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On the Existence and Uniqueness of the Solution of Dirichlet Generalized Problem in Arbitrary Domain of n-dimensional Space $\mathbb{R}^n$ for Magnetic Schrödinger Operator

Sh.Sh. Rajabov

Abstract. In this paper, we consider the first generalized Dirichlet problem in Sobolev and Lebesgue spaces for magnetic Schrödinger operator. The Green operator is introduced, its self-adjointness and positive-definiteness in the space $L_2(G)$ is established. The existence and uniqueness of the solution of the first generalized Dirichlet problem is proved in the first order Sobolev space for the spectral parameter $\lambda \in \mathbb{C}\backslash[\lambda_0, +\infty)$ (λ_0 is determined in the text). L_2-estimation for solving the first generalized Dirichlet problem is established. Continuous dependence of the solution of the first generalized Dirichlet problem on the right hand side of the magnetic Schrödinger equation in the space $L_2(G)$ is proved.

Key Words and Phrases: magnetic Schrödinger operator, generalized Dirichlet problem, Green operator, magnetic Sobolev spaces.

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1. Introduction

Let G be an arbitrary domain in $\mathbb{R}^n$. We consider the magnetic Schrödinger expression

$$H_{a,V} = \sum_{k=1}^{n} \left(\frac{1}{i} \frac{\partial}{\partial x_k} + a_k (x) \right)^2 + V (x)$$

in G, where $a(x) = (a_1(x), a_2(x), ..., a_n(x))$ is a real magnetic potential, $V(x)$ is a real electric potential, $x = (x_1, x_2, ..., x_n) \in \mathbb{R}^n$, i is an imaginary unit.

Consider the following two problems in the domain G.

1.1. The first generalized Dirichlet problem

Let $f(x) \in \overset{0}{W}{}_2^{\prime,1}(G)$, where $\overset{0}{W}{}_2^{\prime,1}(G)$ is a space conjugated to the first order Sobolev space $\overset{0}{W}{}_2^1(G)$. It is required to find a function $u(x)$ from the class $\overset{0}{W}{}_2^1(G)$, satisfying the equation

$$\sum_{k=1}^{n}\left(\frac{1}{i}\frac{\partial}{\partial x_k}+a_k(x)\right)^2 u(x)+V(x)u(x)-\lambda u(x)=f(x) \tag{1}$$

in the sense of distributions (see [1], [2]), i.e. for any function from the main space $C_0^{\infty}(G)$

$$\left\langle \sum_{k=1}^{n}\left(\frac{1}{i}\frac{\partial}{\partial x_k}+a_k(x)\right)^2 u(x)+V(x)u(x)-\lambda u(x),\varphi(x)\right\rangle=\langle f(x),\varphi(x)\rangle,$$

where λ is a spectral parameter, $\langle h,\psi\rangle$ is the value of the distribution at the point ψ.

1.2. The second generalized Dirichlet problem

Let $f(x) \in \overset{0}{W}{}_2^{\prime,1}(G)$. It is required to find a function $u(x)$ from the class $\overset{0}{W}{}_2^1(G)$ that for any function $\varphi(x)$ from the space $\overset{0}{W}{}_2^1(G)$ satisfies the equality

$$\sum_{k=1}^{n}\int_G\left(\frac{\partial u(x)}{\partial x_k}+ia_k(x)u(x)\right)\left(\frac{\partial\overline{\varphi(x)}}{\partial x_k}-ia_k(x)\overline{\varphi(x)}\right)dx+$$

$$+\int_G (V(x)-\lambda)u(x)\overline{\varphi(x)}dx=<f(x),\overline{\varphi(x)}>. \tag{2}$$

Note that such problems for the Schrödinger equation were considered in [3], and the generalized Dirichlet problem in the sense of V.P. Mikhailov for a second order elliptic equation in a self-adjoint form without minor terms was studied in [4].

In [5] it is proved that if the functions $a_k(x), k=1,2,...,n$, have first order partial derivatives continuous and bounded in G, the function $V(x)$ is measurable and bounded, then these two problems are equivalent. In the same place, it is proved that if $\mu \in (-\infty,\lambda_0)$, where

$$\lambda_0=a_0-nB_0-2nA_0^2+V_0,$$

$$a_0 = \inf_{x\in G}\left\{\sum_{k=1}^{n} a_k^2\left(x\right)\right\},$$

$$B_0 = \max\left\{B_1, B_2, ..., B_n\right\},$$

$$B_k = \sup_{x\in G}\left|\frac{\partial a_k\left(x\right)}{\partial x_k}\right|, k = 1, 2, ..., n,$$

$$A_0 = \max\left\{A_1, A_2, ..., A_n\right\},$$

$$A_k = \sup_{x\in G}\left|a_k\left(x\right)\right|, \ k = 1, 2, ..., n,$$

$$V_0 = \inf_{x\in G} V\left(x\right),$$

then the first generalized Dirichlet problem has a unique solution. In [5], the Green operator G_μ associating to each element $f(x)$ of the space $\overset{0}{W}{}_2^{',1}(G)$ a unique element $u\left(x\right)$ from the space $\overset{0}{W}{}_2^1\left(G\right)$, that is the solution of the first generalized Dirichlet problem, is introduced and the following lemma is proved.

Lemma 1. *If the functions $a_k(x), k = 1, 2, ..., n,$ have first order partial derivatives continuous and bounded in G, and $V\left(x\right)$ is a measurable bounded function, then for $\mu \in (-\infty, \lambda_0)$ the Green operator G_μ is a linear continuous operator from the space $\overset{0}{W}{}_2^{',1}(G)$ to $\overset{0}{W}{}_2^1\left(G\right)$.*

The goal of the present paper is to study the Green operator in the space $L_2(G)$ and prove the existence and uniqueness of the solution of the first generalized Dirichlet problem in the subset $\mathbb{C}\backslash[\lambda_0, +\infty)$ of complex numbers $\mathbb{C}$.

2. Main results

Let $\mu \in (-\infty, \lambda_0)$. We introduce to the space of main functions $C_0^\infty\left(G\right)$ the following norm:

$$\|u\|_{a,V,\mu} = \sqrt{h_{a,V}(u) - \mu\left\|u\right\|_{L_2(G)}^2},$$

where

$$h_{a,V}(u) = \sum_{k=1}^{n}\int_G\left|\frac{\partial u\left(x\right)}{\partial x_k} + ia_k(x)u(x)\right|^2 dx + \int_G V(x)\left|u(x)\right|^2 dx.$$

Denote the closure of the space $C_0^\infty\left(G\right)$ by $W_{a,V,\mu}(G)$ and call it a first order magnetic Sobolev space. Note that in recent years the properties of any order magnetic Sobolev spaces have been intensively studied (see [6, 7, 8, 9, 10, 11, 12, 13]).

In the sequel we will need the following

Lemma 2. *(see [5]) The norms* $\|\cdot\|_{a,V,\mu}$ *and* $\|\cdot\|_{W_2^1(G)}$ *are equivalent, i.e. the space* $W_{a,V,\mu}(G)$ *as a topological space coincides with the space* $\overset{0}{W_2^1}(G)$.

Now we consider G_μ as an operator from $L_2(G)$ to $L_2(G)$.

Theorem 1. *The Green operator* G_μ *is a bounded positive self-adjoint operator in the space* $L_2(G)$.

Proof. By Lemma 1 G_μ is an operator bounded from $W'_{a,V,\mu}(G)$ to $W_{a,V,\mu}(G)$. When we consider it as an operator acting from $L_2(G)$ to $L_2(G)$, from the embeddings

$$W_{a,V,\mu}(G) \subset L_2(G) \subset W'_{a,V,\mu}(G)$$

it follows that we narrow its domain of definition and expand its range of values. By the same token, we strengthen topology in the domain of definition G_μ, and at the same time we weaken topology in the range of its values. Taking into account these facts and Lemma 2, we see that if $f(x) \in L_2(G)$, then

$$\|G_\mu f\|_{L_2(G)} \le \|G_\mu f\|_{W_{a,V,\mu}(G)} \le c_1 \|G_\mu f\|_{\overset{0}{W_2^1}(G)} \le c_2 \|G_\mu f\|_{W'_{a,V,\mu}(G)} \le$$

$$\le c_3 \|G_\mu f\|_{\overset{0}{W_2^1}(G)} \le c_4 \|f\|_{L_2(G)}, \tag{3}$$

where c_i, $i = 1, 2, 3, 4$, are positive constant numbers. The boundedness of the operator G_μ from the space $L_2(G)$ to the space $L_2(G)$ follows from the chain of inequalities (3).

Since G_μ is a bounded operator in Hilbert space $L_2(G)$, then from the Hellinger-Teoplits theorem (see [14, p. 136, Theorem 1]) it follows that in order to prove the self-adjoitness of the operator G_μ in the space $L_2(G)$ it suffices to establish its symmetricity in $L_2(G)$.

At first we prove that G_μ is a symmetric operator in the space $W_{a,V,\mu}(G)$, i.e. for any elements $\varphi(x)$and $\psi(x)$ from $W_{a,V,\mu}(G)$ the equality

$$(G_\mu \varphi, \psi)_{W_{a,V,\mu}(G)} = (\varphi, G_\mu \psi)_{W_{a,V,\mu}(G)}. \tag{4}$$

is valid.

Note that by the definition of the Green operator G_μ the equality

$$(G_\mu \varphi, \psi)_{W_{a,V,\mu}(G)} = (\varphi, \psi)_{L_2(G)} \tag{5}$$

is valid.

On the other hand,

$$(\varphi, G_\mu\psi)_{W_{a,V,\mu}(G)} = \overline{(G_\mu\psi, \varphi)}_{W_{a,V,\mu}(G)} = \left(\overline{\psi, \varphi}\right)_{L_2(G)} = (\varphi, \psi)_{L_2(G)}\,, \tag{6}$$

where $\overline{z}$ denotes any complex conjugation of the complex number z. Equality (4) follows from (5) and (6).

We now prove that for any elements $\varphi(x)$ and $\psi(x)$ from the main space $C_0^\infty(G)$ the equality

$$(G_\mu\varphi, \psi)_{L_2(G)} = (\varphi, G_\mu\psi)_{L_2(G)}\,. \tag{7}$$

is valid.

By the definition of the Green operator G_μ

$$(G_\mu\varphi, \psi)_{L_2(G)} = \left(G_\mu^2\varphi, \psi\right)_{W_{a,V,\mu}(G)}\,. \tag{8}$$

On the other hand,

$$(\varphi, G_\mu\psi)_{L_2(G)} = \overline{(G_\mu\psi, \varphi)}_{L_2(G)} = \overline{\left(G_\mu^2\psi, \varphi\right)}_{W_{a,V,\mu}(G)} = \left(\varphi, G_\mu^2\psi\right)_{W_{a,V,\mu}(G)}\,. \tag{9}$$

It follows from the symmetricity of the operator G_μ in the space $W_{a,V,\mu}(G)$ that G_μ^2 is also a symmetric operator in the space $W_{a,V,\mu}(G)$.

Thus, the validity of (7) for any elements $\varphi(x)$ and $\psi(x)$ from the main space $C_0^\infty(G)$ follows from equalities (8) and (9). From the boundedness of the operator G_μ in the space $L_2(G)$ and everywhere density of the space $C_0^\infty(G)$ in the space $L_2(G)$ it follows that equality (7) remains valid in the space $L_2(G)$ as well.

In what follows, we assume$f(x) \in W_{a,V,\mu}(G)$. Then from the definition of the Green operator G_μ we get:

$$(G_\mu f, f)_{L_2(G)} = \left(G_\mu^2 f, f\right)_{W_{a,V,\mu}(G)} = (G_\mu f, G_\mu f)_{W_{a,V,\mu}(G)} \geq 0.$$

From everywhere density of the space $W_{a,V,\mu}(G)$ in $L_2(G)$ it follows that for any element $f(x)$ from the space $L_2(G)$ the inequality

$$(G_\mu f, f)_{L_2(G)} \geq 0$$

is valid.

From the bijectivity of the operator G_μ from $W'_{a,V,\mu}(G)$ to $W_{a,V,\mu}(G)$ it follows that if $(G_\mu f, f)_{L_2(G)} = 0$, then $f = 0$. ◄

Theorem 2. *Let λ be an arbitrary complex number and $f(x)$ be an arbitrary function from the space $L_2(G)$. Then in order the first generalized problem for equation* (1) *have a solution, it is necessary and sufficient that the equation*

$$u(x) - (\mu - \lambda)G_\mu u(x) = G_\mu f(x) \tag{10}$$

have a solution from the space $L_2(G)$ provided $\lambda \in (-\infty, \lambda_0)$.

Proof. Necessity. Let $f(x) \in L_2(G)$ and equation (1) have the solution $u(x)$ from the space $\overset{0}{W}{}_2^1(G)$. Rewriting equation (1) in the form

$$(H_{a,V}u(x) - \mu u(x)) + (\mu - \lambda)u(x) = f(x)$$

and applying the Green operator G_μ to it, we get

$$u(x) - (\mu - \lambda)G_\mu u(x) = G_\mu f(x).$$

Hence the function $u(x)$ is the solution of equation (10).

Sufficiency. Let $f(x) \in L_2(G)$ and equation (10) have the solution $u(x)$ from the space $L_2(G)$. By the definition of the operator G_μ, the functions $G_\mu f(x)$ and $G_\mu u(x)$ are the elements of the space $\overset{0}{W}{}_2^1(G)$. Consequently, from equation (10) it follows that the function $u(x)$ is also an element of the space $\overset{0}{W}{}_2^1(G)$. Now applying the operator $H_{a,V} - \mu E$ to equation (10), we see that the function $u(x)$ is the solution of the first generalized Dirichlet problem for equation (1). ◄

Now we formulate the main theorem on the existence and uniqueness of the solution of the first generalized Dirichlet problem for equation (1).

Theorem 3. *Let G be an arbitrary open domain in $\mathbb{R}^n$ and $\lambda \in \mathbb{C}\backslash[\lambda_0, +\infty)$. Then for an arbitrary function $f(x)$ from the space $L_2(G)$ equation* (1) *has a unique solution $u(x)$ from the space $\overset{0}{W}{}_2^1(G)$ and the following estimation is valid:*

$$\|u(x)\|_{L_2(G)} \leq \frac{1}{k(\lambda)} \|f(x)\|_{L_2(G)}, \tag{11}$$

where $k(\lambda)$ is the distance from the complex number λ to the set $[\lambda_0, +\infty)$.

Proof. Let $f(x) \in L_2(G)$ and $\lambda \in \mathbb{C}\backslash[\lambda_0, +\infty)$. The existence and uniqueness of the solution of equation (1) from the space $\overset{0}{W}{}_2^1(G)$ follow from the self-adjointness of the Green operator and Theorem 2. Let us prove the validity of estimation (11). Let $u(x) \in \overset{0}{W}{}_2^1(G)$ be the solution of the first generalized

Dirichlet problem for equality (1). Then for any function $\varphi(x)$ from $\overset{0}{W_2^1}(G)$ the equality (2) is valid. From this equality in particular we get

$$(f,u)_{L_2(G)} = \sum_{k=1}^{n} \int_G \left\{ \left| \frac{\partial u(x)}{\partial x_k} + i a_k(x) u(x) \right|^2 + (V(x)-\lambda)\,|u(x)|^2 \right\} dx.$$

Hence, using the Schwarz inequality, we have

$$\left| \sum_{k=1}^{n} \int_G \left| \frac{\partial u(x)}{\partial x_k} + i a_k(x) u(x) \right|^2 dx + \int_G (V(x)-\lambda)\,|u(x)|^2\, dx \right| \le$$

$$\le \|f(x)\|_{L_2(G)} \|u(x)\|_{L_2(G)}. \tag{12}$$

Introducing the denotation $\lambda = Re\lambda + iIm\lambda$, we estimate below the left hand side of inequality (12):

$$\left| \sum_{k=1}^{n} \int_G \left| \frac{\partial u(x)}{\partial x_k} + i a_k(x) u(x) \right|^2 dx + \right.$$

$$\left. + \int_G (V(x)-Re\lambda)\,|u(x)|^2\, dx - iIm\lambda \int_G |u(x)|^2 dx \right|^2 =$$

$$= \left(\sum_{k=1}^{n} \int_G \left| \frac{\partial u(x)}{\partial x_k} + i a_k(x) u(x) \right|^2 dx + \int_G (V(x)-Re\lambda)|u(x)|^2 dx \right)^2 +$$

$$+ (Im\lambda)^2 \left(\int_G |u(x)|^2 dx \right)^2 \ge k^2(\lambda)\, \|u(x)\|^4_{L_2(G)}.$$

Thus, we have

$$\sum_{k=1}^{n} \int_G \left| \frac{\partial u(x)}{\partial x_k} + i a_k(x) u(x) \right|^2 dx +$$

$$+ \int_G (V(x)-\lambda)|u(x)|^2 dx \ge k(\lambda)\, \|u(x)\|^2_{L_2(G)}. \tag{13}$$

Taking into account estimation (13) in inequality (12), we get

$$k(\lambda)\, \|u(x)\|^2_{L_2(G)} \le \|f(x)\|_{L_2(G)} \|u(x)\|_{L_2(G)}.$$

Hence it follows that the inequality (11) is valid. ◄

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Shahin Sh. Rajabov
Institute of Mathematics and Mechanics, Azerbaijan National Academy of Sciences, 9 B. Vahabzadeh str., AZ1141, Baku, Azerbaijan.
E-mail: shahin.racabov.88@mail.ru

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