac{|k|}{2\pi} \sum_{p\in I} \left| \int_{G_p} \frac{(\overline{z} - \overline{w})^k}{|z-w|^{k+2}} \Phi_n(w)dm(w) - \int_{G_p} \frac{(\overline{z} - \overline{w})^k}{|z-w|^{k+2}} \Phi_n^*(w)dm(w) \right|. \tag{12}$$

It follows from the integral mean value theorem that for any $p \in I$ there are points $w_{p,i} \in B(z_p; 5r_{z_p}))$, $i = 1, 2, 3, 4$, such that

$$\int_{G_p} \frac{(\overline{z} - \overline{w})^k}{|z-w|^{k+2}} \Phi_n(w)dm(w)$$

$$= \left[Re\frac{(\overline{z}-\overline{w_{p,1}})^k}{|z-w_{p,1}|^{k+2}} + iIm\frac{(\overline{z}-\overline{w_{p,2}})^k}{|z-w_{p,2}|^{k+2}}\right]\cdot\int_{G_p}\Phi_n(w)dm(w),$$

$$\int_{G_p}\frac{(\overline{z}-\overline{w})^k}{|z-w|^{k+2}}\Phi_n^*(w)dm(w)$$

$$= \left[Re\frac{(\overline{z}-\overline{w_{p,3}})^k}{|z-w_{p,3}|^{k+2}} + iIm\frac{(\overline{z}-\overline{w_{p,4}})^k}{|z-w_{p,4}|^{k+2}}\right]\cdot\int_{G_p}\Phi_n^*(w)dm(w),$$

Then, using (9) and (12),

$$|h_n(z)| \le \frac{|k|}{2\pi}\sum_{p\in I}\sum_{i=1}^{2}\left|\frac{(\overline{z}-\overline{w_{p,i}})^k}{|z-w_{p,i}|^{k+2}} - \frac{(\overline{z}-\overline{w_{p,i+2}})^k}{|z-w_{p,i+2}|^{k+2}}\right|\cdot\int_{G_p}\Phi_n(w)dm(w). \quad (13)$$

For any w, $v\in B(z_p;5r_{z_p})$ and $z\in T_n$ the inequality

$$\left|\frac{(\overline{z}-\overline{w})^k}{|z-w|^{k+2}} - \frac{(\overline{z}-\overline{v})^k}{|z-v|^{k+2}}\right| \le \frac{400\cdot 3^{|k|}\cdot r_{z_p}}{|z-z_p|^3}$$

holds. Indeed, if $k>0$, then

$$\left|\frac{(\overline{z}-\overline{w})^k}{|z-w|^{k+2}} - \frac{(\overline{z}-\overline{v})^k}{|z-v|^{k+2}}\right|$$

$$\le \left|\frac{(\overline{z}-\overline{w})^k}{|z-w|^{k+2}} - \frac{(\overline{z}-\overline{w})^k}{|z-v|^{k+2}}\right| + \left|\frac{(\overline{z}-\overline{w})^k}{|z-v|^{k+2}} - \frac{(\overline{z}-\overline{v})^k}{|z-v|^{k+2}}\right|$$

$$= \frac{\left||z-v|^{k+2}-|z-w|^{k+2}\right|}{|z-w|^2\cdot|z-v|^{k+2}} + \frac{\left|(\overline{z}-\overline{w})^k-(\overline{z}-\overline{v})^k\right|}{|z-v|^{k+2}}$$

$$\le \frac{|v-w|\cdot\sum_{l=0}^{k+1}|z-v|^l|z-w|^{k+1-l}}{|z-w|^2\cdot|z-v|^{k+2}} + \frac{|v-w|\cdot\sum_{l=0}^{k-1}|z-v|^l|z-w|^{k-1-l}}{|z-v|^{k+2}}$$

$$= \frac{|v-w|}{|z-w|^2\cdot|z-v|}\cdot\sum_{l=0}^{k+1}\left|\frac{z-w}{z-v}\right|^{k+1-l} + \frac{|v-w|}{|z-v|^3}\cdot\sum_{l=0}^{k-1}\left|\frac{z-w}{z-v}\right|^{k-1-l}$$

$$\le \frac{80r_{z_p}}{|z-z_p|^3}\cdot\left(\sum_{l=0}^{k+1}3^{k+1-l}+\sum_{l=0}^{k-1}3^{k-1-l}\right) \le \frac{400\cdot 3^k\cdot r_{z_p}}{|z-z_p|^3},$$

if $k<0$, then

$$\left|\frac{(\overline{z}-\overline{w})^k}{|z-w|^{k+2}} - \frac{(\overline{z}-\overline{v})^k}{|z-v|^{k+2}}\right| = \left|\frac{(z-w)^{|k|}}{|z-w|^{|k|+2}} - \frac{(z-v)^{|k|}}{|z-v|^{|k|+2}}\right| \le \frac{400\cdot 3^{|k|}\cdot r_{z_p}}{|z-z_p|^3}.$$

Therefore it follows from (13) that

$$|h_n(z)| \leq \frac{|k|}{2\pi} \sum_{p\in I} \frac{800 \cdot 3^{|k|} \cdot r_{z_p}}{|z-z_p|^3} \cdot \int_{G_p} \Phi_n(w)dm(w)$$

$$\leq \frac{|k|}{\pi} \sum_{p\in I} \frac{400 \cdot 3^{|k|} \cdot r_{z_p}}{|z-z_p|^3} \cdot \left(\frac{1}{2}\pi \cdot 25 r_{z_p}^2 \cdot n\right) = \sum_{p\in I} \frac{10000 \cdot 3^{|k|} \cdot |k| \cdot r_{z_p}^3 \cdot n}{|z-z_p|^3}.$$

This implies the inequality

$$\int_{T_n} |h_n(z)|dm(z) \leq 10000 \cdot 3^{|k|} \cdot |k| \cdot n \cdot \sum_{p\in I} r_{z_p}^3 \int_{T_n} \frac{dm(z)}{|z-z_p|^3}$$

$$\leq 10000 \cdot 3^{|k|} \cdot |k| \cdot n \cdot \sum_{p\in I} r_{z_p}^3 \int_{\{z:\, |z-z_p|\geq 10 r_{z_p}\}} \frac{dm(z)}{|z-z_p|^3}$$

$$= 10000 \cdot 3^{|k|} \cdot |k| \cdot n \cdot \sum_{p\in I} r_{z_p}^3 \cdot 2\pi \int_{10 r_{z_p}}^{+\infty} \frac{dr}{r^2} = 2000 \cdot 3^{|k|} \cdot |k| \cdot \pi \cdot n \cdot \sum_{p\in I} r_{z_p}^2$$

$$= 2000 \cdot 3^{|k|} \cdot |k| \cdot \pi \cdot n \cdot \frac{2\alpha_n}{\pi n} = 4000 \cdot 3^{|k|} \cdot |k| \cdot \alpha_n.$$

proving the inequality (11).

We represent the function $f(z)$ in the form

$$f(z) = [f(z)]^n + \Phi_n^*(z) + [\Phi_n - \Phi_n^*](z). \tag{14}$$

Step 2. In this part we prove the equality

$$\lim_{n\to\infty} \int_{T_n} g(z)(R_\Omega^k f)(z)dm(z) = (-1)^k \int_\Omega f(z)(R_\Omega^k g)(z)dm(z). \tag{15}$$

Consider the integral

$$\int_{T_n} g(z)(R_\Omega^k f)(z)dm(z)$$

$$= \int_{T_n} g(z)\{(R_\Omega^k [f]^n)(z) + (R_\Omega^k \Phi_n^*)(z) + R_\Omega^k(\Phi_n - \Phi_n^*)(z)\}dm(z)$$

$$= \int_{T_n} g(z)(R_\Omega^k [f]^n)(z)dm(z) + \int_{T_n} g(z)(R_\Omega^k \Phi_n^*)(z)dm(z)$$

$$+ \int_{T_n} g(z)R_\Omega^k(\Phi_n - \Phi_n^*)(z)dm(z) = S_1 + S_2 + S_3. \tag{16}$$

By (7), we have, for square integrable functions,

$$S_1 = \int_{T_n} g(z)(R^k_\Omega[f]^n)(z)dm(z)$$

$$= \int_\Omega g(z)(R^k_\Omega[f]^n)(z)dm(z) - \int_{L'_n} g(z)(R^k_\Omega[f]^n)(z)dm(z)$$

$$= (-1)^k \int_\Omega [f(z)]^n(R^k_\Omega g)(z)dm(z) - \int_{L'_n} g(z)(R^k_\Omega[f]^n)(z)dm(z) = S_1^{(1)} + S_1^{(2)}.$$

Since

$$|S_1^{(2)}| = \left|\int_{L'_n} g(z)(R^k_\Omega[f]^n)(z)dm(z)\right| \le \int_{L'_n} |(R^k_\Omega[f]^n)(z)|dm(z)$$

$$\le \left[m(L'_n)\cdot\int_\Omega (R^k_\Omega[f]^n)^2(z)dm(z)\right]^{1/2} \le C_2\left[m(L'_n)\cdot\int_\Omega ([f(z)]^n)^2dm(z)\right]^{1/2}$$

$$\le C_2\left[n\cdot m(L'_n)\cdot\int_\Omega f(z)dm(z)\right]^{1/2},$$

it follows from (10) that

$$\lim_{n\to\infty} S_1 = (-1)^k \lim_{n\to\infty}\int_\Omega [f(z)]^n(R^k_\Omega g)(z)dm(z)$$

$$= (-1)^k \int_\Omega f(z)(R^k_\Omega g)(z)dm(z). \tag{17}$$

For the integral S_2 we also have

$$S_2 = \int_{T_n} g(z)(R^k_\Omega \Phi^*_n)(z)dm(z)$$

$$= \int_\Omega g(z)(R^k_\Omega \Phi^*_n)(z)dm(z) - \int_{L'_n} g(z)(R^k_\Omega \Phi^*_n)(z)dm(z)$$

$$= (-1)^k \int_\Omega \Phi^*_n(z)(R^k_\Omega g)(z)dm(z) - \int_{L'_n} g(z)(R^k_\Omega \Phi^*_n)(z)dm(z) = S_2^{(1)} + S_2^{(2)}.$$

The following estimates are valid.

$$|S_2^{(1)}| = \left|\int_\Omega \Phi^*_n(z)(R^k_\Omega g)(z)dm(z)\right| \le \int_\Omega |\Phi^*_n(z)(R^k_\Omega g)(z)|dm(z)$$

$$\leq \int_{\Omega} \Phi_n^*(z)dm(z) = \int_{\Omega} \Phi_n(z)dm(z) = \alpha_n,$$

$$|S_2^{(2)}| = \left| \int_{L_n'} g(z)(R_\Omega^k \Phi_n^*)(z)dm(z) \right| \leq \int_{L_n'} |(R_\Omega^k \Phi_n^*)(z)|dm(z)$$

$$\leq \left[m(L_n') \cdot \int_{\Omega} (R_\Omega^k \Phi_n^*)^2(z)dm(z) \right]^{1/2} \leq C_2 \left[m(L_n') \cdot \int_{\Omega} (\Phi_n^*(z))^2 dm(z) \right]^{1/2}$$

$$\leq C_2 \left[\frac{25}{2} n \cdot m(L_n') \cdot \int_{\Omega} \Phi_n^*(z)dm(z) \right]^{1/2} = C_2 \left[\frac{25}{2} n \cdot m(L_n') \cdot \alpha_n \right]^{1/2}.$$

Then it follows from (10) that

$$\lim_{n\to\infty} S_2 = 0. \tag{18}$$

To estimate the integral S_3, we apply inequality (11). We have

$$|S_3| = \left| \int_{T_n} g(z) R_\Omega^k (\Phi_n - \Phi_n^*)(z)dm(z) \right| \leq \int_{T_n} |g(z) R_\Omega^k (\Phi_n - \Phi_n^*)(z)|dm(z)$$

$$\leq \int_{T_n} |R_\Omega^k (\Phi_n - \Phi_n^*)(z)|dm(z) < 4000 \cdot 3^{|k|} \cdot |k| \cdot \alpha_n.$$

This implies the equality

$$\lim_{n\to\infty} S_3 = 0. \tag{19}$$

Now (15) follows from equalities (16), (17), (18) and (19).

Step 3. In this part we prove the equality

$$(A) \int_{\Omega} g(z)(R_\Omega^k f)(z)dm(z) = \lim_{n\to\infty} \int_{T_n} g(z)(R_\Omega^k f)(z)dm(z). \tag{20}$$

Consider the difference of integrals

$$\int_{T_n} g(z)(R_\Omega^k f)(z)dm(z) - \int_{\Omega} [g(z)(R_\Omega^k f)(z)]^n dm(z)$$

$$= - \int_{L_n'} [g(z)(R_\Omega^k f)(z)]^n dm(z)$$

$$+ \int_{T_n} \{ g(z)(R_\Omega^k f)(z) - [g(z)(R_\Omega^k f)(z)]^n \} dm(z) = S^{(1)} + S^{(2)}. \tag{21}$$

From the inequality $|S^{(1)}| \le n \cdot m(L'_n)$ it follows that

$$\lim_{n\to\infty} S^{(1)} = 0. \tag{22}$$

Denote $\sigma_n = \{z \in \Omega : \ |g(z)(R^k_\Omega f)(z)| > n\}$.

Since $m\{z \in \Omega : \ |(R^k_\Omega f)(z)| > n\} = o\left(\frac{1}{n}\right)$, $n \to \infty$, we have $m(\sigma_n) = o\left(\frac{1}{n}\right)$, $n \to \infty$. Using (11) and (14), we obtain

$$|S^{(2)}| \le \int_{T_n \bigcap \sigma_n} |g(z)(R^k_\Omega f)(z)| dm(z) \le \int_{T_n \bigcap \sigma_n} |(R^k_\Omega f)(z)| dm(z)$$

$$\le \int_{\sigma_n} |(R^k_\Omega [f]^n)(z)| dm(z) + \int_{\sigma_n} |(R^k_\Omega \Phi^*_n)(z)| dm(z) + \int_{T_n} |R^k_\Omega(\Phi_n - \Phi^*_n)(z)| dm(z)$$

$$\le \left[m(\sigma_n) \int_\Omega (R^k_\Omega [f]^n)^2(z) dm(z)\right]^{1/2} + \left[m(\sigma_n) \int_\Omega (R^k_\Omega \Phi^*_n)^2(z) dm(z)\right]^{1/2} + d_k \cdot \alpha_n$$

$$\le C_2 \left[m(\sigma_n) \int_\Omega ([f(z)]^n)^2 dm(z)\right]^{1/2} + C_2 \left[m(\sigma_n) \int_\Omega (\Phi^*_n(z))^2 dm(z)\right]^{1/2} + d_k \cdot \alpha_n$$

$$\le C_2 \left[n m(\sigma_n) \int_\Omega f(z) dm(z)\right]^{1/2} + C_2 \left[\frac{25}{2} n m(\sigma_n) \int_\Omega \Phi^*_n(z) dm(z)\right]^{1/2} + d_k \cdot \alpha_n.$$

It follows that

$$\lim_{n\to\infty} S^{(2)} = 0. \tag{23}$$

Now equality (20) follows from equalities (21), (22) and (23).

From the equalities (15) and (20) we obtain (8). Theorem 1 is proved. ◀

References

[1] A.B. Aleksandrov, *A-integrability of the boundary values of harmonic functions*, Math. Notes, **30(1)**, 1981, 515-523.

[2] R.A. Aliev, *$N^\pm$–integrals and boundary values of Cauchy-type integrals of finite measures*, Sbornik: Mathematics, **205(7)**, 2014, 913-935.

[3] R.A. Aliev, *Riesz's equality for the Hilbert transform of the finite complex measures*, Azerbaijan Journal of Mathematics, **6(1)**, 2016, 126-135.

[4] R.A. Aliev, *Representability of Cauchy-type integrals of finite complex measures on the real axis in terms of their boundary values*, Complex Variables and Elliptic Eq., **62(4)**, 2017, 536-553.

[5] R.A. Aliev, A.F. Amrahova, *Properties of the Discrete Hilbert transform*, Complex Analysis and Operator Theory, **13(8)**, 2019, 3883-3897.

[6] R.A. Aliev, Kh.I. Nebiyeva, *The A-integral and restricted Ahlfors-Beurling transform*, Integral Transforms and Special Functions, **29(10)**, 2018, 820-830.

[7] K. Astala, T. Iwaniec, G.Martin, *Elliptic partial differential equations and quasiconformal mappings in the plane*, Princeton: University Press, 2009.

[8] A.S. Besicovitch, *On a general metric property of summable functions*, J. London Math. Soc., **2(1)**, 1926, 120-128.

[9] M.P. Efimova, *On the properties of the Q-integral*, Math. Notes, **90(3-4)**, 2011, 322-332.

[10] M.P. Efimova, *The sufficient condition for integrability of a generalized Q-integral and points of integrability*, Moscow Univ. Math. Bul., **70(4)**, 2015, 181-184.

[11] L.C. Evans, R.F. Gariepy, *Measure theory and fine properties of functions*, Boca Raton, CRC Press, 1992.

[12] T.S. Salimov, *The A-integral and boundary values of analytic functions*, Math. USSR-Sbornik, **64(1)**, 1989, 23-40.

[13] V.A. Skvortsov, *A-integrable martingale sequences and Walsh series*, Izvestia: Math., **65(3)**, 2001, 607-616.

[14] E.M. Stein, *Singular Integrals and Differentiability Properties of Functions*, Princeton, University Press, 1970.

[15] E.C. Titchmarsh, *On conjugate functions*, Proc. London Math. Soc., **29(2)**, 1928, 49-80.

[16] A. Tumanov, *Commutators of singular integrals, the Bergman projection, and boundary regularity of elliptic equations in the plane*, Math. Research Letters, **23(4)**, 2016, 1221-1246.

[17] P.L. Ul'yanov, *The A-integral and conjugate functions*, Uch. Zap. Mosk. Gos. Univ. Mathematics, **181(7)**, 1956, 139-157 (in Russian).

[18] P.L. Ul'yanov, *Integrals of Cauchy type*, Twelve Papers on Approximations and Integrals. Amer. Math. Soc. Trans., **44(2)**, 1965, 129-150.

Rashid A. Aliev
Baku State University, AZ1148, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, Az1141, Baku, Azerbaijan
E-mail: aliyevrashid@mail.ru

Khanim I. Nebiyeva
Khazar University, AZ1096, Baku, Azerbaijan
E-mail: xanim.nebiyeva@gmail.com

Received 14 November 2019
Accepted 03 December 2019