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Some Remarks on the Local Energy Decay for Wave Equations in the Whole Space

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Abstract. We first consider the Cauchy problem for wave equations in the whole space $\mathbf{R}^n$ with $n = 1, 2$. For a special class of initial data, we derive uniform L^2-bounds of solutions. In the framework of compactly supported initial data, this L^2-bound is an essential ingredient to derive local energy decay estimates as shown in Morawetz [11] in the exterior domain case. We do not assume such compactness of the support of the initial data. Our results seem new in the low dimensional whole space cases ($n = 1, 2$). Furthermore, we discuss local energy decay property for wave equations with space dependent damping coefficient, which vanishes with some rate near spatial infinity.

Key Words and Phrases: wave equation, Cauchy problem, weighted initial data, multiplier method, Fourier analysis, L^2-bound, local energy decay, non-effective damping.

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1. Introduction

We first consider the Cauchy problem for the wave equation in $\mathbf{R}^n$ ($n = 1, 2$)

$$u_{tt}(t,x) - \Delta u(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n, \tag{1}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbf{R}^n, \tag{2}$$

where (u_0, u_1) are initial data chosen as:

$$u_0 \in H^2(\mathbf{R}^n), \quad u_1 \in H^1(\mathbf{R}^n), \tag{3}$$

and

$$u_t = \frac{\partial u}{\partial t}, \quad u_{tt} = \frac{\partial^2 u}{\partial t^2}, \quad \Delta = \sum_{j=1}^{n} \frac{\partial^2}{\partial x_j^2}, \quad x = (x_1, \cdots, x_n).$$

Note that solutions and/or functions considered in this paper are all real valued except for several parts concerning the Fourier transform. Then, it is known that the problem (1)-(2) has a unique strong solution

$$u \in C([0,\infty); H^2(\mathbf{R}^n)) \cap C^1([0,\infty); H^1(\mathbf{R}^n)) \cap C^2([0,\infty); L^2(\mathbf{R}^n)) =: C_2^n.$$

The rather stronger assumption on the initial data (3) is used only to justify the unique existence of solutions and the integration by parts in order to get the local energy decay.

As is well known, the local energy decay estimates for the equation (1) are one of the main questions in the scattering theory. About this problem, Morawetz's estimates in [11] are very famous as one of pioneering works, and she derived the local energy decay estimate such that

$$E_R(t) = O(t^{-1}) \quad (t \to \infty),$$

for each $R > 0$. The problem in [11] was considered in the exterior domain $\Omega \subset \mathbf{R}^n$ of a smooth bounded obstacle for $n \geq 3$, and the Dirichlet null condition on the boundary was assumed. At that process of the proof, she assumed the compactness of the support of initial data in order to use the finite speed of propagation property (FSPP for short) of solutions. In this sense, her theory fully depends on such FSPP. One of essential parts of proof in [11] is to get the L^2-bounds of solutions. To get such L^2-bounds she prepared an auxiliary function as stated in Remark 4 below. We can also cite the paper [14] concerning how to use the Morawetz method to obtain L^2-bounds of solutions in the damped wave equation case. So, a natural question arose in [4] and the references therein whether the local energy decay estimates can be derived or not under the framework of non-compact support condition on the initial data. In [3] it is shown that L^2-boundedness of solutions to the exterior mixed problem

$$u_{tt}(t,x) - \Delta u(t,x) = 0, \quad (t,x) \in (0,\infty) \times \Omega,$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \Omega$$

$$u(t,x) = 0, \quad x \in \partial\Omega$$

can be derived under the non-compact support conditions on the initial data. The method used in [11] was modified in [3] by removing support compact conditions assumed in [11]. Unfortunately, it was not sufficient only to get L^2-bounds of solutions in order to prove local energy decay under non-compact support conditions on the initial data. We need additional estimates, which are completely free from FSPP. After several papers published by Ikehata, by combining the additional weighted energy estimates and such L^2-bounds, the local energy decay estimates have been established in [4] in the final form. The weighted energy estimates used in [4] is a modification of the method, which has its origin in [15]. In [4], the so called Hardy-Sobolev inequality was effectively used to derive L^2-bounds, so one had to restrict the spatial dimension $n \geq 2$. This is a result in the exterior domain case. Therefore, if we apply the method of [4] to the whole space case without FSPP, we have to restrict $n \geq 3$, because of the unavailability of Hardy-Sobolev inequality for $n = 1, 2$. We should emphasize that the previous attempts to remove the support compact condition on the initial data have been made in [13] and [18].

Our aim in this paper is to derive L^2-bounds of solutions without FSPP to problem (1)-(2) considered in the whole space $\mathbf{R}^n$ with $n = 1, 2$. The main difficulty comes from a lack of an effective Hardy-Sobolev type inequality. For this we prepare some inequality in Proposition 1 without proof, which plays a role alternative to that of Hardy-Sobolev one.

Before stating our main results, we set

$$X^n_{1,\gamma} := \left\{ f \in L^{1,\gamma}(\mathbf{R}^n) | \int_{\mathbf{R}^n} f(x)dx = 0 \right\},$$

where

$$L^{p,\gamma}(\mathbf{R}^n) := \left\{ f \in L^p(\mathbf{R}^n) | \, \|f\|_{p,\gamma} := \int_{\mathbf{R}^n} (1 + |x|^\gamma)|f(x)|^p dx < +\infty \right\}.$$

Here are our main results.

Theorem 1. *Let $n = 2$ and $\gamma \in (0, 1]$. If $[u_0, u_1] \in H^2(\mathbf{R}^n) \times (H^1(\mathbf{R}^n) \cap X^n_{1,\gamma})$, then the unique solution $u \in C^n_2$ to problem* (1)-(2) *satisfies*

$$\|u(t, \cdot)\| \leq C(\|u_0\| + \|u_1\| + \|u_1\|_{1,\gamma}),$$

with some constant $C > 0$.

Theorem 2. *Let $n = 1$ and $\gamma \in (\frac{1}{2}, 1]$. If $[u_0, u_1] \in H^2(\mathbf{R}^n) \times (H^1(\mathbf{R}^n) \cap X^n_{1,\gamma})$, then the unique solution $u \in C^n_2$ to problem* (1)-(2) *satisfies*

$$\|u(t, \cdot)\| \leq C(\|u_0\| + \|u_1\| + \|u_1\|_{1,\gamma}),$$

with some constant $C > 0$.

Remark 1. *We still assume rather a stronger assumption on the initial velocity $u_1(x)$ such that $\int_{\mathbf{R}^n} u_1(x)dx = 0$. Even if the results above are derived under* *the special type of initial data, the obtained results seem to be new in the low dimensional case. In the Navier-Stokes equation case, as stated in [17], this vanishing moment condition is not so special. Anyway, it is still open to get the same results as in Theorems 1 and 2 without such vanishing moment condition.*

Remark 2. *We can also get the similar L^2-boundedness of the solutions to the wave equations with variable damping coefficient such as*

$$u_{tt}(t,x) - \Delta u(t,x) + a(x)u_t(t,x) = 0, \quad t > 0, \quad x \in \mathbf{R}^n.$$

For this case, we refer the reader to [5].

This paper is organized as follows. In Section 2 we shall prove Theorems 1 and 2 by relying on a modified method of [3], and in Section 3, we apply it to the local energy decay property of the problem (1)-(2), and generalize the results to another evolution equations with fractional Laplacian. Section 4 is dedicated to the study of local energy decay for wave equations with non-effective damping.

Notation. Throughout this paper, $\|\cdot\|_q$ stands for the usual $L^q(\mathbf{R}^n)$-norm. For simplicity of notation, we use $\|\cdot\|$ instead of $\|\cdot\|_2$. Furthermore, we denote $\|\cdot\|_{H^1}$ as the usual H^1-norm. The local energy $E_R(t)$ on the area $|x| \leq R$ $(R > 0)$ corresponding to the solution $u(t,x)$ of (1) is defined by

$$E_R(t) := \frac{1}{2}\int_{|x|\leq R}(|u_t(t,x)|^2 + |\nabla u(t,x)|^2)dx.$$

where

$$|\nabla f(x)|^2 := \sum_{j=1}^{n}|\frac{\partial f(x)}{\partial x_j}|^2.$$

On the other hand, we denote the Fourier transform of $f(x)$ by

$$\mathcal{F}(f)(\xi) := \hat{f}(\xi) := (\frac{1}{2\pi})^{\frac{n}{2}}\int_{\mathbf{R}^n} e^{-ix\cdot\xi}f(x)dx$$

as usual with $i := \sqrt{-1}$, and the fractional Laplacian is defined by

$$(-\Delta)^\theta f(x) := \mathcal{F}^{-1}(|\xi|^{2\theta}\hat{f}(\xi))(x),$$

where $\mathcal{F}^{-1}$ denotes the usual inverse Fourier transform of $\mathcal{F}$. For later use, we set additionally

$$H^1_1(\mathbf{R}^n) := \left\{f \in H^1(\mathbf{R}^n) \,|\, |\nabla f| \in L^{2,1}(\mathbf{R}^n)\right\},$$

$$BC(\mathbf{R}^n) := \{f \in C(\mathbf{R}^n) \,|\, \sup_{x\in\mathbf{R}^n}|f(x)| < +\infty\}.$$

2. Proof of Theorems 1 and 2

In the course of the proof, the following inequality concerning the Fourier image of the Riesz potential plays a crucial role. This comes from [2, (ii) of Proposition 2.1].

Proposition 1. *Let* $[n, \gamma, \theta]$ *satisfy* $n \geq 1$, $\gamma \in [0, 1]$ *and* $\theta \in [0, \gamma + \frac{n}{2})$. *Then, for all* $f \in L^2(\mathbf{R}^n) \cap X^n_{1,\gamma}$ *the following inequality is true:*

$$\int_{\mathbf{R}^n} \frac{|\hat{f}(\xi)|^2}{|\xi|^{2\theta}} d\xi \leq C(\|f\|^2_{1,\gamma} + \|f\|^2)$$

with some constant $C = C_{n,\theta,\gamma} > 0$.

Proof of Theorems 1 and 2. Let us prove Theorems 1 and 2 at a stroke. By applying the spatial Fourier transform to the both sides of the equation (1), the problem (1)-(2) can be reduced to ODE with the parameter ξ:

$$\hat{u}_{tt}(t, \xi) + |\xi|^2 \hat{u}(t, \xi) = 0, \quad (t, \xi) \in (0, \infty) \times \mathbf{R}^n_\xi, \tag{4}$$

$$\hat{u}(0, \xi) = \hat{u}_0(\xi), \quad \hat{u}_t(0, \xi) = \hat{u}_1(\xi), \quad \xi \in \mathbf{R}^n_\xi. \tag{5}$$

For the solution $\hat{u}(t, \xi)$ to problem (4)-(5), one introduces an auxiliary function

$$\hat{w}(t, \xi) := \int_0^t \hat{u}(s, \xi) ds.$$

Then $\hat{w}(t, \xi)$ satisfies

$$\hat{w}_{tt}(t, \xi) + |\xi|^2 \hat{w}(t, \xi) = \hat{u}_1(\xi), \quad (t, \xi) \in (0, \infty) \times \mathbf{R}^n_\xi, \tag{6}$$

$$\hat{w}(0, \xi) = 0, \quad \hat{w}_t(0, \xi) = \hat{u}_0(\xi), \quad \xi \in \mathbf{R}^n_\xi. \tag{7}$$

Now, we introduce an alternative function $\hat{v}(t, \xi)$ defined on $\mathbf{R}^n_\xi \setminus \{0\}$ as follows:

$$\hat{v}(t, \xi) = \hat{w}(t, \xi) - \frac{\hat{u}_1(\xi)}{|\xi|^2}, \quad \xi \in \mathbf{R}^n_\xi \setminus \{0\}. \tag{8}$$

Note that $\hat{v}(t, \xi)$ includes in its definition the so called Riesz potential of the measurable function $u_1(x)$. Then, the function $\hat{v}(t, \xi)$ satisfies

$$\hat{v}_{tt}(t, \xi) + |\xi|^2 \hat{v}(t, \xi) = 0, \quad t > 0, \quad \xi \in \mathbf{R}^n_\xi \setminus \{0\}, \tag{9}$$

$$\hat{v}(0, \xi) = -\frac{\hat{u}_1(\xi)}{|\xi|^2}, \quad \hat{v}_t(0, \xi) = \hat{u}_0(\xi), \quad \xi \in \mathbf{R}^n_\xi \setminus \{0\}. \tag{10}$$

By multiplying both sided of (9) by $\overline{\hat{v}_t(t,\xi)}$, integrating it over $\{|\xi| \geq \delta\}$ with small $\delta > 0$, and taking real parts of the resulted equality one can get

$$\frac{d}{dt}\int_{|\xi|\geq\delta}(|\hat{v}_t(t,\xi)|^2 + |\xi|^2|\hat{v}(t,\xi)|^2)d\xi = 0,$$

so that by integrating it over $[0,t]$ one has

$$\int_{|\xi|\geq\delta}(|\hat{v}_t(t,\xi)|^2 + |\xi|^2|\hat{v}(t,\xi)|^2)d\xi$$

$$= \int_{|\xi|\geq\delta}(|\hat{u}_0(\xi)|^2 + \frac{|\hat{u}_1(\xi)|^2}{|\xi|^2})d\xi,$$

where we have just used (10). Since $\hat{v}_t(t,\xi) = \hat{w}_t(t,\xi) = \hat{u}(t,\xi)$, one can arrive at

$$\int_{|\xi|\geq\delta}|\hat{u}(t,\xi)|^2 \leq \int_{\mathbf{R}^n_\xi}|\hat{u}_0(\xi)|^2 d\xi + \int_{|\xi|\geq\delta}\frac{|\hat{u}_1(\xi)|^2}{|\xi|^2}d\xi. \tag{11}$$

By applying Proposition 1 (for $n = 2, 1$) to the final term on the right-hand side of (11) one can get

$$\int_{|\xi|\geq\delta}\frac{|\hat{u}_1(\xi)|^2}{|\xi|^2}d\xi \leq \int_{\mathbf{R}^n_\xi}\frac{|\hat{u}_1(\xi)|^2}{|\xi|^2}d\xi$$

$$\leq C(\|u_1\|^2_{1,\gamma} + \|u_1\|^2), \tag{12}$$

so one has

$$\int_{|\xi|\geq\delta}|\hat{u}(t,\xi)|^2 d\xi \leq C(\|u_0\|^2 + \|u_1\|^2_{1,\gamma} + \|u_1\|^2). \tag{13}$$

Since the right-hand side of (13) is independent from any $\delta > 0$, by letting $\delta \downarrow 0$, and by relying on the Plancherel theorem one has the desired estimate. ◄

Remark 3. *It is known (see [17]) that if $u_1 \in X^n_{1,1}$, then $|\hat{u}_1(\xi)| \leq K|\xi|$ with some constant $K > 0$. Unfortunately, this property pointed out in [17] can not be well applied to control the estimate* (12). *Proposition 1 is meaningful.*

Remark 4. *The introduction of* (8) *is an important idea to get L^2-bound of solutions. This is a modification of that of Morawetz [11], which introduced an auxiliary function defined by*

$$v(t,x) := \int_0^t u(s,x)ds + h(x),$$

where the function $h \in H^2_{loc}(\mathbf{R}^n)$ is a unique solution to the Poisson equation

$$\Delta h(x) = u_1(x), \quad x \in \mathbf{R}^n,$$

$$h(x) = O(|x|^{-(n-2)}) \quad as \quad |x| \to \infty. \tag{14}$$

The condition (14) *can be justified (for example) when the support of the function $u_1(x)$ is compact in $\mathbf{R}^n$. We do not assume such compactness of the support of initial data, and we can treat the low dimensional case $n = 1, 2$ as well as $n \geq 3$, which has already been done in [3].*

3. Application to related problems

In this section, we first consider the Cauchy problem (1)-(2) again in order to apply Theorems 1 and 2 to the local energy decay property. By modifying the results of Ikehata-Nishihara [4], and applying Theorems 1 and 2, one can get the following results in the low dimensional whole space case.

Theorem 3. *Let $n = 2$ and $\gamma \in (0, 1]$. If $[u_0, u_1] \in (H^2(\mathbf{R}^n) \cap H^1_1(\mathbf{R}^n)) \times (H^1(\mathbf{R}^n) \cap X^n_{1,\gamma} \cap L^{2,1}(\mathbf{R}^n))$, then for each $R > 0$ the local energy $E_R(t)$ to problem* (1)-(2) *satisfies*

$$E_R(t) = O(t^{-1}) \quad (t \to \infty).$$

Theorem 4. *Let $n = 1$ and $\gamma \in (\frac{1}{2}, 1]$. If $[u_0, u_1] \in (H^2(\mathbf{R}^n) \cap H^1_1(\mathbf{R}^n)) \times (H^1(\mathbf{R}^n) \cap X^n_{1,\gamma} \cap L^{2,1}(\mathbf{R}^n))$, then for each $R > 0$ the local energy $E_R(t)$ to problem* (1)-(2) *satisfies*

$$E_R(t) = O(t^{-1}) \quad (t \to \infty).$$

Remark 5. *The results corresponding to Theorems 3 and 4 have already been obtained in [4] in the exterior domain case $\Omega \subset \mathbf{R}^n$ with $n \geq 2$. In the exterior domain case, the so called Hardy-Sobolev inequality played an important role in [4]. So, $n \geq 2$ is crucial. However, in the whole space case, such Hardy-Sobolev inequality does not hold, unfortunately. Proposition 1 plays a role alternative to that of the Hardy-Sobolev inequality in the low dimensional case $n = 1, 2$.*

Remark 6. *The novelty in Theorem 3 and 4 is that they do not assume any compactness of the support of initial data, and the results can be derived in the low dimensional whole space case.*

Outlines of the proofs of Theorems 3 and 4. We can use Theorems 1 and 2 in place of [4, Lemma 2.2] in order to get the L^2-boundedness of solutions. The other parts of proofs are the same as in [4]. Note that the conditions $u_0 \in H_1^1(\mathbf{R}^n)$ and $u_1 \in L^{2,1}(\mathbf{R}^n)$ can be used to prove another important weighted energy estimate [4, Lemma 2.3] such as

$$\int_{\mathbf{R}^n} \psi(t,x)(|u_t(t,x)|^2 + |\nabla u(t,x)|^2)dx \leq \int_{\mathbf{R}^n} (1+|x|)(|u_1(x)|^2 + |\nabla u_0(x)|^2)dx$$

with some weight function $\psi(t,x)$ satisfying the so called Eikonal equation of (1)

$$\psi_t(t,x) < 0, \quad |\psi_t(t,x)|^2 - |\nabla\psi(t,x)|^2 = 0 \quad t > 0, \quad x \in \mathbf{R}^n. \blacktriangleleft$$

Next, we consider the Cauchy problem for the following generalized evolution equations with the fractional Laplacian in $\mathbf{R}^n$:

$$u_{tt}(t,x) + (-\Delta)^\theta u(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n, \tag{15}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbf{R}^n, \tag{16}$$

where $\theta \geq 1$. $\theta = 2$ corresponds to the so called beam (plate) equation. It is well known that for the initial data $[u_0, u_1] \in H^{2\theta}(\mathbf{R}^n) \times H^\theta(\mathbf{R}^n)$, the problem (15)-(16) admits a unique solution

$$u \in C([0,\infty); H^{2\theta}(\mathbf{R}^n)) \cap C^1([0,\infty); H^\theta(\mathbf{R}^n)) \cap C^2([0,\infty); L^2(\mathbf{R}^n)).$$

Now, for simplicity, we choose $\gamma = 1$, and then by repeating the same procedure as in the proofs of Theorems 1 and 2 one can get the following a priori estimate.

Theorem 5. *Let $\theta \geq 1$ and $n > 2(\theta - 1)$. If $[u_0, u_1] \in H^{2\theta}(\mathbf{R}^n) \times (H^\theta(\mathbf{R}^n) \cap X_{1,1}^n)$, then for the solution $u(t,x)$ to problem* (15)-(16)

$$\|u(t,\cdot)\| \leq C(\|u_0\| + \|u_1\| + \|u_1\|_{1,1}),$$

with some constant $C > 0$.

Outline of proof of Theorem 5. If we use

$$\hat{v}(t,\xi) = \hat{w}(t,\xi) - \frac{\hat{u}_1(\xi)}{|\xi|^{2\theta}}, \quad \xi \in \mathbf{R}_\xi^n \setminus \{0\},$$

in place of (8), and apply the multiplier method as in the proof of Theorems 1 and 2, we can get

$$\int_{|\xi|\geq\delta} |\hat{u}(t,\xi)|^2 \leq \int_{\mathbf{R}_\xi^n} |\hat{u}_0(\xi)|^2 d\xi + \int_{|\xi|\geq\delta} \frac{|\hat{u}_1(\xi)|^2}{|\xi|^{2\theta}} d\xi. \tag{17}$$

By applying Proposition 1 to the last term of (17), one can get the desired estimate. $\blacktriangleleft$

Remark 7. *If $\theta = 2$ (plate equation case), then one can get the L^2-bound for $n = 3, 4$. This is one of the merits of Theorem 5, because people usually have to restrict the spatial dimension to $n \geq 5$ in the plate equation case (see [1], [7] and the references therein).*

4. Wave equation with decaying damping

We next consider the Cauchy problem for damped wave equation in $\mathbf{R}^n$

$$u_{tt}(t,x) - \Delta u(t,x) + V(x)u_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n, \tag{18}$$

$$u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \quad x \in \mathbf{R}^n, \tag{19}$$

where (u_0, u_1) are initial data chosen as (for simplicity):

$$u_0 \in C_0^\infty(\mathbf{R}^n), \quad u_1 \in C_0^\infty(\mathbf{R}^n). \tag{20}$$

Here, $V \in BC(\mathbf{R}^n)$ satisfies

$$(\mathbf{A-1}): \qquad \frac{V_0}{< x >^\alpha} \leq V(x),$$

where $V_0 > 0$, $\alpha \geq 0$, and $< x > := \sqrt{1 + |x|^2}$. Under these conditions, the problem (18)-(19) has a unique strong solution $u \in C_2^n$ (see page 1).

It is well-known that the equation (18) has a diffusive aspect as $t \to \infty$ in the case of $\alpha \in [0,1]$, and in fact, the case for $\alpha \in [0,1)$ was studied in [16] and [19], and the case for $\alpha = 1$ was investigated by [6]. While, in the case when $\alpha > 1$, and $V(x)$ satisfies

$$\frac{V_0}{< x >^\alpha} \geq V(x) \geq 0, \tag{21}$$

it is well-known that the solution $u(t,x)$ to problem (18)-(19) is asymptotically free as $t \to \infty$, and the total energy $E_u(t)$ defined by

$$E_u(t) := \frac{1}{2} \int_{\mathbf{R}^n} (|u_t(t,x)|^2 + |\nabla u(t,x)|^2) dx$$

does not go to 0 as $t \to \infty$, in general. This pioneering work was done in [9] in 1976. This observation implies a hyperbolic structure of the equation (1) under the condition (21). But, it seems still unknown that whether the local energy decays or not (as $t \to \infty$) under such a non-effective damping condition (21). Note that there are overlap parts between (A-1) and (21). In connection with the above results, Mochizuki and Nakazawa [10] investigated decay/non-decay properties of the energy, that is, if $V(x)$ satisfies

$$V_0(e+|x|)^{-1}\{\log(e+|x|)\}^{-1} \leq V(x),$$

then the total energy decays with some rate, while in the case when $V(x)$ satisfies

$$0 \leq V(x) \leq V_0(e+|x|)^{-1}\{\log(e+|x|)\}^{-1-\delta},$$

with $\delta > 0$, the corresponding total energy does not decay, in general, and the corresponding solution is asymptotically free as $t \to \infty$. This implies that a threshold on $V(x)$ between decay and non-decay of the total energy seems to be more delicate. However, until now one has known nothing about the decay of the "local" energy in the latter case.

Now, our final aim is to obtain the following partial answer for the (non-uniform) local energy decay estimate.

Theorem 6. *Let $n \geq 2$, $\alpha \in [0, n-2]$, and assume* (**A-1**). *Then, for any $R > 0$*

$$\lim_{t\to\infty}\int_{|x|\leq R}(|u_t(t,x)|^2 + |\nabla u(t,x)|^2)dx = 0.$$

Remark 8. *In the case where $n = 2, 3$, we have $\alpha \in [0, 1]$, so in this case the results are not new, and are included in [16] and [6]. But, in case of $n \geq 4$ we can choose $\alpha > 1$, so these cases are essentially new, and the local energy decay property for such cases was not discussed in [9].*

Remark 9. By density argument, one can choose the initial data from a more general class such as $[u_0, u_1] \in H^2(\mathbf{R}^n) \times H^1(\mathbf{R}^n)$. This argument is standard because the equation (18) is just linear.

Now, let us prove Theorem 6. We first prove the integrability of the local energy.

Lemma 1. *Under the same assumptions as in Theorem 6, for any $R > 0$, there exist a constant C, which depends on $R > 0$, and α such that*

$$\int_0^\infty\int_{|x|\leq R}(|u_t(t,x)|^2 + |\nabla u(t,x)|^2)dxdt \leq C(\|u_0\|_{H^1}^2 + \|u_1\|^2).$$

Proof. We use the multiplier technique borrowed from [6]. In the course of computation below, one can assume $u(t,x)$ is smooth enough in x and t, and vanishes for large $|x|$ for each t. This allows to use the integration by parts.

Now, we define an auxiliary function

$$W(x) := \frac{V_0}{2} < x >^{-\alpha} .$$

Then, it is easy to see that

$$-\Delta W(x) = \frac{V_0}{2} < x >^{-(\alpha+2)} \frac{\alpha n + \alpha(n-\alpha-2)|x|^2}{1+|x|^2} > 0 \quad (x \in \mathbf{R}^n). \tag{22}$$

Next, let us multiply both sides of (18) by $2u_t + W(x)u$. Then, it follows that

$$\frac{d}{dt}\int_{\mathbf{R}^n} \big(|u_t(t,x)|^2 + |\nabla u(t,x)|^2 + u_t(t,x)u(t,x)W(x)\big)\big)dx - \int_{\mathbf{R}^n} |u_t(t,x)|^2 W(x)dx$$

$$+2\int_{\mathbf{R}^n} V(x)|u_t(t,x)|^2 dx + \int_{\mathbf{R}^n} (\nabla W(x)\cdot \nabla u(t,x))u(t,x))\, dx$$

$$+\int_{\mathbf{R}^n} W(x)|\nabla u(t,x)|^2 dx + \frac{1}{2}\frac{d}{dt}\int_{\mathbf{R}^n} V(x)W(x)|u(t,x)|^2 dx = 0,$$

where one has just used the fact that

$$\Delta u(t,x)W(x)u(t,x)$$

$$= \nabla\cdot(W(x)u(t,x)\nabla u(t,x)) - (\nabla W(x)\cdot\nabla u(t,x))u(t,x) - W(x)|\nabla u(t,x)|^2.$$

Since

$$(\nabla W(x)\cdot\nabla u(t,x))u(t,x) = \frac{1}{2}\nabla\cdot(|u(t,x)|^2\nabla W(x)) - \frac{1}{2}|u(t,x)|^2\Delta W(x),$$

one has the equality

$$\frac{d}{dt}G(t) + \int_{\mathbf{R}^n} \big\{(2V(x) - W(x))|u_t(t,x)|^2 + W(x)|\nabla u(t,x)|^2 +$$

$$+\frac{1}{2}(-\Delta W(x))|u(t,x)|^2\Big\}\, dx = 0, \tag{23}$$

where we set

$$G(t) := \int_{\mathbf{R}^n} \big(|u_t(t,x)|^2 + |\nabla u(t,x)|^2 +$$

$$+u_t(t,x)u(t,x)W(x) + \frac{1}{2}V(x)W(x)|u(t,x)|^2 dx\Big)\, dx.$$

Integrating both sides of (23) over $[0,t]$, one has

$$G(0)-G(t)$$
$$= \int_0^t \int_{\mathbf{R}^n} \Big\{ (2V(x)-W(x))|u_s(s,x)|^2 + W(x)|\nabla u(s,x)|^2 + $$
$$+\frac{1}{2}(-\Delta W(x))|u(s,x)|^2 \Big\} dxds. \tag{24}$$

Since

$$2V(x)-W(x) \geq \frac{3V_0}{2} < x >^{-\alpha},$$

by using (22) one can get

$$G(t) + \frac{V_0}{2}\int_0^t \int_{\mathbf{R}^n} < x >^{-\alpha} (|u_s(s,x)|^2 + |\nabla u(s,x)|^2) dxds \leq G(0). \tag{25}$$

On the other hand, in order to check $G(t) > 0$, for any $\varepsilon > 0$ we use the following inequality

$$|W(x)u_t(t,x)u(t,x)| \leq \frac{\varepsilon}{2}|u_t(t,x)|^2 + \frac{1}{2\varepsilon}W(x)^2|u(t,x)|^2$$
$$\leq \frac{\varepsilon}{2}|u_t(t,x)|^2 + \frac{1}{2\varepsilon}V(x)W(x)|u(t,x)|^2. \tag{26}$$

By choosing $\epsilon := 3/2$ in (26) it follows that

$$G(t) \geq \int_{\mathbf{R}^n} \left(\frac{1}{4}|u_t(t,x)|^2 + |\nabla u(t,x)|^2 + \frac{1}{6}V(x)W(x)|u(t,x)|^2 \right) dx > 0,$$

which implies the positivity of the function $G(t)$. Thus, from (25) one has

$$\frac{V_0}{2}\int_0^t \int_{\mathbf{R}^n} < x >^{-\alpha} (|u_s(s,x)|^2 + |\nabla u(s,x)|^2) dxds \leq G(0).$$

Here, by assumption on the initial data we can check

$$G(0) = \int_{\mathbf{R}^n} (|u_1(x)|^2 + |\nabla u_0(x)|^2 + W(x)u_0(x)u_1(x) + \frac{1}{2}V(x)W(x)|u_0(x)|^2) dx$$
$$\leq \int_{\mathbf{R}^n} (|u_1(x)|^2 + |\nabla u_0(x)|^2 + V_0|u_0(x)u_1(x)| + V_0^2|u_0(x)|^2) dx < +\infty.$$

This implies the desired estimate

$$\int_0^t \int_{\mathbf{R}^n} < x >^{-\alpha} (|u_s(s,x)|^2 + |\nabla u(s,x)|^2) dxds \leq C(\|u_0\|_{H^1}^2 + \|u_1\|^2) \tag{27}$$

with some constant $C > 0$. ◄

Remark 10. *The essential part of the statement of Lemma 1 is in the estimate* (27)*. However, the boundedness of the part of* (27) *defined by* $\int_0^t \int_{\mathbf{R}^n} < x >^{-\alpha}$ $|u_s(s,x)|^2 dxds$ *is trivial because of (A-1) and the energy identity for* (18)-(19)*:*

$$E_u(t) + \int_0^t \int_{\mathbf{R}^n} V(x)|u_s(s,x)|^2 dxds = E_u(0).$$

Now, since the function $v(t,x) := u_t(t,x)$ satisfies

$$v_{tt}(t,x) - \Delta v(t,x) + V(x)v_t(t,x) = 0, \quad (t,x) \in (0,\infty) \times \mathbf{R}^n, \tag{28}$$

$$v(0,x) = u_1(x), \quad v_t(0,x) = \Delta u_0(x) - u_1(x), \quad x \in \mathbf{R}^n, \tag{29}$$

by applying Lemma 1 to problem (28)-(29) one can get the following result, which means the local decay of the second local energy of the equation (18). Note that it may be sufficient only to assume $[u_0, u_1] \in H^2(\mathbf{R}^n) \times H^1(\mathbf{R}^n)$ to get the result below.

Lemma 2. *Under the same assumptions as in Theorem 6, for any $R > 0$, there exist a constant C, which depends on $R > 0$, and α such that*

$$\int_0^\infty \int_{|x| \le R} (|u_{tt}(t,x)|^2 + |\nabla u_t(t,x)|^2) dxdt \le C(\|u_0\|_{H^2}^2 + \|u_1\|_{H^1}^2).$$

Using Lemmas 1 and 2, we now prove Theorem 6.

Proof of Theorem 6. We borrow the idea from [12] (see also [8, section 4]). Let $0 < t_1 < t$. Then, first of all, one has

$$(t - t_1)E_R(t) = \int_{t_1}^t \frac{d}{ds}(s - t_1 E_R(s))ds$$

$$= \int_{t_1}^t E_R(s)ds + \int_{t_1}^t (s - t_1) \int_{|x| \le R} (u_s(s,x)u_{ss}(s,x) + \nabla u(s,x) \cdot \nabla u_s(s,x))\, dxds.$$

For fixed $t > 1$, by choosing $t_1 := t - 1$ one has

$$E_R(t) \le \int_{t-1}^t E_R(s)ds + \int_{t-1}^t \int_{|x| \le R} (|u_s(s,x)||u_{ss}(s,x)| +$$

$$+ |\nabla u(s,x)||\nabla u_s(s,x)|)\, dxds$$

$$\le \int_{t-1}^t E_R(s)ds + \frac{1}{2} \int_{t-1}^t \int_{|x| \le R} \big(|u_s(s,x)|^2 + |u_{ss}(s,x)|^2 + |\nabla u(s,x)|^2 +$$

$$+|\nabla u_s(s,x)|^2\big)\, dxds$$

$$\leq 2\int_{t-1}^{\infty} E_R(s)ds + \frac{1}{2}\int_{t-1}^{\infty}\int_{|x|\leq R}\left(|u_{ss}(s,x)|^2 + |\nabla u_s(s,x)|^2\right) dxds,$$

where one has just used the fact that $s-(t-1)\leq 1$ for $s\in[t-1,t]$. By Lemmas 1 and 2, one can get

$$\lim_{t\to\infty}\int_{t-1}^{\infty} E_R(s)ds = 0,$$

and

$$\lim_{t\to\infty}\int_{t-1}^{\infty}\int_{|x|\leq R}\left(|u_{ss}(s,x)|^2 + |\nabla u_s(s,x)|^2\right) dxds = 0$$

for each $R>0$. This completes the proof of Theorem 6. ◀

Example 1. *In the case of $n=4$ and $V(x) := V_0 < x >^{-2}$, due to the result of [9], the corresponding Cauchy problem* (18)-(19) *has a non-diffusive structure, and in general, one has the non-decay nature of the total energy: $\lim_{t\to\infty} E_u(t) > 0$ (in spite of the monotone decreasing property of the mapping $t\mapsto E_u(t)$). From Theorem 6 we can additionally see that the local energy necessarily decays. This observation seems new.*

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References

[1] R.C. Charão, C.R. da Luz, *Asymptotic properties for a semilinear plate equation in unbounded domains*, J. Hyperbolic Diff. Eqns., **6**, 2009, 269-294.

[2] R. Ikehata, *Fast energy decay for wave equations with a localized damping in the n-D half space*, Asymptotic Anal., **103**, 2017, 77-94.

[3] R. Ikehata, T. Matsuyama, *L^2-behaviour of solutions to the linear heat and wave equations in exterior domains*, Sci. Math. Japon., **55**, 2002, 33-42.

[4] R. Ikehata, K. Nishihara, *Local energy decay for wave equations with initial data decaying slowly near infinity*, GAKUTO International series, Math. Sci. Appl., **22**, 2005, 265-275.

[5] R. Ikehata, H. Takeda, *Uniform energy decay for wave equations with unbounded damping coefficients*, Funk. Ekvac., **63(1)**, 2020 (in press).

[6] R. Ikehata, G. Todorova, B. Yordanov, *Optimal decay rate of the energy for wave equations with critical potential*, J. Math. Soc. Japan, **65**, 2013, 183-236.

[7] S. Jiang, *L^p-L^q estimates for solutions to the damped plate equation in exterior domains*, Results in Math., **18**, 1990, 231-253.

[8] J.E. Lin, *Local time decay for a nonlinear beam equation*, Methods and Applications of Analysis, **11**, 2004, 65-68.

[9] K. Mochizuki, *Scattering theory for wave equations with dissipative terms*, Publ. Res. Inst. Math. Sci., **12**, 1976, 383-390.

[10] K. Mochizuki, H. Nakazawa, *Energy decay and asymptotic behavior of solutions to the wave equations with linear dissipation*, Publ. Res. Inst. Math. Sci., **32**, 1996, 401-414.

[11] C. Morawetz, *The decay of solutions of the exterior initial-boundary value problem for the wave equation*, Comm. Pure Appl. Math., **14**, 1961, 561-568.

[12] C. Morawetz, *Time decay for the nonlinear Klein-Gordon equation*, Proc. Roy. Soc. A, **306**, 1968, 291-296.

[13] L.A. Muraveĭ, *The wave equation in an unbounded domain with a star-shaped boundary*, Soviet Math. Dokl., **38**, 1989, 527-530.

[14] M. Nakao, *Stabilization of local energy in an exterior domain for the wave equation with a localized dissipation*, J. Diff. Eqns., **148**, 1998, 388-406.

[15] G. Todorova, B. Yordanov, *Critical exponent for a nonlinear wave equation with damping*, J. Diff. Eqns., **174**, 2001, 464-489.

[16] G. Todorova, B. Yordanov, *Weighted L^2-estimates of dissipative wave equations with variable coefficients*, J. Diff. Eqns., **246**, 2009, 4497-4518.

[17] M. Wiegner, *Decay of the L_∞-norm of solutions of Navier-Stokes equations in unbounded domains*, Acta Appl. Math., **37**, 1994, 215-219.

[18] E.C. Zachmanoglou, *The decay of solutions of the initial-boundary value problem for hyperbolic equations*, J. Math. Anal. Appl., **13**, 1966, 504-515.

[19] Y. Wakasugi, *On diffusion phenomena for the linear wave equation with space-dependent damping*, J. Hyperbolic Diff. Eqns., **11**, 2014, 795-819.

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