Azerbaijan Journal of Mathematics
V. 10, No 1, 2020, January
ISSN 2218-6816

On a Boundary Value Problem for Fourth-Order Operator-Differential Equations with a Variable Coefficient

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Abstract. In this work, conditions for the regular solvability of one boundary-value problem for fourth-order operator-differential equations with a variable coefficient on the semi-axis are found. The obtained conditions are expressed by the properties of the coefficients of the considered operator-differential equation. Moreover, the norms of intermediate derivatives operators are estimated in terms of the norm of the right-hand side of the equation and these estimates are related to the solvability conditions of the boundary value problem.
Key Words and Phrases: Hilbert space, self-adjoint operator, boundary value problem, operator-differential equation.
2010 Mathematics Subject Classifications: 34G10, 35J40, 47D03

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1. Introduction

Interest in studying the solvability of differential equations in Banach and Hilbert spaces and the spectral problems associated with them is growing every year (see, for example, [1, 2] and the references therein). First of all, this is due to the fact that many problems for differential operators with partial derivatives can be reduced to the study of such equations.

Let H be a separable Hilbert space and A be a positive definite self-adjoint operator in H with domain of definition $D(A)$. Obviously, the domain of definition $D(A^\gamma)$ of the operator A^γ becomes a Hilbert space H_γ with respect to the scalar product $(x,y)_\gamma = (A^\gamma x, A^\gamma y)$, $\gamma \geq 0$. For $\gamma = 0$ we assume $H_0 = H$.

Denote by $L_2(R_+;H)$ the space of all vector functions defined on $R_+ = (0,+\infty)$ almost everywhere with values in H and with a norm

$$\|f\|_{L_2(R_+;H)} = \left(\int_0^{+\infty} \|f(t)\|^2 \, dt\right)^{\frac{1}{2}} < +\infty.$$

Following the monograph [3], we define a Hilbert space

$$W_2^4\left(R_+;H\right)=\left\{u:u^{(4)}\in L_2\left(R_+;H\right),A^4u\in L_2\left(R_+;H\right)\right\}$$

with a norm

$$\|u\|_{W_2^4(R_+;H)}=\left(\left\|u^{(4)}\right\|^2_{L_2(R_+;H)}+\left\|A^4u\right\|^2_{L_2(R_+;H)}\right)^{\frac{1}{2}}.$$

Throughout this work, derivatives are understood in the sense of the theory of distributions [3].

Obviously, from the trace theorem [3] it follows that

$$\overset{\circ}{W}{}_2^4\left(R_+;H\right)=\left\{u:u\in W_2^4\left(R_+;H\right),u\left(0\right)=0,u'\left(0\right)=0\right\}$$

is a complete subspace of a space $W_2^4\left(R_+;H\right)$.

Consider the following boundary value problem in a Hilbert space H:

$$L\left(d/dt\right)u\left(t\right)=\frac{d^4u\left(t\right)}{dt^4}+\rho\left(t\right)A^4u\left(t\right)+\sum_{j=0}^{4}A_{4-j}u^{(j)}\left(t\right)=f\left(t\right),t\in R_+, \tag{1}$$

$$u\left(0\right)=\varphi_0,\ u'\left(0\right)=\varphi_1, \tag{2}$$

where $f\left(t\right)\in L_2\left(R_+;H\right)$, $u\left(t\right)\in W_2^4\left(R_+;H\right)$, $\varphi_0\in H_{\frac{7}{2}}$, $\varphi_1\in H_{\frac{5}{2}}$, and operator coefficients satisfy the following conditions:

1. A is a positive definite self-adjoint operator;
2. $\rho\left(t\right)$ is a scalar measurable function in R_+ and $0<\alpha<f\left(t\right)<\beta<+\infty$;
3. the operators $B_j=A_jA^{-j},\ j=\overline{0,4}$, are bounded in H.

Definition 1. *If for any* $f\left(t\right)\in L_2\left(R_+;H\right)$, $\varphi_0\in H_{\frac{7}{2}}$, $\varphi_1\in H_{\frac{5}{2}}$ *there exists a vector function* $u\left(t\right)\in W_2^4\left(R_+;H\right)$ *that satisfies equation* (1) *almost everywhere in* R_+, *the boundary conditions* (2) *in the sense of convergence*

$$\lim_{t\to+0}\left\|u\left(t\right)-\varphi\right\|_{\frac{7}{2}}=0,\lim_{t\to+0}\left\|u'\left(t\right)-\varphi_1\right\|_{\frac{5}{2}}=0$$

and an estimate

$$\|u\|_{W_2^4(R_+;H)}\leq const\left(\|f\|_{L_2(R_+;H)}+\|\varphi_0\|_{\frac{7}{2}}+\|\varphi_1\|_{\frac{5}{2}}\right),$$

then the boundary value problem (1), (2) *is called regularly solvable.*

It should be noted that operator-differential equations and boundary value problems of type (1), (2) for them have been studied by many authors.

In these works, basically, $\varphi_0 = \varphi_1 = 0$ and $\rho(t)$ takes only two positive values (see, for example, [4–11]), i.e. $\rho(t) = \alpha$ for $t \in (0, t_0)$ and $\rho(t) = \beta$ for $t \in (t_0, \infty)$.

In this paper, $\rho(t)$ can take any positive value and have discontinuity points of the first kind.

Further, it should be noted that if $u(t) \in W_2^4(R_+; H)$, then by the trace theorem and by the intermediate derivative theorem $u^{(k)}(0) \in H_{4-k-\frac{1}{2}}$, $k = \overline{0,3}$, $A^{4-k}u^{(k)} \in L_2(R_+; H)$, $k = \overline{0,4}$. Moreover

$$\left\|u^{(k)}(0)\right\|_{4-k-\frac{1}{2}} \leq const \left\|u\right\|_{W_2^4(R_+;H)}, \left\|A^{4-k}u^{(k)}\right\|_{L_2(R_+;H)} \leq const \left\|u\right\|_{W_2^4(R_+;H)}.$$

2. Main results

First let us prove the following

Lemma 1. *Let $\omega_1 = -\frac{1}{\sqrt{2}}(1+i)$, $\omega_2 = -\frac{1}{\sqrt{2}}(1-i)$, and e^{-At} be a semigroup of bounded operators generated by the operator $-A$. Then the vector-function $u_0(t) = e^{\omega_1 tA}x_1 + e^{\omega_2 tA}x_2$ belongs to the space $W_2^4(R_+; H)$ if and only if $x_1 \in H_{\frac{7}{2}}$, $x_2 \in H_{\frac{7}{2}}$.*

Proof. The necessity of the statement follows from the trace theorem, since $x_1 + x_2 \in H_{\frac{7}{2}}$, $\omega_1 x_1 + \omega_2 x_2 = A^{-1}u'(0) \in H_{\frac{7}{2}}$. Hence we have $x_1, x_2 \in H_{\frac{7}{2}}$. On the other hand, for $x_1, x_2 \in H_{\frac{7}{2}}$ we have $u_0(t) \in W_2^4(R_+; H)$. For $x_1 \in H_{\frac{7}{2}}$ let us show that $e^{\omega_1 A}x_1 \in W_2^4(R_+; H)$. Obviously, there exists a vector $y \in H$ such that $A^{7/2}x_1 = y$. Then

$$\left\|e^{\omega_1 tA}x_1\right\|^2_{W_2^4(R_+;H)} =$$

$$= 2\left\|e^{\omega_1 tA}A^{1/2}y\right\|^2_{L_2(R_+;H)} = 2\left(A^{1/2}e^{\omega_1 tA}y, A^{1/2}e^{\omega_1 tA}y\right)_{L_2(R_+;H)} =$$

$$= 2\int_0^{\mu} \mu \int_{\mu}^{\infty} e^{-\sqrt{2}t\mu}\left(dE_\mu y, y\right) = \sqrt{2}\left\|y\right\|^2 = \sqrt{2}\left\|x_1\right\|^2_{7/2}.$$

Hence, we have $u_0(t) \in W_2^4(R_+; H)$. Lemma is proved. ◄

In problem (1), (2) we make the change

$$u(t) = v(t) + u_0(t), v(t) \in W_2^4(R_+; H), u_0(t) \in W_2^4(R_+; H).$$

Moreover, we choose vectors x_1 and x_2 from $u_0(t) = e^{\omega_1 tA}x_1 + e^{\omega_2 tA}x_2$ so that $v(0) = 0$, $v'(0) = 0$, i.e.

$$x_1 + x_2 = \varphi_0, \omega_1 x_1 + \omega_2 x_2 = A^{-1}\varphi_1.$$

It is obvious that φ_0, $A^{-1}\varphi_1 \in H_{7/2}$ and are uniquely determined from this system, and

$$\|x_1\|_{\frac{7}{2}} \leq const\left(\|\varphi_0\|_{\frac{7}{2}} + \|\varphi_1\|_{\frac{5}{2}}\right),$$

$$\|x_2\|_{\frac{7}{2}} \leq const\left(\|\varphi_0\|_{\frac{7}{2}} + \|\varphi_1\|_{\frac{5}{2}}\right),$$

i.e.

$$\|u_0\|_{W_2^4(R_+;H)} \leq const\left(\|\varphi_0\|_{\frac{7}{2}} + \|\varphi_1\|_{\frac{5}{2}}\right).$$

Thus, from (1), (2) we obtain the boundary value problem

$$L\left(d/dt\right)v\left(t\right) = L\left(d/dt\right)u\left(t\right) - L\left(d/dt\right)u_0\left(t\right), \tag{3}$$

$$v\left(0\right) = 0, v'\left(0\right) = 0. \tag{4}$$

Denote

$$g\left(t\right) = L\left(d/dt\right)u\left(t\right) - L\left(d/dt\right)u_0\left(t\right) = f\left(t\right) - L\left(d/dt\right)u_0\left(t\right).$$

Let us show that $g\left(t\right) \in L_2\left(R_+;H\right)$. In fact,

$$\|g\|_{L_2(R_+;H)} \leq \|f\|_{L_2(R_+;H)} + \|L\left(d/dt\right)u_0\|_{L_2(R_+;H)} \leq$$

$$\leq \|f\|_{L_2(R_+;H)} + \left\|\frac{d^4u_0}{dt^4} + \rho\left(t\right)A^4u_0\right\|_{L_2(R_+;H)} +$$

$$+\sum_{j=0}^{4}\left\|A_{4-j}A^{-(4-j)}\right\|\left\|A^{4-j}u_0^{(j)}\right\|_{L_2(R_+;H)} \leq$$

$$\leq \|f\|_{L_2(R_+;H)} + \left\|\frac{d^4u_0}{dt^4}\right\|_{L_2(R_+;H)} + \beta\left\|A^4u_0\right\|_{L_2(R_+;H)} +$$

$$+\sum_{j=0}^{4}\|B_{4-j}\|\left\|A^{4-j}u_0^{(j)}\right\|_{L_2(R_+;H)}.$$

Further, taking into account the intermediate derivatives theorem, we obtain

$$\|g\|_{L_2(R_+;H)} \leq \|f\|_{L_2(R_+;H)} + const\,\|u_0\|_{W_2^4(R_+;H)} \leq$$

$$\leq \|f\|_{L_2(R_+;H)} + const\left(\|\varphi_0\|_{7/2} + \|\varphi_1\|_{5/2}\right),$$

i.e. $g\left(t\right) \in L_2\left(R_+;H\right)$.

Thus, from (1), (2) we obtain the boundary value problem

$$L\,(d/dt)\,v\,(t) = g\,(t)\,,$$

$$v\,(0) = 0, v'\,(0) = 0.$$

Let us define in space $L_2\,(R_+;H)$ an operator L_0 with a domain of definition $D\,(L_0) = \overset{\circ}{W}{}_2^4\,(R_+;H)$, and

$$L_0 v = \frac{d^4v}{dt^4} + \rho\,(t)\,A^4 v.$$

It is obvious that the operator $L_0 : \overset{\circ}{W}{}_2^4\,(R_+;H) \to L_2\,(R_+;H)$ is self-adjoint. On the other hand, for every $v \in D\,(L_0)$ $(v\,(0) = v'\,(0) = 0)$

$$(L_0 v, v)_{L_2(R_+;H)} = \left(\frac{d^4v}{dt^4}, v\right)_{L_2(R_+;H)} + \left(\rho\,(t)\,A^4 v, v\right)_{L_2(R_+;H)} \geq$$

$$\geq \left\|v''\right\|^2_{L_2(R_+;H)} + \alpha\lambda_0^4 \left\|v\right\|^2_{L_2(R_+;H)}\,,$$

where λ_0 is a lower bound for the spectrum of the operator A. Then we obtain $KerL_0 = \{0\}$ and $ImL_0 = L_2\,(R_+;H)$. Thus, it is proved.

Theorem 1. *Operator L_0 maps the domain of definition of the operator L_0 isomorphically onto $L_2\,(R_+;H)$.*

First, let us study some properties of solutions of the equation $L_0 v = h$, where $v \in D\,(L_0)$, $h \in L_2\,(R_+;H)$.

The following theorem is valid.

Theorem 2. *Let the vector function $v(t)$ be a solution of the equation*

$$\frac{d^4v}{dt^4} + \rho\,(t)\,A^4 v\,(t) = h\,(t)\,. \tag{5}$$

Then the following inequalities hold for this solution:

$$\left\|A^4 v\right\|_{L_2(R_+;H)} \leq \alpha^{-1} \left\|h\right\|_{L_2(R_+;H)}\,, \tag{6}$$

$$\left\|A^3 v'\right\|_{L_2(R_+;H)} \leq 2^{-\frac{1}{2}} \alpha^{-\frac{3}{4}} \left\|h\right\|_{L_2(R_+;H)}\,, \tag{7}$$

$$\left\|A^2 v''\right\|_{L_2(R_+;H)} \leq 2^{-1} \alpha^{-\frac{1}{2}} \left\|h\right\|_{L_2(R_+;H)}\,, \tag{8}$$

$$\left\|A v'''\right\|_{L_2(R_+;H)} \leq 2^{-\frac{1}{4}} \alpha^{-\frac{1}{2}} \beta^{\frac{1}{4}} \left\|h\right\|_{L_2(R_+;H)}\,, \tag{9}$$

$$\left\|\frac{d^4v}{dt^4}\right\|_{L_2(R_+;H)} \leq \alpha^{-\frac{1}{2}} \beta^{\frac{1}{2}} \left\|h\right\|_{L_2(R_+;H)}\,. \tag{10}$$

Proof. Multiplying both sides of equation (5) by a function $\rho^{-1/2}(t)$, we obtain

$$\rho^{-\frac{1}{2}}(t)\,\frac{d^4v(t)}{dt^4}+\rho^{\frac{1}{2}}(t)\,A^4v(t)=\rho^{-\frac{1}{2}}(t)\,h(t)\,.$$

Hence we have

$$\left\|\rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}+\rho^{\frac{1}{2}}A^4v\right\|^2_{L_2(R_+;H)}=\left\|\rho^{-\frac{1}{2}}h\right\|^2_{L_2(R;H)}.$$

On the other hand, we obtain

$$\left\|\rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}+\rho^{\frac{1}{2}}A^4v\right\|^2_{L_2(R_+;H)}=\left\|\rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}\right\|^2_{L_2(R_+;H)}+$$

$$\left\|\rho^{\frac{1}{2}}A^4v\right\|^2_{L_2(R_+;H)}+2Re\left(\frac{d^4v}{dt^4},A^4v\right)_{L_2(R_+;H)}. \tag{11}$$

After integration by parts, we have

$$\left(\frac{d^4v}{dt^4},A^4v\right)_{L_2(R_+:H)}=\left(A^2\frac{d^2v}{dt^2},A^2\frac{d^2v}{dt^2}\right)_{L_2(R_+:H)}. \tag{12}$$

Taking into account (12) in equality (11), we obtain

$$\left\|\rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}\right\|^2_{L_2(R_+;H)}+\left\|\rho^{\frac{1}{2}}A^4v\right\|^2_{L_2(R_+;H)}+2\left\|A^2\frac{d^2v}{dt^2}\right\|^2_{L_2(R_+;H)}=\left\|\rho^{-\frac{1}{2}}h\right\|^2_{L_2(R_+;H)}. \tag{13}$$

From equality (13) it follows that

$$\left\|\rho^{\frac{1}{2}}A^4v\right\|^2_{L_2(R_+;H)}\le\left\|\rho^{-\frac{1}{2}}h\right\|^2_{L_2(R_+;H)}. \tag{14}$$

Therefore, taking inequality (14) into account, we obtain

$$\left\|A^4v\right\|^2_{L_2(R_+;H)}=\left\|\rho^{-\frac{1}{2}}\rho^{\frac{1}{2}}A^4v\right\|^2_{L_2(R_+;H)}\le\alpha^{-1}\left\|\rho^{\frac{1}{2}}A^4v\right\|^2_{L_2(R_+;H)}\le$$

$$\le\alpha^{-1}\left\|\rho^{-\frac{1}{2}}h\right\|^2_{L_2(R_+;H)}\le\alpha^{-2}\left\|h\right\|^2_{L_2(R_+;H)},$$

i.e.

$$\left\|A^4v\right\|_{L_2(R_+;H)}\le\alpha^{-1}\left\|h\right\|_{L_2(R_+;H)}.$$

Inequality (6) is proved.

Further, from inequality (13), we similarly obtain

$$\left\|\rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}\right\|^2_{L_2(R_+;H)} \le \left\|\rho^{-\frac{1}{2}}h\right\|^2_{L_2(R_+;H)},$$

i.e.

$$\left\|\frac{d^4v}{dt^4}\right\|^2_{L_2(R_+;H)} = \left\|\rho^{\frac{1}{2}}\rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}\right\|^2_{L_2(R_+;H)} \le \beta\left\|\rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}\right\|^2_{L_2(R_+;H)} \le$$

$$\le \beta\left\|\rho^{-\frac{1}{2}}h\right\|^2_{L_2(R_+;H)} \le \alpha^{-1}\beta\,\|h\|^2_{L_2(R_+;H)}.$$

Therefore, inequality (10) is also proved.

On the other hand, $v\,(t)$ is a solution of the equation $L_0v = h$ $(v\,(0) = v'\,(0) = 0)$. Then it is obvious that after integration by parts we have:

$$\left\|A^2v''\right\|^2_{L_2(R_+;H)} = \left(A^2v'', A^2v''\right)_{L_2(R_+;H)} = \left(A^4v, \frac{d^4v}{dt^4}\right)_{L_2(R_+;H)} =$$

$$= \left(\rho^{\frac{1}{2}}A^4v, \rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}\right)_{L_2(R_+;H)} \le$$

$$\le \left\|\rho^{\frac{1}{2}}A^4v\right\|_{L_2(R_+;H)}\left\|\rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}\right\|_{L_2(R_+;H)} \le$$

$$\le \frac{1}{2}\left(\left\|\rho^{\frac{1}{2}}A^4v\right\|^2_{L_2(R_+;H)} + \left\|\rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}\right\|^2_{L_2(R_+;H)}\right). \tag{15}$$

From equality (13) it follows that

$$\left\|\rho^{\frac{1}{2}}A^4v\right\|^2_{L_2(R_+;H)} + \left\|\rho^{-\frac{1}{2}}\frac{d^4v}{dt^4}\right\|^2_{L_2(R_+;H)} = \left\|\rho^{-\frac{1}{2}}h\right\|^2_{L_2(R_+;H)} - 2\left\|A^2v''\right\|^2_{L_2(R_+;H)}.$$

Considering this equality in (15), we have

$$\left\|A^2v''\right\|^2_{L_2(R_+;H)} \le \frac{1}{2}\left(\left\|\rho^{-\frac{1}{2}}h\right\|^2_{L_2(R_+;H)} - 2\left\|A^2v''\right\|^2_{L_2(R_+;H)}\right).$$

Consequently,

$$2\left\|A^2v''\right\|^2_{L_2(R_+;H)} \le \frac{1}{2}\left\|\rho^{-\frac{1}{2}}h\right\|^2_{L_2(R_+;H)},$$

i.e.

$$\left\|A^2v''\right\|^2_{L_2(R_+;H)} \le \frac{1}{4}\left\|\rho^{-\frac{1}{2}}h\right\|^2_{L_2(R_+;H)}.$$

Hence we have

$$\left\|A^2 v''\right\|_{L_2(R_+;H)} \le \frac{1}{2}\left\|\rho^{-\frac{1}{2}} h\right\|_{L_2(R_+;H)} \le \frac{1}{2}\alpha^{-\frac{1}{2}} \|h\|_{L_2(R_+;H)}.$$

Inequality (8) is proved.

Let us prove the remaining inequalities. We have

$$\left\|A^3 v'\right\|_{L_2(R_+;H)} = \left(A^3 v', A^3 v'\right)_{L_2(R_+;H)} =$$

$$= -\left(A^4 v, A^2 v''\right)_{L_2(R_+;H)} \le \left\|A^4 v\right\|_{L_2(R_+;H)} \left\|A^2 v''\right\|_{L_2(R_+;H)}.$$

Taking into account the proved inequalities (6) and (8), we obtain

$$\left\|A^3 v'\right\|^2_{L_2(R_+;H)} \le \alpha^{-1} 2^{-1} \alpha^{-\frac{1}{2}} \|h\|^2_{L_2(R_+;H)} \le 2^{-1}\alpha^{-\frac{3}{2}} \|h\|^2_{L_2(R_+;H)}.$$

Consequently,

$$\left\|A^3 v'\right\|_{L_2(R_+;H)} \le 2^{-\frac{1}{2}}\alpha^{-\frac{3}{4}} \|h\|_{L_2(R_+;H)}.$$

Inequality (7) is proved.

Now let us prove inequality (9). For this purpose, consider the norm

$$N = \left\|\tau^2 v^{(4)} + A v''' + \frac{1}{\tau^2} A^2 v''\right\|^2_{L_2(R_+;H)}$$

for $v \in D(L_0)$ $(v(0) = v'(0) = 0)$, and $\tau > 0$. It is clear that

$$N = \tau^4 \left\|v^{(4)}\right\|^2_{L_2(R_+;H)} + \frac{1}{\tau^4}\left\|A^2 v''\right\|^2_{L_2(R_+;H)} +$$

$$+\left\|A v'''\right\|^2_{L_2(R_+;H)} + 2Re\left(v^{(4)}, A^2 v''\right)^2_{L_2(R_+;H)} +$$

$$+2\tau^2 Re\left(v^{(4)}, A^2 v'''\right)_{L_2(R_+;H)} + 2\frac{1}{\tau^2} Re\left(A v''', A^2 v''\right)_{L_2(R_+;H)}. \tag{16}$$

After integration by parts, we have:

$$\left(v^{(4)}, A^2 v''\right)_{L_2(R_+;H)} = -\left\|A v'''\right\|^2_{L_2(R_+;H)} - \left(A^{\frac{1}{2}} v'''(0), A^{\frac{3}{2}} v''(0)\right), \tag{17}$$

$$2Re\left(v^{(4)}, A v'''\right)_{L_2(R_+;H)} = -\left\|A^{\frac{1}{2}} v'''(0)\right\|^2, \tag{18}$$

$$2Re\left(A v''', A^2 v''\right)_{L_2(R_+;H)} = -\left\|A^{\frac{3}{2}} v''(0)\right\|^2. \tag{19}$$

Therefore, taking into account equalities (17)-(19) in (16), we obtain

$$N=\tau^{4}\left\|v^{(4)}\right\|^{2}_{L_2(R_+;H)}+\frac{1}{\tau^{4}}\left\|A^{2}v''\right\|^{2}_{L_2(R_+;H)}-$$

$$-\left\|Av'''\right\|^{2}_{L_2(R_+;H)}-\left\|\tau A^{\frac{1}{2}}v'''\left(0\right)+\tau^{-1}A^{\frac{3}{2}}v''\left(0\right)\right\|^{2}.$$

Hence we have

$$\left\|Av'''\right\|^{2}_{L_2(R_+;H)}+\left\|\tau A^{\frac{1}{2}}v'''\left(0\right)+\tau^{-1}A^{\frac{3}{2}}v''\left(0\right)\right\|^{2}+N=$$

$$=\tau^{4}\left\|v^{(4)}\right\|^{2}_{L_2(R_+;H)}+\frac{1}{\tau^{4}}\left\|A^{2}v''\right\|^{2}_{L_2(R_+;H)}.$$

Consequently,

$$\left\|Av'''\right\|^{2}_{L_2(R_+;H)}\leq\tau^{4}\left\|v^{(4)}\right\|^{2}_{L_2(R_+;H)}+\tau^{-4}\left\|A^{2}v''\right\|^{2}_{L_2(R_+;H)}.$$

Putting $\tau^{2}=\left(\left\|A^{2}v''\right\|_{L_2(R_+;H)}\left\|v^{(4)}\right\|^{-1}_{L_2(R_+;H)}\right)^{\frac{1}{2}}$, we obtain for any $\varepsilon>0$

$$\left\|Av'''\right\|^{2}_{L_2(R_+;H)}\leq 2\left\|A^{2}v''\right\|_{L_2(R_+;H)}\left\|v^{(4)}\right\|_{L_2(R_+;H)}\leq$$

$$\leq 2\beta^{\frac{1}{2}}\left\|\rho^{-\frac{1}{2}}v^{(4)}\right\|_{L_2(R_+;H)}\left\|A^{2}v''\right\|_{L_2(R_+;H)}\leq$$

$$\leq\beta^{\frac{1}{2}}\left(\varepsilon\left\|\rho^{-\frac{1}{2}}v^{(4)}\right\|^{2}_{L_2(R_+;H)}+\frac{1}{\varepsilon}\left\|A^{2}v''\right\|^{2}_{L_2(R_+;H)}\right).$$

Taking $\varepsilon=2^{-\frac{1}{2}}$, we obtain

$$\left\|Av'''\right\|^{2}_{L_2(R_+;H)}\leq\beta^{\frac{1}{2}}\left(2^{-\frac{1}{2}}\left\|\rho^{-\frac{1}{2}}v^{(4)}\right\|^{2}_{L_2(R_+;H)}+2^{\frac{1}{2}}\left\|A^{2}v''\right\|^{2}_{L_2(R_+;H)}\right).$$

Taking into account the equality (13) we have:

$$\left\|Av'''\right\|^{2}_{L_2(R_+;H)}\leq 2^{-\frac{1}{2}}\beta^{\frac{1}{2}}\left\|\rho^{-\frac{1}{2}}h\right\|^{2}_{L_2(R_+;H)}\leq 2^{-\frac{1}{2}}\beta^{\frac{1}{2}}\alpha^{-1}\left\|h\right\|^{2}_{L_2(R_+;H)},$$

i.e.

$$\left\|Av'''\right\|_{L_2(R_+;H)}\leq 2^{-\frac{1}{4}}\beta^{\frac{1}{4}}\alpha^{-\frac{1}{2}}\left\|h\right\|_{L_2(R_+;H)}.$$

Theorem is proved. ◄

Now, let us prove the main theorem.

Theorem 3. *Let the conditions 1) -3) be satisfied, and moreover, the operators $B_j = A_j A^{-j}$, $j = \overline{0,4}$, be such that the inequality*

$$q = \sum_{j=0}^{4} c_j \|B_{4-j}\| < 1$$

holds, where $c_0 = \alpha^{-1}$, $c_1 = 2^{-\frac{1}{2}}\alpha^{-\frac{3}{4}}$, $c_2 = 2^{-1}\alpha^{-\frac{1}{2}}$, $c_3 = 2^{-\frac{1}{4}}\alpha^{-\frac{1}{2}}\beta^{\frac{1}{4}}$, $c_4 = \alpha^{-\frac{1}{2}}\beta^{\frac{1}{2}}$. Then the boundary value problem (1), (2) *is regularly solvable.*

Proof. After replacing $u(t) = v(t) + u_0(t)$, where $v(t) \in W_2^4(R_+; H)$, $u_0(t) = e^{\omega_1 tA} x_1 + e^{\omega_2 tA} x_2$, problem (3), (4) can be written in the form $L_0 v(t) + L_1 v(t) = g(t)$, where $L_0 v(t) = \frac{d^4 v(t)}{dt^4} + \rho(t) A^4 v(t)$, $L_1 v = \sum_{j=0}^4 A_j v^{(4-j)}(t)$, $g(t) \in L_2(R_+; H)$. Since L_0 is an invertible operator, then, assuming $L_0 v = w$, we obtain an equation $w + L_1 L_0^{-1} w = g$ in the space $L_2(R_+; H)$. On the other hand, for any $w(t) \in L_2(R_+; H)$ we have:

$$\left\|L_1 L_0^{-1} w\right\|_{L_2(R_+;H)} = \|L_1 v\|_{L_2(R_+;H)} \le \sum_{j=0}^{4} \|B_{4-j}\| \left\|A^{4-j} v^{(j)}\right\|_{L_2(R_+;H)}.$$

Taking into account the inequalities (6)-(10) from Theorem 2, we obtain:

$$\left\|L_1 L_0^{-1} w\right\|_{L_2(R_+;H)} \le \sum_{j=0}^{4} c_j \|B_{4-j}\| \|w\|_{L_2(R_+;H)} = q \|w\|_{L_2(R_+;H)}.$$

Since $0 < q < 1$, we have $v = L_0^{-1}\left(E + L_1 L_0^{-1}\right)^{-1} g$ and

$$\|v\|_{W_2^4(R_+;H)} \le const \|g\|_{L_2(R_+;H)}.$$

Then the solution of the boundary value problem (1), (2) is representable in the form $u(t) = v(t) + u_0(t)$. Moreover

$$\|u\|_{W_2^4(R_+;H)} \le \|v\|_{L_2(R_+;H)} + \|u_0\|_{L_2(R_+;H)} \le$$

$$\le const\left(\|g\|_{L_2(R_+;H)} + \|\varphi_0\|_{\frac{7}{2}} + \|\varphi_1\|_{\frac{5}{2}}\right) \le$$

$$\le const\left(\|f\|_{L_2(R_+;H)} + \|\varphi_0\|_{\frac{7}{2}} + \|\varphi_1\|_{\frac{5}{2}}\right).$$

Theorem is proved. ◄

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Received 15 July 2019
Accepted 11 November 2019