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Multiple Solutions to a $(p_1(x), \ldots, p_n(x))$-Laplacian-type Systems in Unbounded Domain

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Abstract. In this paper, we establish the existence of at least three weak solutions for a parametric problem for doubly eigenvalue elliptic systems involving the $(p_1(x), \ldots, p_n(x))$-Laplacian operator. Our technical approach is based on variational methods and recent three critical points theorem obtained by Bonanno and Marano.

Key Words and Phrases: weak solutions, variable exponent spaces, $(p_1(x), \ldots, p_n(x))$-Laplacian operator, three critical points theorem.

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1. Introduction

In this work, we deal with the multiplicity of weak solutions for nonlinear elliptic system:

$$-\Delta_{p_i(x)}u_i + a_i(x)|u_i|^{p_i(x)-2}u_i = \lambda F_{u_i}(x, u_1, u_2, ..., u_n) \quad \text{in } \mathbb{R}^N, \tag{1}$$

for $1 \leq i \leq n$, where $\Delta_{p_i(x)}u_i := \operatorname{div}(|\nabla u_i|^{p_i(x)-2}\nabla u_i)$ is the $p_i(x)$-Laplacian operator for all $1 \leq i \leq n$, $p_i(x)$ are continuous real-valued functions such that $1 < p_i^- = \inf_{x\in\mathbb{R}^N} p_i(x) \leq p_i(x) \leq p_i^+ = \sup_{x\in\mathbb{R}^N} p_i(x) < N$ $(N \geq 2)$ for all $x \in \mathbb{R}^N$, λ is a positive parameter, $a_i \in L^\infty(\mathbb{R}^N)$ such that $a_i := ess\inf_{x\in\mathbb{R}^N} a_i(x) > 0$, the real function F belongs to $C^1(\mathbb{R}^N \times \mathbb{R}^n)$, F_{u_i} denotes the partial derivative of F with respect to u_i.

The $p(x)$-Laplacian operator possesses more complicated non-linearities than p-Laplacian operator, mainly due to the fact that it is not homogeneous. The study of various mathematical problems involving $p(x)$ growth condition have seen a strong rise of interest in recent years, we can, for example, refer to

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[8, 9, 10, 11, 14, 17, 19, 22, 23, 28]. This great interest may be justified by various physical applications of this operator. In fact, there are applications concerning nonlinear elasticity theory [30], electro-rheological fluids [26, 27], stationary thermorheological viscous flows [2] and continuum mechanics [3], etc. It also has wide applications in different research fields, such as image processing model [12], and the mathematical description of the filtration process of an ideal barotropic gas through a porous medium [4].

The goal of this work is to establish the existence of some interval which includes λ, where the system (1) admits at least three weak solutions, by means of a very recent abstract critical points result of G. Bonanno and S.A. Marano [7], which is a more precise version of Theorem 3.2 of [6]. For other basic notations and definitions we refer to [29].

Lemma 1 (see [7, Theorem 3.6]). *Let X be a reflexive real Banach space; $\Phi : X \to \mathbb{R}$ be a coercive, continuously Gâteaux differentiable and sequentially weakly lower semicontinuous functional whose Gâteaux derivative admits a continuous inverse on X^*; $\Psi : X \to \mathbb{R}$ be a continuously Gâteaux differentiable functional whose Gâteaux derivative is compact such that*

$$\Phi(0) = \Psi(0) = 0.$$

Assume that there exist $r > 0$ and $\overline{x} \in X$, with $r < \Phi(\overline{x})$, such that

(a$_1$) $\frac{\sup_{\Phi(x)\le r}\Psi(x)}{r} < \frac{\Psi(\overline{x})}{\Phi(\overline{x})}$;

(a$_2$) *for each $\lambda \in \Lambda_r := \left]\frac{\Phi(\overline{x})}{\Psi(\overline{x})}, \frac{r}{\sup_{\Phi(x)\le r}\Psi(x)}\right[$ the functional $\Phi - \lambda\Psi$ is coercive.*

Then, for each $\lambda \in \Lambda_r$ the functional $\Phi - \lambda\Psi$ has at least three distinct critical points in X.

The rest of the paper is organized as follows. Section 2 contains some basic preliminary knowledge of the variable exponent spaces and some results which will be needed later. In Section 3, we establish our main results and give an example of potential function F satisfying the assumptions requested in our main results.

2. Preliminaries and basic notations

In this section, we introduce some definitions and results which will be used in the next section. Firstly, we introduce some theories of Lebesgue-Sobolev spaces with variable exponent. The details can be found in [16, 20, 21]. Denote by $S(\mathbb{R}^N)$ the set of all measurable real functions on $\mathbb{R}^N$. Set

$$C_+(\mathbb{R}^N) = \{p \in C(\mathbb{R}^N) : \inf_{x\in\mathbb{R}^N} p(x) > 1\}.$$

For any $p \in C_+(\mathbb{R}^N)$ we define

$$p^- := \inf_{x\in\mathbb{R}^N} p(x) \quad \text{ and } p^+ := \sup_{x\in\mathbb{R}^N} p(x).$$

For any $p \in C_+(\mathbb{R}^N)$, we define the variable exponent Lebesgue space as

$$L^{p(x)}(\mathbb{R}^N) = \left\{ u \in S(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u(x)|^{p(x)} dx < \infty \right\},$$

endowed with the Luxemburg norm

$$|u|_{p(x)} := |u|_{L^{p(x)}(\mathbb{R}^N)} = \inf \left\{ \mu > 0 : \int_{\mathbb{R}^N} \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

Let $a \in S(\mathbb{R}^N)$, and $a(x) > 0$ for a.e $x \in \mathbb{R}^N$. Define the weighted variable exponent Lebesgue space $L_a^{p(x)}(\mathbb{R}^N)$ by

$$L_a^{p(x)}(\mathbb{R}^N) = \left\{ u \in S(\mathbb{R}^N) : \int_{\mathbb{R}^N} a(x)|u(x)|^{p(x)} dx < \infty \right\},$$

with the norm

$$|u|_{p(x),a(x)} := |u|_{L_a^{p(x)}(\mathbb{R}^N)} = \inf \left\{ \mu > 0 : \int_{\mathbb{R}^N} a(x) \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

From now on, we suppose that $a \in L^\infty(\mathbb{R}^N)$ with $a := ess \inf_{x\in\mathbb{R}^N} a(x) > 0$. Then obviously $L_a^{p(x)}$ is a Banach space (see [13] for details).

On the other hand, the variable exponent Sobolev space $W^{1,p(x)}(\mathbb{R}^N)$ is defined by

$$W^{1,p(x)}(\mathbb{R}^N) = \{u \in L^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L^{p(x)}(\mathbb{R}^N)\},$$

and is endowed with the norm

$$\|u\|_{1,p(x)} := \|u\|_{W^{1,p(x)}(\mathbb{R}^N)} = |u|_{p(x)} + |\nabla u|_{p(x)}, \qquad \forall u \in W^{1,p(x)}(\mathbb{R}^N).$$

Next, the weighted-variable exponent Sobolev space $W_a^{1,p(x)}(\mathbb{R}^N)$ is defined by

$$W_a^{1,p(x)}(\mathbb{R}^N) = \{u \in L_a^{p(x)}(\mathbb{R}^N) : |\nabla u| \in L_a^{p(x)}(\mathbb{R}^N)\},$$

with the norm

$$\|u\|_a := \inf\left\{\mu > 0 : \int_{\mathbb{R}^N} \left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + a(x)\left|\frac{u(x)}{\mu}\right|^{p(x)} dx \le 1\right\}, \forall u \in W_a^{1,p(x)}(\mathbb{R}^N).$$

Then the norms $\|u\|_a$ and $\|u\|_{p(x)}$ are equivalent in $W_a^{1,p(x)}(\mathbb{R}^N)$.

If $p^- > 1$, then the spaces $L^{p(x)}(\mathbb{R}^N)$, $W^{1,p(x)}(\mathbb{R}^N)$ and $W_a^{1,p(x)}(\mathbb{R}^N)$ are separable, reflexive and uniformly convex Banach spaces.

Here we display some facts which will be used later.

Proposition 1 (see [16, 20])**.** *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where*

$$\frac{1}{p(x)} + \frac{1}{p'(x)} = 1.$$

Moreover, for any $(u,v) \in L^{p(x)}(\Omega) \times L^{p'(x)}(\Omega)$, we have

$$|\int_\Omega uvdx| \le (\frac{1}{p^-} + \frac{1}{(p')^-})|u|_{p(x)}|v|_{p'(x)} \le 2|u|_{p(x)}|v|_{p'(x)}.$$

Proposition 2 (see [16, 20])**.** *Denote $\rho(u) = \int_{\mathbb{R}^N} |u|^{p(x)}dx$, for all $u \in L^{p(x)}(\mathbb{R}^N)$. We have*

$$\min\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\} \le \rho(u) \le \max\{|u|_{p(x)}^{p^-}, |u|_{p(x)}^{p^+}\},$$

and the following implications are true:

(i) $|u|_{p(x)} < 1$ *(resp.* $= 1, > 1$*)* $\Leftrightarrow \rho(u) < 1$ *(resp.* $= 1, > 1$*),*

(ii) $|u|_{p(x)} > 1 \Rightarrow |u|_{p(x)}^{p^-} \le \rho(u) \le |u|_{p(x)}^{p^+}$,

(iii) $|u|_{p(x)} < 1 \Rightarrow |u|_{p(x)}^{p^+} \le \rho(u) \le |u|_{p(x)}^{p^-}$.

From Proposition (2), we have:

$$\|u\|_a^{p^-} \le \int_{\mathbb{R}^N} \left(|\nabla u(x)|^{p(x)} + a(x)\,|u(x)|^{p(x)}\right) dx \le \|u\|_a^{p^+} \quad \text{if} \quad \|u\|_a \ge 1. \tag{2}$$

$$\|u\|_a^{p^+} \le \int_{\mathbb{R}^N} \left(|\nabla u(x)|^{p(x)} + a(x)\,|u(x)|^{p(x)}\right) dx \le \|u\|_a^{p^-} \quad \text{if} \quad \|u\|_a \le 1. \tag{3}$$

Proposition 3 (see [18]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p \in L^\infty(\mathbb{R}^N)$ and $1 \le p(x), q(x) \le \infty$ almost everywhere in $\mathbb{R}^N$. If $u \in L^{q(x)}(\mathbb{R}^N)$, $u \ne 0$. Then, we have*

$$|u|_{p(x)q(x)} \le 1 \Rightarrow |u|^{p^-}_{p(x)q(x)} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|^{p^+}_{p(x)q(x)},$$
$$|u|_{p(x)q(x)} \ge 1 \Rightarrow |u|^{p^+}_{p(x)q(x)} \le \left||u|^{p(x)}\right|_{q(x)} \le |u|^{p^-}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ is constant, then

$$||u|^p|_{q(x)} = |u|^p_{pq(x)}.$$

For all $x \in \mathbb{R}^N$ denote by

$$p^*(x) = \begin{cases} \frac{Np(x)}{N-p(x)} & \text{for } p(x) < N \\ +\infty & \text{for } p(x) \ge N \end{cases}$$

the critical Sobolev exponent of $p(x)$.

Proposition 4 (see [16, 18]). *Let $p \in C^{0,1}_+(\mathbb{R}^N)$ be the space of Lipschitz-continuous functions defined on $\mathbb{R}^N$. Then, there exists a positive constant c such that*

$$|u|_{p^*(x)} \le c\|u\|_a, \quad \forall u \in W^{1,p(x)}_a(\mathbb{R}^N).$$

Proposition 5 (see [16, 18]). *Assume that $p \in C(\mathbb{R}^N)$ with $p(x) > 1$ for each $x \in \mathbb{R}^N$. If $q \in C(\mathbb{R}^N)$ and $1 < q(x) < p^*(x)$ for each $x \in \Omega$, then there exists a continuous and compact embedding $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{q(x)}(\mathbb{R}^N)$.*

In the following discussions, we will use the product space

$$X := \prod_{i=1}^{n} W^{1,p_i(x)}_{a_i}(\mathbb{R}^N),$$

which is equipped with the norm

$$\|u\| := \sum_{i=1}^{n} \|u\|_{a_i}, \qquad \forall u = (u_1, u_2, ..., u_n) \in X,$$

where $\|u\|_{a_i}$ is the norm of $W^{1,p_i(x)}_{a_i}(\mathbb{R}^N)$. The space $X^\star$ denotes the dual space of X equipped with the usual dual norm.

Definition 1. *$u = (u_1, u_2, ..., u_n) \in X$ is called a weak solution of the system* (1) *if*

$$\sum_{i=1}^{n} \int_{\mathbb{R}^N} \left(|\nabla u_i(x)|^{p_i(x)-2} \nabla u_i \nabla v_i + a_i(x) \, |u_i|^{p_i(x)-2} \, u_i v_i \right) dx -$$

$$-\lambda \sum_{i=1}^{n} F_{u_i}(x, u_1, ... u_n) v_i \, dx = 0$$

for all $v = (v_1, v_2, ..., v_n) \in X$.

We denote by E_λ the energy functional associated with the problem (1)

$$E_\lambda(.) := \Phi(.) - \lambda \Psi(.),$$

where $\Phi, \Psi : X \longrightarrow \mathbb{R}$ are defined as follows:

$$\Phi(u) = \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} \left(|\nabla u_i(x)|^{p_i(x)} dx + a_i(x) |u_i(x)|^{p_i(x)} \right) dx,$$
$$\Psi(u) = \int_{\mathbb{R}^N} F(x, u_1(x), ..., u_n(x)) dx.$$

for any $u = (u_1, ..., u_n)$ in X.

It is well known that $E_\lambda \in C^1(X, \mathbb{R})$ and a critical point of E_λ corresponds to a weak solution of problem (1).

Hypotheses. We assume some growth conditions:

(H1) $F \in C^1(\mathbb{R}^N \times \mathbb{R}^n, \mathbb{R})$ and $F(x, 0, ..., 0) = 0$.

(H2) There exist positive functions b_{ij} $(1 \leq i, j \leq n)$, such that

$$\left| \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n) \right| \leq \sum_{j=1}^{n} b_{ij}(x) |u_j|^{\mu_{ij}-1},$$

where $1 < \mu_{ij} < \inf_{x \in \mathbb{R}^N} p_i(x)$, and $p_i(x) > \frac{N}{2}$, for all $x \in \mathbb{R}^N$ and for all $i \in \{1, 2, ..., n\}$. The weight-functions b_{ii} (resp b_{ij} if $i \neq j$) belong to the generalized Lebesgue spaces $L^{\alpha_i}(\mathbb{R}^N)$ (resp $L^{\alpha_{ij}}(\mathbb{R}^N)$), with

$$\alpha_i(x) = \frac{p_i(x)}{p_i(x) - 1}, \quad \alpha_{ij}(x) = \frac{p_i^*(x) p_j^*(x)}{p_i^*(x) p_j^*(x) - p_i^*(x) - p_j^*(x)}.$$

(H3) Assume that there exist $r>0$ and $w=(w_1,...,w_n)\in X$ such that the following conditions are satisfied:

(c1) $\sum_{i=1}^{n}\frac{\min\left\{\|w_i\|_{a_i}^{p^-},\|w_i\|_{a_i}^{p^+}\right\}}{p_i^+}>r$

(c2) $\frac{\int_{\mathbb{R}^N}\sup_{(\xi_1,...\xi_n)\in K(sr)}F(x,\xi_1,...\xi_n)dx}{r}<\frac{\int_{\mathbb{R}^N}F(x,w_1,...,w_n)dx}{\sum_{i=1}^{n}\max\left\{\|w_i\|_{a_i}^{p^-},\|w_i\|_{a_i}^{p^+}\right\}}$

where $K(t):=\left\{(\xi_1,...\xi_n)\in\mathbb{R}^N:\sum_{i=1}^{n}\min\left\{|\xi_i|_{p^\star(x)}^{(p^\star)^-},|\xi_i|_{p^\star(x)}^{(p^\star)^-}\right\}\le t\right\}$ with $t>0$
and $s=\min\left\{p_i^+\min\left\{c_{p_i(x)}^{(p_i^\star)^-},c_{p_i(x)}^{(p_i^\star)^+}\right\}\right\}$, such that $c_{p_i(x)}$ represent the constants associated with Proposition (4).

3. The main results

We will use the three critical points theorem obtained by Bonanno and Marano together with the following lemmas to get our main results.

Lemma 2. *The functional Φ is continuously Gâteaux differentiable and sequentially weakly lower semicontinuous, coercive whose Gâteaux derivative admits a continuous inverse on $X^\star$.*

Proof. It is well-known that the functional Φ is well defined and is continuously Gâteaux differentiable functional whose derivative at the point $u=(u_1,...,u_n)\in X$ is the functional $\Phi^{'}(u)$ given by

$$\Phi'(u)(v)=\int_\Omega\sum_{i=1}^{n}\left(|\nabla u_i(x)|^{p_i(x)-2}\nabla u_i(x)\nabla v_i(x)+a_i(x)|u_i(x)|^{p_i(x)}u_i(x)v_i(x)\right)dx,$$

for every $v=(v_1,...,v_n)\in X$.

Let us show that it is coercive. By using (2) and (3), we have for all $u=(u_1,...,u_n)\in X$

$$\begin{aligned}\Phi(u)&=\int_{\mathbb{R}^N}\sum_{i=1}^{n}\frac{1}{p_i(x)}\left(|\nabla u_i(x)|^{p_i(x)}dx+a_i(x)|u_i(x)|^{p_i(x)}\right)dx,\\&\ge\sum_{i=1}^{n}\frac{1}{p_i^+}\int_{\mathbb{R}^N}\left(|\nabla u_i(x)|^{p_i(x)}dx+a_i(x)|u_i(x)|^{p_i(x)}\right)dx,\\&\ge\sum_{i=1}^{n}\frac{1}{p_i^+}\min\left\{\|u_i\|_{a_i}^{p_i^-},\|u_i\|_{a_i}^{p_i^+}\right\}.\end{aligned}$$

This shows that $\Phi(u) \to +\infty$ as $\|u\| \to \infty$; that is , Φ is coercive on X.

Now we recall the elementary inequality (see, e.g., Chapter I in [15]) for any $\alpha, \beta \in \mathbb{R}^N$

$$\begin{cases} |\alpha-\beta|^\gamma \leq 2^\gamma \left(|\alpha|^{\gamma-2}\alpha - |\beta|^{\gamma-2}\beta\right)\cdot(\alpha-\beta) & \text{if } \gamma \geq 2 \\ |\alpha-\beta|^2 \leq (|\alpha|+|\beta|)^{2-\gamma} \left(|\alpha|^{\gamma-2}\alpha - |\beta|^{\gamma-2}\beta\right)\cdot(\alpha-\beta) & \text{if } 1<\gamma<2 \end{cases} \tag{4}$$

where $\cdot$ denotes the standard inner product in $\mathbb{R}^N$.

Let us define the sets of $\mathbb{R}^N$ dependent on p_i

$$U_{p_i} := \left\{x \in \mathbb{R}^N \quad p_i(x) \geq 2\right\},$$

$$V_{p_i} := \left\{x \in \mathbb{R}^N \quad 1 < p_i(x) < 2\right\}.$$

Now we show that Φ' is uniformly monotone. Indeed,

$$\left(\Phi'(u^1) - \Phi'(u^2)\right)(u^1 - u^2) =$$

$$= \sum_{i=1}^n \int_{\mathbb{R}^N} \left(|\nabla u_i^1|^{p_i(x)-2}\nabla u_i^1 - |\nabla u_i^2|^{p_i(x)-2}\nabla u_i^2\right)(\nabla u_i^1 - \nabla u_i^2)\,dx$$

$$+ \sum_{i=1}^n \int_{\mathbb{R}^N} \left(a_i(x)|u_i^1|^{p_i(x)}u_i^1 - a_i(x)|u_i^2|^{p_i(x)}u_i^2\right)(u_i^1 - u_i^2)\,dx,$$

Using the elementary inequality (4), we get

$$\left(\Phi'(u^1) - \Phi'(u^2)\right)(u^1 - u^2) \geq$$

$$\sum_{i=1}^n \int_{U_{p_i}} \frac{1}{2^{p(x)}} \left(|\nabla(u_i^1 - u_i^2)|^{p_i(x)} + a_i(x)|u_i^1 - u_i^2|^{p_i(x)}\right) dx$$
$$+ \sum_{i=1}^n \int_{V_{p_i}} |\nabla(u_i^1 - u_i^2)|^{p_i(x)} \left(\frac{|\nabla u_i^1 - \nabla u_i^2|}{|\nabla u_i^1| + |\nabla u_i^2|}\right)^{2-p_i(x)} dx$$
$$+ \sum_{i=1}^n \int_{V_{p_i}} a_i(x)|u_i^1 - u_i^2|^{p_i(x)} \left(\frac{|u_i^1 - u_i^2|}{|u_i^1| + |u_i^2|}\right)^{2-p_i(x)} dx.$$

Due to the facts

$$0 \leq \left(\frac{|\nabla u_i^1 - \nabla u_i^2|}{|\nabla u_i^1| + |\nabla u_i^2|}\right)^{2-p_i(x)} \leq 1$$

and

$$0 \le \left(\frac{|u_i^1-u_i^2|}{|u_i^1|+|u_i^2|}\right)^{2-p_i(x)} \le 1,$$

we have

$$|(\Phi'(u^1)-\Phi'(u^2))(u^1-u^2)| \ge$$

$$\ge \sum_{i=1}^{n}\int_{\mathbb{R}^N}\frac{1}{2^{p(x)}}\left(|\nabla(u_i^1-u_i^2)|^{p_i(x)}+a_i(x)|u_i^1-u_i^2|^{p_i(x)}\right)dx,$$

in other words, we have

$$(\Phi'(u^1)-\Phi'(u^2))(u^1-u^2) \ge$$

$$\ge \min\left\{\frac{1}{2^{p_i(x)}}\right\}\sum_{i=1}^{n}\int_{\mathbb{R}^N}\left(|\nabla(u_i^1-u_i^2)|^{p_i(x)}+a_i(x)|u_i^1-u_i^2|^{p_i(x)}\right)dx$$

$$\ge \frac{1}{2^N}\sum_{i=1}^{n}\min\left\{\|u_i^1-u_i^2\|_{a_i}^{p_i^-},\|u_i^1-u_i^2\|_{a_i}^{p_i^+}\right\}\ge 0$$

for all $u^1=(u_1^1,...,u_n^1), u^2=(u_1^2,...,u_n^2)\in X$. Hence Φ' is uniformly monotone and therefore coercive (see (2)). Since Φ' is semicontinuous in X, by applying Minty-Browder theorem (Theorem 26.A of [29]), Φ' admits a continuous inverse on $X^\star$. Moreover, the monotony of Φ' on $X^\star$ assures us that Φ is sequentially lower semicontinuous on X (see [29], Proposition 25. 20). ◄

Lemma 3. *Under the assumptions $(H1)$ and $(H2)$, the functional Ψ is well defined, and it is of class C^1 on X. Moreover, its derivative is*

$$\Psi'(u)h=\sum_{i=1}^{n}\int_{\mathbb{R}^N}\frac{\partial F}{\partial u_i}(x,u_1(x),...,u_n(x)h_i(x)\,dx$$

$$\forall u=(u_1,...,u_n), h=(h_1,...,h_n)\in X.$$

Proof. For all $u=(u_1,...,u_n)\in X$, under the assumptions $(H1)$ and $(H2)$, we can write

$$F(x,u_1,...,u_n)=\sum_{i=1}^{n}\int_0^{u_i}\frac{\partial F}{\partial s}(x,u_1,...,s,...,u_n)\,ds+F(x,0,...,0),$$

$$F(x,u_1,...,u_n)\le c_1\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{n}b_{ij}(x)|u_j(x)|^{\mu_{ij}-1}|u_i(x)|\right)\right]. \tag{5}$$

Then,

$$\int_{\mathbb{R}^N} F(x,u_1,...,u_n)\,dx \le c_2\left[\sum_{i=1}^{n}\left(\int_{\mathbb{R}^N}\sum_{j=1}^{n} b_{ij}(x)|u_j(x)|^{\mu_{ij}-1}|u_i(x)|\,dx\right)\right]. \quad (6)$$

If we consider the fact that $W^{1,p(x)}(\mathbb{R}^N) \hookrightarrow L^{\mu(x)}(\mathbb{R}^N)$, for $\mu(x)>1$, then there exists $c>0$ such that

$$||u|^{\mu}|_{p(x)} = |u|^{\mu}_{\mu p(x)} \le c\|u\|^{\mu}_{p(x)},$$

and if we apply Propositions 1, 3 and 4 and take $b_{ii}\in L^{\alpha_i(x)}$, $b_{ij}\in L^{\alpha_{ij}(x)}$ if $i\neq j$, then we have

$$\begin{aligned}\int_{\mathbb{R}^N} F(x,u_1,...,u_n) &\le c_3\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|b_{ij}|_{\alpha_{ij}(x)}||u_j|^{\mu_{ij}-1}|_{p_j^\star(x)}|u_i|_{p_i^\star(x)}\Big)\Big] && (7)\\ &\le c_3\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|b_{ij}|_{\alpha_{ij}(x)}|u_j|^{\mu_{ij}-1}_{(\mu_{ij}-1)p_j^\star(x)}|u_i|_{p_i^\star(x)}\Big)\Big] && (8)\\ &\le c_3\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|b_{ij}|_{\alpha_{ij}(x)}\|u_j\|^{\mu_{ij}-1}_{p_j(x)}\|u_i\|_{p_i(x)}\Big)\Big] < \infty. && (9)\end{aligned}$$

Hence, Ψ is well defined. Moreover, one can easily see that Ψ' is also well defined on X. Indeed, using $(F2)$ for all $h=(h_1,...,h_n)\in X$, we have

$$\begin{aligned}\Psi'(u)h &= \sum_{i=1}^{n}\int_{\mathbb{R}^N}\frac{\partial F}{\partial u_i}(x,u_1,...,u_n)h_i\,dx\\ &\le \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} b_{ij}(x)|u_j(x)|^{\mu_{ij}-1}\Big)|h_i(x)|dx.\end{aligned}$$

Following Hölder inequality, we obtain

$$\Psi'(u)h \le c_4\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|b_{ij}|_{\alpha_{ij}(x)}||u_j|^{\mu_{ij}-1}|_{p_j^\star(x)}|h_i|_{p_i^\star(x)}\Big)\Big].$$

The above propositions yield

$$\Psi'(u)h \le c\Big[\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}|b_{ij}|_{\beta_{ij}(x)}\|u_j\|^{\mu_{ij}-1}_{p_j(x)}\|h_i\|_{p_i(x)}\Big)\Big] < \infty.$$

Now let us show that Ψ is differentiable in the sense of Frechet, that is, for fixed $u=(u_1,...,u_n)\in X$ and given $\varepsilon>0$, there must be a $\delta=\delta_{\varepsilon,u_1,...,u_n}>0$ such that

$$|\Psi(u_1+h_1,...,u_n+h_n)-\Psi(u_1,...,u_n)-\Psi^{'}(u_1,...,u_n)(h_1,...,h_n)|\leq\varepsilon\sum_{i=1}^{n}(\|h_i\|_{p_i(x)})$$

for all $h=(h_1,...,h_n)\in X$ with $\sum_{i=1}^{n}(\|h_i\|_{p_i(x)})\leq\delta$.

Let B_R be the ball of radius R which is centered at the origin of $\mathbb{R}^N$ and denote $B_R^{'}=\mathbb{R}^N-B_R$. Moreover, let us define the functional Ψ_R on $\prod_{i=1}^{n}W_{a_i}^{1,p_i(x)}(B_R)$ as follows:

$$\Psi_R(u)=\int_{B_R}F(x,u_1(x),...,u_n(x))\;dx.$$

If we consider $(H1)$ and $(H2)$, it is easy to see that $\Psi_R\in C^1(\prod_{i=1}^{n}W_{a_i}^{1,p_i(x)}(B_R))$, and in addition for all $h=(h_1,...,h_n)\in\prod_{i=1}^{n}W_{a_i}^{1,p_i(x)}(B_R)$, we have

$$\Psi_R^{'}(u)h=\sum_{i=1}^{n}\int_{B_R}\frac{\partial F}{\partial u_i}(x,u_1(x),...,u_n(x))h_i(x)\,dx.$$

Also as we know, the operator $\Psi_R^{'}:X\to X^\star$ is compact [20]. Then, for all $u=(u_1,...,u_n),h=(h_1,...,h_n)\in X$, we can write

$$\begin{aligned}&|\Psi(u_1+h_1,...,u_n+h_n)-\Psi(u_1,...,u_n)-\Psi^{'}(u_1,...,u_n)(h_1,...,h_n)|\\&\leq|\Psi_R(u_1+h_1,...,u_n+h_n)-\Psi_R(u_1,...,u_n)-\Psi_R^{'}(u_1,...,u_n)(h_1,...,h_n)|\\&\quad+\left|\int_{B_R^{'}}\Big(F(x,u_1+h_1,...,u_n+h_n)-F(x,u_1,...,u_n)-\right.\\&\quad\left.-\sum_{i=1}^{n}\int_{B_R}\frac{\partial F}{\partial u_i}(x,u_1,...,u_n)h_i\Big)\,dx\right|.\end{aligned}$$

According to a classical theorem, there exist $\xi_1,...,\xi_n\in]0,1[$ such that

$$\begin{aligned}&\left|\int_{B_R^{'}}\Big(F(x,u_1+h_1,...,u_n+h_n)-F(x,u_1,...,u_n)\Big)-\right.\\&\quad\left.-\sum_{i=1}^{n}\int_{B_R}\frac{\partial F}{\partial u_i}(x,u_1,...,u_n)h_i\,dx\right|\\&=\left|\int_{B_R^{'}}\left(\sum_{i=1}^{n}\frac{\partial F}{\partial u_i}(x,u_1,...,u_i+\xi_ih_i,...,u_n)h_i-\sum_{i=1}^{n}\frac{\partial F}{\partial u_i}(x,u_1,...,u_n)\right)\,dx\right|.\end{aligned}$$

Using the condition $(H2)$, we have

$$\Big|\int_{B'_R}\Big(F(x,u_1+h_1,...,u_n+h_n)-F(x,u_1,...,u_n)\Big)-\sum_{i=1}^{n}\int_{B_R}\frac{\partial F}{\partial u_i}(x,u_1,...,u_n)h_i\,dx\Big|$$

$$\leq\Big|\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\int_{B'_R}b_{ij}(x)(|u_j+\xi_j h_j|^{\mu_{ij}-1}-|u_j|^{\mu_{ij}-1})h_i dx\Big)\Big|.$$

Using the elementary inequality $|a+b|^s\leq 2^{s-1}(|a|^s+|b|^s)$ for $a,b\in\mathbb{R}^N$, we can write

$$\leq\sum_{i=1}^{n}\Big(\sum_{j=1}^{n}\Big((2^{\mu_{ij}-1}-1)\int_{B'_R}b_{ij}(x)|u_j|^{\mu_{ij}-1}|h_i|dx+$$

$$+(\xi_j 2)^{\mu_{ij}-1}\int_{B'_R}b_{ij}(x)|h_j|^{\mu_{ij}-1}|h_i|dx\Big)\Big).$$

Then, applying Propositions 1, 3, and 4, we have

$$\leq\sum_{i=1}^{n}c\Big(\sum_{j=1}^{n}\Big(|b_{ij}(x)|_{\alpha_{ij}}\|u_j\|^{\mu_{ij}-1}_{p_1^\star(x)}+|b_{ij}(x)|_{\alpha_{ij}}\|h_j\|^{\mu_{ij}-1}_{p_j^\star(x)}\Big)\Big)\|h_i\|_{p_i(x)},$$

and by the fact that

$$|b_{ii}(x)|_{L^{\alpha_i}(B'_R)}\longrightarrow 0,$$

$$|b_{ij}(x)|_{L^{\alpha_{ij}}(B'_R)}\longrightarrow 0$$

for $1\leq i,j\leq n$, as $R\to\infty$, and for R sufficiently large, we obtain the estimate

$$\Bigg|\int_{B'_R}\Big(F(x,u_1+h_1,...,u_n+h_n)-F(x,u_1,...,u_n)-$$

$$-\sum_{i=1}^{n}\frac{\partial F}{\partial u_i}(x,u_1,...,u_n)h_i\Big)\,dx\Bigg|\leq\varepsilon\sum_{i=1}^{n}(\|h_i\|_{p_i(x)}).$$

It remains only to show that Ψ' is continuous on X. Let $u^m=(u_1^m,...,u_n^m)$ be such that $u^m\to u$ as $m\to\infty$. Then, for $h=(h_1,...,h_n)\in X$, we have

$$|\Psi'(u^m)h-\Psi'(u)h|\leq|\Psi'_R(u^m)h-\Psi'_R(u)h|$$

$$+\sum_{i=1}^{n}\int_{B'_R}\Big|(\frac{\partial F}{\partial u_i}(x,u_1^m,...,u_n^m)h_i-\frac{\partial F}{\partial u_i}(x,u_1,...,u_n)h_i)\,dx\Big|.$$

Since Ψ'_R is continuous on $\prod_{i=1}^n W_{a_i}^{1,p_i(x)}(B_R)$(see [20]), we have

$$|\Psi'_R(u^m)h - \Psi'_R(u)h| \longrightarrow 0,$$

as $m \to \infty$. Now, using $(H2)$ once again and taking into account that the other terms on the right-hand side of the above inequality tend to zero, we conclude that Ψ' is continuous on X. ◄

Lemma 4. *Under the assumptions* $(H1)$ *and* $(H2)$*,* Ψ' *is compact from* X *to* $X^\star$*.*

Proof. Let $u^m = (u_1^m, ..., u_n^m)$ be a bounded sequence in X. Then, there exists a subsequence (we denote it also as $u^m = (u_1^m, ..., u_n^m)$) which converges weakly in X to $u = (u_1, ..., u_n) \in X$. Then, if we use the same arguments as above, we have

$$\begin{aligned}|\Psi'(u^m)h - \Psi'(u)h| &\le |\Psi'_R(u^m)h - \Psi'_R(u)h| \\ &+ \sum_{i=1}^n \int_{B'_R} |(\frac{\partial F}{\partial u_i}(x, u_1^m, ..., u_n^m)h_i - \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n)h_i)\, dx|\end{aligned}$$

Since the restriction operator is continuous, we have $u^m \rightharpoonup u$ in $\prod_{i=1}^n W_{a_i}^{1,p_i(x)}(B_R)$. Because of the compactness of Ψ', the first expression on the right-hand side of the inequality tends to 0, as $m \longrightarrow \infty$, and, as above, for sufficiently large R we obtain

$$\sum_{i=1}^n \int_{B'_R} \Big|\Big(\frac{\partial F}{\partial u_i}(x, u_1^m, ..., u_n^m)h_i - \frac{\partial F}{\partial u_i}(x, u_1, ..., u_n)h_i\Big)\, dx\Big| \longrightarrow 0.$$

This implies Ψ' is compact from X to $X^\star$. ◄

Theorem 1. *Under the assumptions* $(H1) - (H3)$*, for each*

$$\lambda \in \left] \frac{\sum_{i=1}^n \max\left\{ \|w_i\|_{a_i}^{p_i^-}, \|w_i\|_{a_i}^{p_i^+} \right\}}{\int_{\mathbb{R}^N} F(x, w_1(x), ..., w_n(x))dx}, \frac{r}{\int_{\mathbb{R}^N} \sup_{(\xi_1,...\xi_n)\in K(sr)} F(x, \xi_1, ..., \xi_n)dx} \right[,$$

the system (1) *admits at least three distinct weak solutions in* X*.*

Proof. By Lemma 2, Φ is coercive and $\Phi(0) = \Psi(0) = 0$. Also, we see that the required hypothesis $\Phi(\overline{x}) > r$ follows from (c_1) and the definition of Φ by choosing $\overline{x} = w = (w_1, ..., w_n)$. Moreover, by applying Proposition 4, for each $u_i \in W^{1,p_i(x)}_{a_i}$

$$|u_i|_{p_i^\star} \leq c\|u_i\|_{a_i},$$

with $1 \leq i \leq n$, we have for each $u = (u_1, ..., u_n) \in X$

$$\frac{1}{s}\sum_{i=1}^{n} \min\left\{|u_i|^{p^-}_{p_i^\star}, |u_i|^{p^+}_{p_i^\star}\right\} \leq \sum_{i=1}^{n} \frac{1}{p_i^+} \min\left\{\|u_i\|^{p^-}_{a_i}, \|u_i\|^{p^+}_{a_i}\right\} \tag{10}$$

with $s = \min\left\{p_i^+ \min\left\{c^{(p_i^\star)^-}_{p_i(x)}, c^{(p_i^\star)^+}_{p_i(x)}\right\}\right\}$. From (10), for each $r > 0$ we obtain

$$\begin{aligned}
\Phi^{-1}(]-\infty, r[) &= \{u = (u_1, ..., u_n) \in X : \Phi(u) \leq r\} \\
&= \left\{u = (u_1, ..., u_n) \in X : \sum_{i=1}^{n} \frac{1}{p_i^+} \min\left\{\|u_i\|^{p_i^-}_{a_i}, \|u_i\|^{p_i^+}_{a_i}\right\} \leq r\right\} \\
&\subseteq \left\{u = (u_1, ..., u_n) \in X : \sum_{i=1}^{n} \min\left\{|u_i|^{p_i^-}_{p_i^\star}, |u_i|^{p_i^+}_{p_i^\star}\right\} \leq sr\right\}.
\end{aligned}$$

Then,

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) &= \sup_{u\in\Phi^{-1}(]-\infty,r[)} \int_{\mathbb{R}^N} F(x, u_1, ..., u_n)dx, \\
&\leq \int_{\mathbb{R}^N} \sup_{(\xi_1,...\xi_n)\in K(sr)} F(x, \xi_1, ..., \xi_n)dx.
\end{aligned}$$

Therefore, from the condition (c_2), we have

$$\begin{aligned}
\sup_{u\in\Phi^{-1}(]-\infty,r[)} \Psi(u) &\leq r\frac{\int_{\mathbb{R}^N} F(x, w_1(x), ..., w_n(x))dx}{\sum_{i=1}^{n} \max\left\{\|w_i\|^{p_i^-}_{a_i}, \|w_i\|^{p_i^+}_{a_i}\right\}}, \\
&\leq r\frac{\Psi(w)}{\Phi(w)}.
\end{aligned}$$

from which (a_1) of Lemma 1 follows.

To show that the functional $\Phi - \lambda\Psi$ is coercive, we use the inequallity (9). Then for all $u \in X$, we have by virtue of $(H1)$ and $(H2)$

$$\Phi(u) - \lambda\Psi(u) = \sum_{i=1}^{n} \int_{\mathbb{R}^N} \frac{1}{p_i(x)} \left(|\nabla u_i(x)|^{p_i(x)}dx + a_i(x)|u_i(x)|^{p_i(x)}\right) dx$$

$$
\begin{aligned}
&- \lambda \int_{\mathbb{R}^N} F(x, u_1(x), ..., u_n(x))dx,\\
&\geq \sum_{i=1}^{n} \frac{1}{p_i^+} \int_{\mathbb{R}^N} \Big(|\nabla u_i(x)|^{p_i(x)} dx + a_i(x)|u_i(x)|^{p_i(x)} \Big) dx\\
&- c_3 \Big[\sum_{i=1}^{n} \Big(\sum_{i=j}^{n} |b_{ij}|_{\alpha_{ij}(x)} \|u_j\|_{p_j(x)}^{\mu_{ij}-1} \|u_i\|_{p_i(x)} \Big) \Big].
\end{aligned}
$$

Using Young's inequality, we obtain

$$
\begin{aligned}
\Phi(u) - \lambda\Psi(u) &\geq \sum_{i=1}^{n} \frac{1}{p_i^+} \|u_i\|_{a_i}^{p_i^-} - c_3 \Big[\sum_{i=1}^{n} \Big(\sum_{j=1}^{n} |b_{ij}|_{\alpha_{ij}(x)} \Big(\frac{\mu_{ij}-1}{\mu_{ij}} \|u_j\|_{p_j(x)}^{\mu_{ij}}\\
&\quad + \frac{1}{\mu_{ij}} \|u_i\|_{p_i(x)}^{\mu_{ij}} \Big) \Big) \Big],\\
&\geq \sum_{i=1}^{n} \frac{1}{p_i^+} \|u_i\|_{a_i}^{p_i^-} - c_4 \Big[\sum_{i=1}^{n} \Big(\sum_{j=1}^{n} \Big(|b_{ij}|_{\alpha_{ij}(x)} \|u_j\|_{p_j(x)}^{\mu_{ij}}\\
&\quad + |b_{ij}|_{\alpha_{ij}(x)} \|u_i\|_{p_i(x)}^{\mu_{ij}} \Big) \Big) \Big].
\end{aligned}
$$

This shows that $\Phi - \lambda\Psi \to +\infty$ as $\|u\|_X \to +\infty$, since $1 < \mu_{ij} < \inf_{x\in\mathbb{R}^N} p_i(x)$; that is, $\Phi - \lambda\Psi$ is coercive on X for every parameter λ, in particular, for every $\lambda \in \Lambda_r :=]\frac{\Phi(w)}{\Psi(w)}, \frac{r}{\sup_{\Phi(w)\leq r} \Psi((w)}[$. Then, condition (a_2) also holds. Now all the hypotheses of Lemma 1 are satisfied. Also note that the solutions of the equation $\Phi^{'}(u) - \lambda\Psi^{'}(u) = 0$ are exactly the weak solutions of(1). Thus for each

$$
\lambda \in \left] \frac{\sum_{i=1}^{n} \max\left\{ \|w_i\|_{a_i}^{p_i^-}, \|w_i\|_{a_i}^{p_i^+} \right\}}{\int_{\mathbb{R}^N} F(x, w_1(x), ..., w_n(x))dx}, \frac{r}{\int_{\mathbb{R}^N} \sup_{(\xi_1,...\xi_n)\in K(sr)} F(x, \xi_1, ..., \xi_n)dx} \right[
$$

the system (1) admits at least three weak solutions in X. ◄

Example

Let

$$
F(x, u, v) = a(x)|u|^{\beta(x)}|v|^{\gamma(x)},
$$

where $\frac{\beta(x)}{p(x)} + \frac{\gamma(x)}{q(x)} < 1$ and a is a positive function in $L^{s(x)}(\mathbb{R}^N)$ such that

$s(x) = \dfrac{p^{\star(x)}q^{\star(x)}}{p^{\star(x)}q^{\star(x)} - \beta(x)q^{\star(x)} - \gamma(x)p^{\star(x)}}$ for each $x \in \mathbb{R}^N$.

We can easily verify that $F(x,u,v)$ satisfies the conditions (H_1) and (H_3). Moreover, by using Young inequality we easily check that the condition (H_2) is also satisfied, and then the conclusion of Theorem 1 holds true.

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