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Generalized Morrey Spaces over Unbounded Domains

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Abstract. We study generalized Morrey type spaces $M^p_\omega(\Omega, d)$ over unbounded domains. Our goal is to describe the main properties of these spaces and some functional subspaces defined as a closure of the L^∞ and C_0^∞ functions with respect to the norm in $M^p_\omega(\Omega, d)$.

Key Words and Phrases: generalized Morrey spaces, vanishing spaces, unbounded domain.

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1. Introduction

In his celebrated work [11] Morrey studied the regularity of the solutions of a kind of elliptic systems. He estimated the L^p-norm of the gradient Du of the solution in a ball via a power of the radius of the same ball. That estimate permitted him to obtain local Hölder regularity of u. This result gave rise to the introduction of new functional spaces named after him. The *classical Morrey spaces* have been formulated and studied in the 60's by Campanato, Peetre and Brudneii independently, using similar notations. Precisely, a function $f \in L^p_{\mathrm{loc}}(\mathbb{R}^n)$ belongs to the *Morrey space* $L^{p,\lambda}(\mathbb{R}^n)$ with $p \geq 1$ and $\lambda \in (0, n)$ if

$$\|f\|_{L^{p,\lambda}(\mathbb{R}^n)} = \sup_{B_r(x)} \left(\frac{1}{r^\lambda} \int_{B_r(x)} |f(y)|^p \, dy \right)^{\frac{1}{p}} < +\infty \tag{1}$$

and the supremum is taken over all balls in $\mathbb{R}^n$ (see [1, 2, 3]).

A natural question that arises is what happens if we consider f defined in some domain $\Omega \subset \mathbb{R}^n$ bounded or unbounded. In the first case it is enough to take

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the integral in (1) over the intersection $\Omega \cap B_r(x)$ and take the supremum over balls centered at $x \in \Omega$ with radius $r \in (0, \operatorname{diam} \Omega]$. The situation becomes a little bit different if we consider unbounded domain Ω. This case requires additional condition over the radius of the balls. Such condition is given by Transirico, Troisi and Vitolo in [18] where the authors study elliptic boundary value problems in unbounded domains. Precisely, they consider spaces $M^{p,\lambda}(\Omega, d)$ that consist of locally integrable functions $f \in L^p_{\mathrm{loc}}(\Omega)$ for which the following norm is finite

$$\|f\|_{M^{p,\lambda}(\Omega,d)} = \sup_{\substack{B_r(x)\\ r\in(0,d]\\ x\in\Omega}} \left(\frac{1}{r^\lambda} \int_{B_r(x)\cap\Omega} |f(y)|^p \, dy \right)^{\frac{1}{p}} < +\infty, \tag{2}$$

where $x \in \Omega$ and $d > 0$ is a fixed real number. The properties of these spaces are studied in [6, 18] where the authors show that for two different positive numbers d_1 and d_2 the spaces $M^{p,\lambda}(\Omega, d_1)$ and $M^{p,\lambda}(\Omega, d_2)$ are equivalent.

The first generalization of the classical Morrey spaces is made by Mizuhara [12] who takes a *weight* $\varphi(r), r > 0$, increasing positive measurable function, satisfying a doubling condition, instead of r^λ in (1). The new *generalized Morrey spaces* $L^{p,\varphi}(\mathbb{R}^n)$ have been deeply studied by Nakai (see, e.g. [13, 16, 17]) supposing that $\varphi(B_r(x)) \equiv \varphi(x, r) : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ satisfies the following conditions

$$\kappa_1 \le \frac{\varphi(B_s(y))}{\varphi(B_r(y))} \le \kappa_2 \quad \text{for all} \quad r \le s \le 2r,$$

$$\int_r^\infty \frac{\varphi(B_s(y))}{s^{n+1}} \, ds \le \kappa_3 \, \frac{\varphi(B_r(y))}{r^n} \, .$$

for any fixed $x \in \mathbb{R}^n$, any $r > 0$ and some constants $\kappa_1, \kappa_2, \kappa_3 > 0$.

The question that we are interested is if such a generalization is possible for Morrey type spaces defined over unbounded domain and what kind of conditions are necessary to impose on the weight function. The present work treats a kind of *generalized Morrey spaces* $M^p_\omega(\Omega, d)$ where the weight function ω satisfies *doubling and monotonicity condition* (see (W1) and (W2) in Section 2). Our goal is to give a description of these spaces through some of their subsets. Precisely, we fix our attention on three subspaces. The first one consists of functions for which the $M^p_\omega(\Omega, d)$ norm vanishes over shrinking balls, the second one is a closure of $L^\infty(\Omega)$ functions with respect to the norm in $M^p_\omega(\Omega, d)$, while the third one is a closure of the $C^\infty_0(\Omega)$ functions with respect to the same norm. In the last two cases we show decomposition of the functions from the corresponding spaces.

Our goal is twofold: to extend the results obtained in [6] for the Morrey spaces over unbounded domain $M^{p,\lambda}(\Omega, d)$ to spaces with some weight ω and to give basic tools for studying Dirichlet boundary problem for elliptic PDEs in unbounded domains as in [4, 5, 7, 9, 10, 15, 16, 17].

We use the following notations:

- Ω is an unbounded domain in $\mathbb{R}^n$, $B_r(x)$ is a ball in $\mathbb{R}^n$ and $\Omega(x,r) = \Omega \cap B_r(x)$ with $x \in \Omega, r > 0$;
- $\Sigma(\Omega)$ is the Lebesgue σ-algebra on A; for $E \in \Sigma(\Omega)$, we denote by χ_E the characteristic function of E and by $|E|$ the Lebesgue measure of E;
- $\mathcal{D}(E)$ is the restriction of the $C_0^\infty(\mathbb{R}^n)$ functions on E, that is

$$\mathcal{D}(E) = \left\{\zeta = \eta_{|_E} : \quad \eta \in C_0^\infty(\mathbb{R}^n),\ \operatorname{supp}\zeta = \operatorname{supp}\eta \cap E \subseteq E\right\};$$

For $p \in [1, +\infty)$ define

$$L^p_{\text{loc}}(E) = \{g : E \to \mathbb{R} :\ \zeta\, g \in L^p(E),\ \zeta \in \mathcal{D}(E)\}.$$

The paper is organized as follows: we start with the definition and main properties of the spaces $M^p_\omega(\Omega, d)$, in Section 3 we introduce the main subspaces of $M^p_\omega(\Omega, d)$ while Section 4 describes decompositions of the functions from the corresponding subspaces.

2. Spaces $M^p_\omega(\Omega, d)$, definition and main properties

We call *weight* a measurable function $\omega : \mathbb{R}^n \times \mathbb{R}_+ \to \mathbb{R}_+$ and for $B_r(x) \subset \mathbb{R}^n$ we write $\omega(x,r) = \omega(B_r(x))$. In what follows we suppose $p \in [1, +\infty)$ and $d > 0$.

Definition 1. *A function $f \in L^p_{\text{loc}}(\Omega)$ belongs to $M^p_\omega(\Omega, d)$ if*

$$\|f\|_{M^p_\omega(\Omega,d)} = \sup_{\substack{x\in\Omega \\ \tau\in(0,d]}} \left(\frac{1}{\omega(x,\tau)} \int_{\Omega(x,\tau)} |f(y)|^p\, dy\right)^{\frac{1}{p}} < +\infty, \tag{3}$$

where the supremum is taken over all balls $B_\tau(x) \subset \mathbb{R}^n$ centered at $x \in \Omega$.

Obviously $M^p_\omega(\Omega, d)$ is a Banach space with a norm defined by (3).

Let us note that $L^{p,\omega}(\mathbb{R}^n) \subset M^p_\omega(\mathbb{R}^n, d)$ and if Ω is bounded, then $L^{p,\omega}(\Omega) \equiv M^p_\omega(\Omega, d)$, where $L^{p,\omega}$ denotes the generalized Morrey space studied by Nakai with $\varphi \equiv \omega$. If $\omega(x,\tau) = \tau^\lambda$, with $0 < \lambda < n$, then the spaces $M^p_\omega(\Omega, d)$ become the Morrey spaces $M^{p,\lambda}(\Omega, d)$ considered in [6, 18].

We assume that the weight ω verifies a *doubling condition* with a positive constant C_ω independent of x, r, s

$$\frac{1}{C_\omega} \le \frac{\omega(x,s)}{\omega(x,r)} \le C_\omega\,, \qquad \forall\, x \in \mathbb{R}^n, \qquad r \le s \le 2r, \tag{W1}$$

and *monotonicity condition*

$$\omega(y,r) \le \omega(x,s) \qquad \forall\, x, y \in \mathbb{R}^n\,, \qquad B_r(y) \subseteq B_s(x)\,. \tag{W2}$$

Example 1. The following weights satisfy the conditions (W1) and (W2):

E_1) $\omega(x,r) = \left(\int_{B_r(x)} u(y)\,dy\right)^\alpha$, where $u \in A_p$, $p > 1$, is a Muckenhoupt weight, with $0 < \alpha \le 1/p < 1$ (see for example [13]);

E_2) $\omega(x,r) = \phi(r)$, where $\phi(r)$ is an increasing function such that $\frac{\phi(r)}{r}$ is decreasing;

E_3) $\omega(x,r) = \Phi(r)$, where $\Phi(r)$ is a Young function satisfying the so-called $\triangle_2$-condition (see [14, 2]).

The dependence of (3) on d would seem to be quite restrictive. It turns out instead that all the spaces $M^p_\omega(\Omega,d)$ on varying of $d \in \mathbb{R}_+$ are equivalent.

Theorem 1. *Let* ω *satisfy* (W1) *and* (W2), *and* $d_1, d_2 \in \mathbb{R}_+$. *Then* $f \in M^p_\omega(\Omega,d_1)$ *iff* $f \in M^p_\omega(\Omega,d_2)$ *and*

$$\|f\|_{M^p_\omega(\Omega,d_1)} \le \|f\|_{M^p_\omega(\Omega,d_2)} \le c\,\|f\|_{M^p_\omega(\Omega,d_1)}, \tag{4}$$

where $c > 0$ *depends on* n, p, C_ω, d_1, d_2.

Proof. Without loss of generality suppose that $d_1 \le d_2$ and fix $f \in M^p_\omega(\Omega,d_1)$.

$$\|f\|_{M^p_\omega(\Omega,d_1)} \le \|f\|_{M^p_\omega(\Omega,d_2)}\,.$$

In order to prove the second inequality in (4), we observe that

$$\begin{aligned}\|f\|_{M^p_\omega(\Omega,d_2)} &\le \|f\|_{M^p_\omega(\Omega,d_1)} + \sup_{\substack{x\in\Omega\\ \tau\in(d_1,d_2]}} \omega(x,\tau)^{-\frac{1}{p}}\,\|f\|_{L^p(\Omega(x,\tau))}\\ &\le \|f\|_{M^p_\omega(\Omega,d_1)} + \sup_{x\in\Omega} \omega(x,d_1)^{-\frac{1}{p}}\,\|f\|_{L^p(\Omega(x,d_2))}\,. \end{aligned} \tag{5}$$

Fix $x \in \Omega$ and take a cube $Q(x,2d_2)$ centered at x and with length of the edge $2d_2$. Let $k \in \mathbb{N}$ be such that

$$\frac{2d_2}{2^{k+1}} \le d_1 < \frac{2d_2}{2^k}\,. \tag{6}$$

Take a dyadic decomposition and choose points $x_1, \cdots, x_{2^{n(k+2)}} \in \mathbb{R}^n$ such that

$$B(x, d_2) \subset Q(x, 2d_2) = \bigcup_{i=1}^{2^{n(k+2)}} Q\big(x_i, \frac{2d_2}{2^{k+2}}\big) \subset \bigcup_{i=1}^{m} B\big(y_i, \frac{d_1}{2}\big)\,, \tag{7}$$

where $m \geq 2^{n(k+2)}$ and $y_i \in \Omega$ for any i. Hence

$$\Omega(x, d_2) \subset \bigcup_{i=1}^{m} \Omega\big(y_i, \frac{d_1}{2}\big)\,.$$

On the other hand, since $B(y_i, \frac{d_1}{2}) \subset B(x, 4d_2)$ for any i, from (W1), (W2) and (6) one gets

$$\omega\big(y_i, \frac{d_1}{2}\big) \leq \omega(x, 4d_2) \leq C_\omega^{k+2}\, \omega(x, d_1)\,, \tag{8}$$

independently of i. Thus, in view of (7), (8) and (W1), for any $x \in \Omega$ we have

$$\begin{aligned}
\left(\frac{1}{\omega(x, d_1)} \int_{\Omega(x,d_2)} |f(y)|^p\, dy\right)^{\frac{1}{p}} &\leq \left(\frac{1}{\omega(x, d_1)} \sum_{i=1}^{m} \int_{\Omega(y_i, \frac{d_1}{2})} |f(y)|^p\, dy\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^{m} \frac{C_\omega^{k+2}}{\omega\big(y_i, \frac{d_1}{2}\big)} \int_{\Omega(y_i, \frac{d_1}{2})} |f(y)|^p\, dy\right)^{\frac{1}{p}} \\
&\leq \left(\sum_{i=1}^{m} \frac{C_\omega^{k+3}}{\omega(y_i, d_1)} \int_{\Omega(y_i,d_1)} |f(y)|^p\, dy\right)^{\frac{1}{p}} \leq C\|f\|_{M_\omega^p(\Omega,d_1)}
\end{aligned} \tag{9}$$

with a positive constant C depending on n, p, C_ω, d_1, d_2. The second inequality in (4) easily follows from (5) and (9). ◄

Because of the equivalence of the norms, from now on, we take $d = 1$, writing $M_\omega^p(\Omega) = M_\omega^p(\Omega, 1)$. Assume in addition that the function ω verifies the assumption

$$\sup_{\substack{x\in\Omega \\ \tau\in(0,1]}} \frac{|\Omega(x, \tau)|}{\omega(x, \tau)} = D < +\infty\,, \tag{W3}$$

which is equivalent to

$$\|\chi_\Omega\|_{M_\omega^p(\Omega)} = D^{\frac{1}{p}} < +\infty\,.$$

Using Hölder's inequality and (W3), it is easy to prove that

$$M_\omega^p(\Omega) \subseteq M_\omega^q(\Omega) \qquad \forall\, 1 \leq q \leq p\,. \tag{10}$$

In addition, (W3) ensures the inclusion $L^\infty(\Omega) \subset M_\omega^p(\Omega)$. In fact, if $f \in L^\infty(\Omega)$, then

$$\|f\|_{M_\omega^p(\Omega)} \leq \|f\|_{L^\infty(\Omega)} \|\chi_\Omega\|_{M_\omega^p(\Omega)} \leq C\,. \tag{11}$$

3. The subspaces $VM^p_\omega(\Omega)$, $\widetilde{M}^p_\omega(\Omega)$ and $\overset{\circ}{M}{}^p_\omega(\Omega)$

In this section we study the properties and the structure of some subspaces of $M^p_\omega(\Omega)$ starting with functions with *vanishing norm.*

Definition 2. *The space* $VM^p_\omega(\Omega)$ *with* ω *satisfying* (W1), (W2) *and* (W3), *consists of all functions* $g \in M^p_\omega(\Omega)$ *such that*

$$\lim_{t\to 0} \|g\|_{M^p_\omega(\Omega,t)} = 0\,. \tag{12}$$

As in (10), the Hölder inequality and (W3) imply the inclusion

$$VM^p_\omega(\Omega) \subseteq VM^q_\omega(\Omega) \qquad \forall\, 1 \le q \le p\,. \tag{13}$$

Suppose that the norm of the characteristic function of Ω satisfies the following *vanishing condition*

$$\lim_{t\to 0} \|\chi_\Omega\|_{M^p_\omega(\Omega,t)} = 0 \tag{V}$$

which is equivalent to $\displaystyle\lim_{t\to 0} \sup_{\substack{x\in\Omega\\ \tau\in(0,t]}} \frac{|\Omega(x,\tau)|}{\omega(x,\tau)} = 0$. Then $L^\infty(\Omega) \subset VM^p_\omega(\Omega)$ as a direct consequence of (11). Moreover, the assumption (V) allows us to improve the inclusion (13) as it is shown in the following lemma.

Lemma 1. *Suppose that* ω *verifies* (W1), (W2), (W3) *and* (V). *Then*

$$M^p_\omega(\Omega) \subset VM^q_\omega(\Omega) \qquad \forall\, 1 \le q < p < +\infty\,. \tag{14}$$

Proof. From (10) and Hölder's inequality we easily get

$$\|g\|_{M^q_\omega(\Omega,t)} \le \|g\|_{M^p_\omega(\Omega)} \cdot \|\chi_\Omega\|^{\frac{p}{q}-1}_{M^p_\omega(\Omega,t)}\,.$$

Then condition (V) proves our claim. ◀

Example 2. E_1) Let $u \in A_p, p > 1$ and $u > 0$ a.e. in Ω. The weight

$$\omega(x,r) = \left(\int_{B_r(x)} u(y)\,dy\right)^\alpha \qquad \text{with} \qquad 0 < \alpha \le 1/p < 1$$

verifies (V). By the Lebesgue Differentiation Theorem

$$\lim_{t\to 0} \frac{1}{|B_t(x)|} \int_{B_t(x)} u(y)dy = u(x) \qquad \text{for a.a. } x \in \Omega.$$

$$\lim_{t\to 0}\sup_{\substack{x\in\Omega\\ \tau\in(0,t]}}\frac{|\Omega(x,\tau)|}{(\int_{B_\tau(x)}u(y)\,dy)^\alpha}\le C\lim_{t\to 0}\sup_{\substack{x\in\Omega\\ \tau\in(0,t]}}\frac{\tau^{n(1-\alpha)}}{(\frac{1}{|B_\tau(x)|}\int_{B_\tau(x)}u(y)\,dy)^\alpha}=0\,.$$

Let us note that since $\omega(x,\tau)>0$ for every ball $B_\tau(x)$, the weight ω verifies also (W3);

E_2) The weight $\omega(x,r)=\phi(r)$, where $\lim_{r\to 0}\phi(r)=0$ and $\lim_{r\to 0}\dfrac{\phi(r)}{r}=+\infty$, verifies (V) and (W3).

In what follows, we are going to give some properties of the spaces $VM^p_\omega(\Omega)$ which are similar to those of the classical vanishing Morrey spaces (see [8, Lemma 1.2] and [19, Proposition 3]).

We say that Ω is of (A)-*type* or *satisfies the condition* (A), if

$$\sup_{\substack{x\in\Omega\\ \tau\in(0,1]}}\frac{|B(x,\tau)|}{|\Omega(x,\tau)|}=A\ <+\infty\,. \tag{A}$$

It is easy to see that the condition (A) implies the *external cone condition*

$$|\Omega(x,\tau)|\ge\frac{1}{A}\,\tau^n\,,\qquad \forall\, x\in\Omega\,,\quad \forall\,\tau\in(0,1]\,. \tag{15}$$

Remark 1. We point out that if the domain Ω is bounded, then the condition (A) is equivalent to the well-known *Campanato type* condition. In the case of unbounded domain the radius τ could be arbitrary. Since the property of the boundary of Ω is a local property, we can add the restriction $\tau\in(0,1]$ without loss of generality.

Remark 2. Comparing (A) with (W3), it is easy to see that if $\omega(x,\tau)\equiv\tau^\lambda$, then we get $\lambda\in(0,n]$, while the condition (V) implies $\lambda<n$.

Let $\{J_h\}_{h\in\mathbb{N}}$ be a sequence of mollifiers, that is, $J\in C_0^\infty(\mathbb{R}^n)$, $\operatorname{supp} J\subset B(0,1)$, $0\le J(x)\le 1$, $\int_{\mathbb{R}^n}J(x)\,dx=1$ and $J_h(x)=h^n\,J(hx)$. Then the following approximation properties are valid.

Lemma 2. *Let* (A), (W1), (W2) *and* (W3) *hold. If* $g\in VM^p_\omega(\Omega)$ *with* $\operatorname{supp} g\Subset\Omega$, *then*

$$\lim_{y\to 0}\|g(\cdot-y)-g(\cdot)\|_{M^p_\omega(\Omega)}=0\,, \tag{16}$$

$$\lim_{h\to+\infty}\|J_h*g-g\|_{M^p_\omega(\Omega)}=0\,, \tag{17}$$

where $\{J_h\}_{h\in\mathbb{N}}$ *is a sequence of mollifiers in* $\mathbb{R}^n$.

Proof. Since $g \in VM^p_\omega(\Omega)$, for any $\epsilon > 0$ there exists $0 < t_\epsilon < 1$ such that

$$\|g\|_{M^p_\omega(\Omega,t_\epsilon)} < \frac{\epsilon}{4}\,. \tag{18}$$

Take $\delta(\epsilon)$ small and let $y \in B_{\delta(\epsilon)}(0) \subset B_1(0)$. By (15), (18) and (W3) we get

$$\begin{aligned}
&\|g(\cdot - y) - g(\cdot)\|_{M^p_\omega(\Omega)} \le \|g(\cdot - y) - g(\cdot)\|_{M^p_\omega(\Omega,t_\epsilon)} \\
&\quad + \sup_{\substack{x\in\Omega \\ \tau\in(t_\epsilon,1]}} \left(\frac{1}{\omega(x,\tau)} \int_{\Omega(x,\tau)} |g(z-y) - g(z)|^p\, dz \right)^{\frac{1}{p}} \\
&\quad < \frac{\epsilon}{2} + D^{\frac{1}{p}} \sup_{x\in\Omega} \left(\frac{1}{|\Omega(x,t_\epsilon)|} \int_{\Omega(x,1)} |g(z-y) - g(z)|^p\, dz \right)^{\frac{1}{p}} \\
&\quad < \frac{\epsilon}{2} + C \left(\frac{1}{t^n_\epsilon} \int_{\operatorname{supp} g + B(0,1)} |g(z-y) - g(z)|^p\, dz \right)^{\frac{1}{p}} < \epsilon
\end{aligned} \tag{19}$$

where in the last step we use the smallness of y and the continuity with respect to translation of the Lebesgue integral. The constant C depends on p, D, and A. In order to prove (17) we use the classical properties of the mollifiers. Let

$$I := \int_{\Omega(x,\tau)} |J_h * g(z) - g(z)|^p\, dz = \int_{\Omega(x,\tau)} \left| \int_{\mathbb{R}^n} J_h(z-y) \cdot (g(y) - g(z))\; dy \right|^p dz\,.$$

Since $\operatorname{supp} g$ is a compact in Ω, we have

$$I \le \int_{\Omega(x,\tau)} \int_{\mathbb{R}^n} J_h(z-y) \cdot |g(y) - g(z)|^p\; dydz\,. \tag{20}$$

Hence, by (20) and Fubini theorem, we get

$$\begin{aligned}
&\frac{1}{\omega(x,\tau)} \int_{\Omega(x,\tau)} |J_h * g(z) - g(z)|^p\, dz \\
&\quad \le \frac{1}{\omega(x,\tau)} \int_{\Omega(x,\tau)} \int_{\mathbb{R}^n} J_h(z-y) \cdot |g(y) - g(z)|^p\, dydz \\
&\quad \le \int_{\mathbb{R}^n} J_h(y) \left(\frac{1}{\omega(x,\tau)} \int_{\Omega(x,\tau)} |g(z-y) - g(z)|^p\, dz \right) dy \\
&\quad \le \int_{\mathbb{R}^n} J_h(y) \cdot \|g(\cdot - y) - g(\cdot)\|^p_{M^p_\omega(\Omega)}\, dy\,.
\end{aligned}$$

Then, since $\operatorname{supp} J \subset B(0,1)$, we have

$$\|J_h * g - g\|_{M^p_\omega(\Omega)} \le \int_{|y|\le \frac{1}{h}} J_h(y) \cdot \|g(\cdot - y) - g(\cdot)\|^p_{M^p_\omega(\Omega)}\, dy\,.$$

The last estimate, along with (16), gives (17). ◄

Definition 3. *Denote by* $\widetilde{M}^p_\omega(\Omega)$ *the class of functions* $g \in M^p_\omega(\Omega)$ ω *satisfying* (W1)*,* (W2) *and* (W3)*, such that*

$$\lim_{h\to+\infty} \Big(\sup_{\substack{E\in\Sigma(\Omega)\\ \|\chi_E\|_{M^p_\omega(\Omega)}\le \frac{1}{h}}} \|g\,\chi_E\|_{M^p_\omega(\Omega)} \Big) = 0\,.$$

The main feature of these spaces is given by the following lemma.

Lemma 3. *A function* $g \in M^p_\omega(\Omega)$ *belongs to* $\widetilde{M}^p_\omega(\Omega)$ *iff* g *is in the closure of* $L^\infty(\Omega)$ *in* $M^p_\omega(\Omega)$*.*

Proof. Because of (11) we have the inclusion $L^\infty(\Omega) \subset M^p_\omega(\Omega)$. Take a function g in the closure of $L^\infty(\Omega)$ w.r.t. the norm in $M^p_\omega(\Omega)$. Hence for each $\epsilon > 0$ there exists a function $g_\epsilon \in L^\infty(\Omega)$ such that

$$\|g - g_\epsilon\|_{M^p_\omega(\Omega)} < \frac{\epsilon}{2}\,. \tag{21}$$

Let us take $E \in \Sigma(\Omega)$ such that

$$\|\chi_E\|_{M^p_\omega(\Omega)} < \frac{1}{h_\epsilon} \qquad \text{with} \qquad h_\epsilon = \frac{2\,\|g_\epsilon\|_{L^\infty(\Omega)}}{\epsilon}\,. \tag{22}$$

Then from (21) and (22) it follows that

$$\begin{aligned} \|g\,\chi_E\|_{M^p_\omega(\Omega)} &\le \|(g - g_\epsilon)\,\chi_E\|_{M^p_\omega(\Omega)} + \|g_\epsilon\,\chi_E\|_{M^p_\omega(\Omega)} \\ &< \frac{\epsilon}{2} + \|g_\epsilon\,\chi_E\|_{M^p_\omega(\Omega)} < \frac{\epsilon}{2} + \|g\|_{L^\infty(\Omega)} \cdot \frac{1}{h_\epsilon} < \epsilon, \end{aligned} \tag{23}$$

which implies $g \in \widetilde{M}^p_\omega(\Omega)$.

To prove the inverse inclusion we take a function $g \in \widetilde{M}^p_\omega(\Omega)$. By the definition, for any $\epsilon > 0$ there exists $h_\epsilon > 0$ large enough such that

$$\|g\,\chi_E\|_{M^p_\omega(\Omega)} < \epsilon \qquad \forall\, E \in \Sigma(\Omega) \qquad \text{satisfying} \qquad \|\chi_E\|_{M^p_\omega(\Omega)} < \frac{1}{h_\epsilon}\,. \tag{24}$$

We are going to construct a sequence of L^∞ - functions converging to g w.r.t. the norm in $M^p_\omega(\Omega)$.For each $k \in \mathbb{R}_+$, we define the set

$$E_k = \{x \in \Omega : |g(x)| \geq k\}, \tag{E}$$

and put $g_k = g(1 - \chi_{E_k})$. Then

$$g_k = \begin{cases} 0 & \text{if } |g(x)| \geq k \\ g & \text{if } |g(x)| < k, \end{cases} \tag{25}$$

and $g_k \in L^\infty(\Omega)$. Let us note that

$$\|g\|_{M^p_\omega(\Omega)} \geq \|g\,\chi_{E_k}\|_{M^p_\omega(\Omega)} \geq k \cdot \|\chi_{E_k}\|_{M^p_\omega(\Omega)}. \tag{26}$$

Taking $k > k_\epsilon = h_\epsilon \cdot \|g\|_{M^p_\omega(\Omega)}$, from (26) we get

$$\|g\|_{M^p_\omega(\Omega)} > h_\epsilon \cdot \|g\|_{M^p_\omega(\Omega)} \|\chi_{E_k}\|_{M^p_\omega(\Omega)}, \tag{27}$$

and hence $\|\chi_{E_k}\|_{M^p_\omega(\Omega)} < \dfrac{1}{h_\epsilon}$. Then for $k > k_\epsilon$, since $g \in \widetilde{M}^p_\omega(\Omega)$, we have

$$\|g - g_k\|_{M^p_\omega(\Omega)} = \|g\,\chi_{E_k}\|_{M^p_\omega(\Omega)} < \epsilon.$$

◄

The following result shows that the subspace $VM^p_\omega(\Omega)$ is larger than $\widetilde{M}^p_\omega(\Omega)$.

Lemma 4. *Suppose that ω satisfies* (W1), (W2), (W3) *and* (V). *Then*

$$\widetilde{M}^p_\omega(\Omega) \subset VM^p_\omega(\Omega).$$

Proof. Analogously to (11), the condition (V) implies the inclusion $L^\infty(\Omega) \subset VM^p_\omega(\Omega)$. Fix now $g \in \widetilde{M}^p_\omega(\Omega)$. As in Lemma 3, for each $\epsilon > 0$, we consider upper level sets E_k (see (E)) and functions g_k with $k > k_\epsilon = h_\epsilon \cdot \|g\|_{M^p_\omega(\Omega)}$ (see (25)) and $h_\epsilon > 0$ such that $\|\chi_{E_k}\|_{M^p_\omega(\Omega)} < \frac{1}{h_\epsilon}$. Then

$$\|g\,\chi_{E_{k_\epsilon}}\|_{M^p_\omega(\Omega,t)} \leq \|g\,\chi_{E_{k_\epsilon}}\|_{M^p_\omega(\Omega)} < \frac{\epsilon}{2} \tag{28}$$

and for $0 < t < \delta_\epsilon \leq 1$ we get

$$\|g\|_{M^p_\omega(\Omega,t)} \leq \|g\,\chi_{E_k}\|_{M^p_\omega(\Omega,t)} + \|g_k\|_{M^p_\omega(\Omega,t)} < \frac{\epsilon}{2} + \frac{\epsilon}{2}, \tag{29}$$

where the last estimate holds since $g_k \in VM^p_\omega(\Omega)$ is an essentially bounded function. ◄

Now we are able to improve the inclusion (14).

Lemma 5. *Suppose that* ω *satisfies the conditions* (W1), (W2) *and* (W3). *Then*

$$M^p_\omega(\Omega) \subset \widetilde{M}^q_\omega(\Omega) \qquad \forall\, 1 \le q < p < +\infty\,. \tag{30}$$

Proof. Fix $g \in M^p_\omega(\Omega)$ and choose $E \in \Sigma(\Omega)$ such that $\|\chi_E\|_{M^p_\omega(\Omega)} < \delta_\epsilon$. From (10) and Hölder's inequality we get

$$\begin{aligned}\|g\,\chi_E\|_{M^q_\omega(\Omega)} &= \sup_{\substack{x\in\Omega\\ \tau\in(0,1]}} \left(\frac{1}{\omega(x,\tau)} \int_{E(x,\tau)} |g(y)|^q\, dy\right)^{\frac{1}{q}} \\ &\le \sup_{\substack{x\in\Omega\\ \tau\in(0,1]}} \left(\frac{1}{\omega(x,\tau)} \int_{E(x,\tau)} |g(y)|^p\, dy\right)^{\frac{1}{p}} \left(\frac{|E(x,\tau)|}{\omega(x,\tau)}\right)^{\frac{1}{q}-\frac{1}{p}} \\ &\le \|g\|_{M^p_\omega(\Omega)} \cdot \|\chi_E\|^{\frac{p}{q}-1}_{M^p_\omega(\Omega)} < \epsilon\end{aligned}$$

for suitable choice of δ_ϵ and E. ◀

Now we introduce a class of mappings necessary for the definition of the subspace $\overset{\circ}{M}{}^p_\omega(\Omega)$. For $h \in \mathbb{R}_+$, define the cut off functions $\zeta_h \in C^\infty_0(\mathbb{R}^n)$ such that

$$\zeta_h(x) = \begin{cases} 1 & x \in B(0,h) \\ 0 & x \notin B(0,2\,h)\,. \end{cases}$$

Definition 4. *Suppose that* ω *satisfies* (W1), (W2) *and* (W3). *Then a function* $g \in M^p_\omega(\Omega)$ *belongs to* $\overset{\circ}{M}{}^p_\omega(\Omega)$ *iff*

$$g \in \widetilde{M}^p_\omega(\Omega) \qquad \textit{and} \qquad \lim_{h\to+\infty} \|(1-\zeta_h)\,g\|_{M^p_\omega(\Omega)} = 0\,. \tag{31}$$

We can describe $\overset{\circ}{M}{}^p_\omega(\Omega)$ by means of the following density result.

Lemma 6. *Let* (A) *and* (V) *hold. A function* $g \in M^p_\omega(\Omega)$ *belongs to* $\overset{\circ}{M}{}^p_\omega(\Omega)$ *if and only if* g *is in the closure of* $C^\infty_0(\Omega)$ *w.r.t. the norm in* $M^p_\omega(\Omega)$.

Proof. Let $g \in \overset{\circ}{M}{}^p_\omega(\Omega)$. In view of (31) and Lemma (4), for each $\epsilon > 0$ there exist $h_\epsilon > 0$ and $0 < t_\epsilon < 1$ such that

$$\|(1-\zeta_{h_\epsilon})\,g\|_{M^p_\omega(\Omega)} < \frac{\epsilon}{3}\,, \qquad \|g\|_{M^p_\omega(\Omega,t_\epsilon)} < \frac{\epsilon}{6}\,. \tag{32}$$

We consider the sequence of functions $\{\zeta_{h_\epsilon}(1-\chi_{\Omega_k})\,g\}_{k\in\mathbb{N}}$, where

$$\Omega_k = \left\{x \in \Omega\ :\ \operatorname{dist}(x,\partial\Omega) > \frac{1}{k}\right\}.$$

From (W3) it follows that

$$
\begin{aligned}
\|\zeta_{h_\epsilon}(1-\chi_{\Omega_k})\, g\|_{M^p_\omega(\Omega)} &\le \sup_{\substack{x\in\Omega \\ \tau\in(0,t_\epsilon]}} \left(\frac{1}{\omega(x,\tau)}\int_{\Omega(x,\tau)} |g(y)|^p\,dy\right)^{\frac{1}{p}} \\
&+ \sup_{\substack{x\in\Omega \\ \tau\in(t_\epsilon,1]}} \left(\frac{1}{\omega(x,\tau)}\int_{\Omega(x,\tau)} |(\zeta_{h_\epsilon}(1-\chi_{\Omega_k})g)\,(y)|^p\;dy\right)^{\frac{1}{p}} \qquad (33) \\
&\le \|g\|_{M^p_\omega(\Omega,t_\epsilon)} + D^{\frac{1}{p}} \sup_{\substack{x\in\Omega \\ \tau\in(t_\epsilon,1]}} \left(\frac{1}{|\Omega(x,t_\epsilon)|}\;\int_{\Omega(x,\tau)} |(\zeta_{h_\epsilon}(1-\chi_{\Omega_k})\,g)\,(y)|^p\;dy\right)^{\frac{1}{p}}.
\end{aligned}
$$

Hence, by (32) and (A)

$$
\left\|\zeta_{h_\epsilon}(1-\chi_{\Omega_k})\,g\right\|_{M^p_\omega(\Omega)} < \frac{\epsilon}{6} + C\;t_\epsilon^{-\frac{n}{p}}\left\|\zeta_{h_\epsilon}(1-\chi_{\Omega_k})\,g\right\|_{L^p(\Omega)} \qquad (34)
$$

with $C = C(p,D,A)$. By the Lebesgue Dominated Convergence theorem, there exists $k_\epsilon \in \mathbb{N}$ such that

$$
\left\|\zeta_{h_\epsilon}(1-\chi_{\Omega_{k_\epsilon}})\,g\right\|_{L^p(\Omega)} < \frac{\epsilon}{6}\,\frac{t_\epsilon^{\frac{n}{p}}}{C}\,. \qquad (35)
$$

Hence, by (34) and (35), we have

$$
\left\|\zeta_{h_\epsilon}(1-\chi_{\Omega_{k_\epsilon}})\,g\right\|_{M^p_\omega(\Omega)} < \frac{\epsilon}{3}\,. \qquad (36)
$$

Now we put $\psi_\epsilon = \zeta_{h_\epsilon}\,\chi_{\Omega_{k_\epsilon}}\,g$, and observe that

$$
\operatorname{supp}\psi_\epsilon \subset \Omega_{k_\epsilon} \cap B(0,2\,h_\epsilon) \Subset \Omega.
$$

Since $g \in VM^p_\omega(\Omega)$, we have $\psi_\epsilon \in VM^p_\omega(\Omega)$. Let us consider now a sequence $\{J_m\}_{m\in\mathbb{N}}$ of mollifiers in $\mathbb{R}^n$. By Lemma 2, there exists $m_\epsilon \in \mathbb{N}$ such that

$$
\|\psi_\epsilon - J_{m_\epsilon} * \psi_\epsilon\|_{M^p_\omega(\Omega)} < \frac{\epsilon}{3}\,. \qquad (37)
$$

Finally, if we put $\varphi_\epsilon = J_{m_\epsilon} * \psi_\epsilon \in C_0^\infty(\Omega)$, from (32), (36) and (37) we deduce

$$
\begin{aligned}
\|g-\varphi_\epsilon\|_{M^p_\omega(\Omega)} &\le \|g-\psi_\epsilon\|_{M^p_\omega(\Omega)} + \|\psi_\epsilon-\varphi_\epsilon\|_{M^p_\omega(\Omega)} \\
&\le \|(1-\zeta_{h_\epsilon})\,g\|_{M^p_\omega(\Omega)} + \|\zeta_{h_\epsilon}(1-\chi_{\Omega_{k_\epsilon}})\,g\|_{M^p_\omega(\Omega)} + \|\psi_\epsilon - J_{m_\epsilon} * \psi_\epsilon\|_{M^p_\omega(\Omega)} < \epsilon\,,
\end{aligned}
$$

hence g belongs to the closure of $C_0^\infty(\Omega)$ w.r.t. the norm in $M^p_\omega(\Omega)$.

To prove the inverse inclusion we take a convergent sequence of functions $\{\varphi_k\}_{k\in\mathbb{N}}$ in $C_0^\infty(\Omega)$ such that $\lim\limits_{k\to+\infty} \|g-\varphi_k\|_{M_\omega^p(\Omega)} = 0$ and $\operatorname{supp}\varphi_k \Subset \Omega$. For each k there exists $h_k > 0$ such that $\operatorname{supp}\varphi_k \subset B(0,h_k)$. In order to show (31) we consider

$$\|(1-\zeta_{h_k})\, g\|_{M_\omega^p(\Omega)} = \|(1-\zeta_{h_k})\,(g-\varphi_k)\|_{M_\omega^p(\Omega)} \le \|g-\varphi_k\|_{M_\omega^p(\Omega)}\,.$$

We see that the last term here goes to 0 as $k\to+\infty$.
On the other hand, for $E\in\Sigma(\Omega)$ we have

$$\|g\,\chi_E\|_{M_\omega^p(\Omega)} \le \|g-\varphi_k\|_{M_\omega^p(\Omega)} + \|\varphi_k\,\chi_E\|_{M_\omega^p(\Omega)}\,. \tag{38}$$

Hence, by the inclusions $C_0^\infty(\Omega)\subset L^\infty(\Omega)\subset \widetilde{M}_\omega^p(\Omega)$, we deduce from (38) that if $\|\chi_E\|_{M_\omega^p(\Omega)}$ is small enough, then $g\in\widetilde{M}_\omega^p(\Omega)$, and this concludes the proof. ◄

4. Decompositions of functions in $\widetilde{M}_\omega^p(\Omega)$ and $\overset{\circ}{M}{}_\omega^p(\Omega)$

In this section, we are going to construct suitable decompositions for functions belonging to $\widetilde{M}_\omega^p(\Omega)$ and $\overset{\circ}{M}{}_\omega^p(\Omega)$. To this aim, we introduce *modulus of continuity of a function* in the corresponding space. Let $g\in\widetilde{M}_\omega^p(\Omega)$. We call *modulus of continuity of g in $\widetilde{M}_\omega^p(\Omega)$* a map $\widetilde{\sigma}_\omega^p[g]:\mathbb{R}_+\to\mathbb{R}_+$ defined as

$$\widetilde{\sigma}_\omega^p[g](h) := \sup_{\substack{E\in\Sigma(\Omega)\\ \|\chi_E\|_{M_\omega^p(\Omega)}\le\frac{1}{h}}} \|g\,\chi_E\|_{M_\omega^p(\Omega)} \tag{39}$$

Then Definition 3 gives $\lim_{h\to+\infty}\widetilde{\sigma}_\omega^p[g](h) = 0$. For the functions in $\overset{\circ}{M}{}_\omega^p(\Omega)$ we define the *modulus of continuity* as a map acting in $\mathbb{R}_+$ and defined in the following way:

$$\overset{\circ}{\sigma}{}_\omega^p[g](h) := \|(1-\zeta_h)\,g\|_{M_\omega^p(\Omega)} + \sup_{\substack{E\in\Sigma(\Omega)\\ \|\chi_E\|_{M_\omega^p(\Omega)}\le\frac{1}{h}}} \|g\,\chi_E\|_{M_\omega^p(\Omega)}\,. \tag{40}$$

Obviously $\lim_{h\to+\infty}\overset{\circ}{\sigma}{}_\omega^p[g](h) = 0$ by (31). We are going to show that any function g in $\widetilde{M}_\omega^p(\Omega)$ or $\overset{\circ}{M}{}_\omega^p(\Omega)$ can be represented as a sum $g=g_1+g_2$, where g_2 is essentially bounded, while the norm of g_1 can be controlled by the modulus of continuity of g in the corresponding space.

Lemma 7. *Let $g\in\widetilde{M}_\omega^p(\Omega)$. Then for any $h>0$ we have $g=g'_h+g''_h$, where $g''_h\in L^\infty(\Omega)$ and*

$$\|g'_h\|_{M_\omega^p(\Omega)} \le \widetilde{\sigma}_\omega^p[g](h), \qquad\qquad \|g''_h\|_{L^\infty(\Omega)} \le h^{\frac{1}{p}}\,\|g\|_{M_\omega^p(\Omega)}\,. \tag{41}$$

Proof. For any $g \in \widetilde{M}^p_\omega(\Omega)$ consider the upper level sets

$$E_h = \left\{ x \in \Omega \, : \, |g(x)| \geq h^{\frac{1}{p}} \, \|g\|_{M^p_\omega(\Omega)} \right\} . \tag{42}$$

Then for $x \in \Omega$ and $\tau \in (0,1]$ we have

$$\begin{aligned} \frac{|E_h(x,\tau)|}{\omega(x,\tau)} &\leq \frac{1}{\omega(x,\tau)} \int_{E_h(x,\tau)} \frac{|g(y)|^p}{h \, \|g\|^p_{M^p_\omega(\Omega)}} \, dy \\ &= \frac{1}{h \, \|g\|^p_{M^p_\omega(\Omega)}} \, \frac{1}{\omega(x,\tau)} \int_{E_h(x,\tau)} |g(y)|^p \, dy \leq \frac{1}{h \, \|g\|^p_{M^p_\omega(\Omega)}} \, \|g\|^p_{M^p_\omega(\Omega)} = \frac{1}{h}. \end{aligned} \tag{43}$$

Define the functions

$$g'_h = g \, \chi_{E_h} = \begin{cases} g & \text{if } \; x \in E_h \, , \\ 0 & \text{if } \; x \in \Omega \setminus E_h \, , \end{cases} \qquad g''_h = (1 - \chi_{E_h}) g = \begin{cases} 0 & \text{if } \; x \in E_h \, , \\ g & \text{if } \; x \in \Omega \setminus E_h \, . \end{cases}$$

Obviously $g''_h \in L^\infty(\Omega)$ and the second estimate in (41) holds. In view of (43) and (39), we have

$$\|g'_h\|_{M^p_\omega(\Omega)} = \|g \, \chi_{E_h}\|_{M^p_\omega(\Omega)} \leq \sup_{\substack{E \in \Sigma(\Omega) \\ \|\chi_E\|_{M^p_\omega(\Omega)} \leq \frac{1}{h}}} \|g \, \chi_E\|_{M^p_\omega(\Omega)} \leq \widetilde{\sigma}^p_\omega[g](h), \tag{44}$$

which implies the first inequality in (41). ◄

Lemma 8. *Let* $g \in \overset{\circ}{M}{}^p_\omega(\Omega)$*. Then for any* $h > 0$ *we have* $g = g'_h + g''_h$*, where* $g''_h \in L^\infty(\Omega)$ *and*

$$\|g'_h\|_{M^p_\omega(\Omega)} \leq \overset{\circ}{\sigma}{}^p_\omega[g](h), \qquad \|g''_h\|_{L^\infty(\Omega)} \leq \zeta_h \, h^{\frac{1}{p}} \, \|g\|_{M^p_\omega(\Omega)} \, . \tag{45}$$

Proof. Fix $g \in \overset{\circ}{M}{}^p_\omega(\Omega)$, consider the upper level sets E_h given by (42) and define the functions

$$g'_h = (1 - \zeta_h) g + \zeta_h \chi_{E_h} g = \begin{cases} g & \text{if } \; x \in E_h \, , \\ (1 - \zeta_h) g & \text{if } \; x \in \Omega \setminus E_h \, , \end{cases}$$

$$g''_h = \zeta_h (1 - \chi_{E_h}) g = \begin{cases} 0 & \text{if } \; x \in E_h \, , \\ g \zeta_h & \text{if } \; x \in \Omega \setminus E_h \, . \end{cases}$$

It is easy to see that

$$\|g'_h\|_{M^p_\omega(\Omega)} \leq \|(1 - \zeta_h) g\|_{M^p_\omega(\Omega)} + \|g \chi_{E_h}\|_{M^p_\omega(\Omega)} \, .$$

Thus by (40) we get the first inequality in (45). The second one follows from (42). ◄

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