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Unified Approach to Fully Fourth Order Nonlinear Problems

Q.A. Dang*, T.H. Nguyen, H.H. Truong

Abstract. In this paper we propose a unified approach to investigate boundary value problems for fully fourth order nonlinear differential equation. It is based on the reduction of the problems to operator equations for the nonlinear terms, but not for the sought functions. By this approach we establish the existence, uniqueness, positivity and convexity of solutions of the problems using different methods under the conditions which are much simpler and weaker than those in known works. The theoretical results are illustrated on examples.

Key Words and Phrases: fully fourth order nonlinear equation, existence and uniqueness of solution, positivity of solution, fixed point theorems.

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1. Introduction

In recent years the boundary value problems for fully fourth order nonlinear differential equations

$$u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \quad 0 < x < 1, \tag{1}$$

have attracted attention from many researchers. A number of works have been dedicated to the existence of solutions of problems with different boundary conditions. The methods employed in these works are the method of Leray-Schauder degree theory [21]; the fixed point index theory in cones [14]; the method of lower and upper solutions with the use of the Schauder fixed point theorem [1, 11, 12], the degree theory [20] or new maximum principles [24]; Fourier analysis [15] and the reproducing kernel theorem [10]. It should be emphasized that in all the above works except for [10] there is an essential assumption that the function

*Corresponding author.

$f : [0,1] \times \mathbb{R}^4 \to \mathbb{R}$ satisfies a Nagumo-type condition. Meanwhile the work [10] requires the boundedness of the function f and its partial derivatives in $[0,1] \times \mathbb{R}^4$.

Very recently, in [3, 4, 5, 6] we proposed a new method for investigating simultaneously existence and uniqueness of a solution of the equation (1) with one of the following sets of boundary conditions:

$$u(0) = u'(0) = u''(1) = u'''(1) = 0, \tag{2}$$

$$u(0) = u''(0) = u(1) = u''(1) = 0 \tag{3}$$

$$u(0) = 0, u'(1) = 0, au''(0) - bu'''(0) = 0, cu''(1) + du'''(1) = 0, \tag{4}$$

where $a, b, c, d \geq 0$. The idea of the method is to reduce the boundary value problems to operator equations for the nonlinear term or the right-hand side of the equation and to apply the contraction principle to the latter ones. There we needed the Lipschitz condition for the function $f(x, u, y, v, z)$ only in a bounded domain. Note that the idea of the reduction of boundary value problems to operator equations for the right-hand sides originated from our work [2] when considering a boundary value problem for biharmonic type equation.

In this paper, we study the solvability and properties of solutions for the equation (1) with the more general boundary conditions, which include as particular cases the boundary conditions (2), (3), (4) and the sets of the conditions

$$u(0) = u'(1) = u''(0) = u'''(1) = 0, \tag{5}$$

$$u(0) = u'(1) = u''(1) = u'''(0) = 0, \tag{6}$$

$$u(1) = u'(0) = u''(0) = u'''(1) = 0, \tag{7}$$

$$u(1) = u'(0) = u''(1) = u'''(0) = 0, \tag{8}$$

$$u(0) = u'(0) = u''(0) = u'''(1) = 0, \tag{9}$$

$$u(0) = u'(0) = u''(1) = u'''(0) = 0, \tag{10}$$

$$u(1) = u'(1) = u''(0) = u'''(1) = 0, \tag{11}$$

$$u(1) = u'(1) = u''(1) = u'''(0) = 0 \tag{12}$$

mentioned in [20]. The tools that we use are the Schauder fixed point theorem for compact operators and the contraction mapping principle. The existence, uniqueness, positivity and convexity of solutions are established under easily verified conditions. Many examples illustrate the obtained theoretical results.

2. Solvability and properties of solution

For brevity we denote

$$\begin{aligned}\bar{u} &= \big(u(0), u(1), u'(0), u'(1)\big),\\ \bar{\bar{u}} &= \big(u''(0), u''(1), u'''(0), u'''(1)\big).\end{aligned}$$

Consider the boundary value problem

$$u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \quad 0 < x < 1, \tag{13}$$

$$\begin{aligned}B_1(\bar{u}) = 0,\ B_2(\bar{u}) = 0,\\ B_3(\bar{\bar{u}}) = 0,\ B_4(\bar{\bar{u}}) = 0,\end{aligned} \tag{14}$$

where B_1, B_2, B_3, B_4 are linear combinations of the components of arguments.
We shall associate this problem with a fixed point problem as follows.
For functions $\varphi(x) \in C[0,1]$ consider the nonlinear operator defined by

$$(A\varphi)(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \tag{15}$$

where $u(x)$ is a solution of the equation

$$u^{(4)}(x) = \varphi(x), \quad 0 < x < 1, \tag{16}$$

satisfying the boundary conditions (14).
It is easy to verify the following

Proposition 1. *If the function $\varphi(x)$ is a fixed point of the operator A, i.e., $\varphi(x)$ is a solution of the operator equation*

$$\varphi = A\varphi, \tag{17}$$

then the function $u(x)$ determined from the boundary value problem (16), (14) *solves the problem* (13)-(14)*. Conversely, if $u(x)$ is a solution of the boundary value problem* (13)-(14)*, then the function*

$$\varphi(x) = f(x, u(x), u'(x), u''(x), u'''(x))$$

is a fixed point of the operator A defined above by (15)*,* (16) *and* (14)*.*

Thus, the solution of the original problem (13)-(14) is reduced to the solution of the operator equation (17). This is the technique used in our previous works [3, 4, 5] for some particular cases of the problem (13)-(14) but not formulated as a separate proposition.

Now in the problem (13)-(14) we set $v(x) = u''(x)$. Then it is decomposed into two second order problems

$$\begin{cases} v''(x) = \varphi(x), \quad 0 < x < 1, \\ B_3(\bar{v}) = 0,\ B_4(\bar{v}) = 0, \end{cases} \tag{18}$$

$$\begin{cases} u''(x) = v(x), \quad 0 < x < 1, \\ B_1(\bar{u}) = 0,\ B_2(\bar{u}) = 0, \end{cases}$$

where $\bar{v} = \big(v(0), v(1), v'(0), v'(1)\big)$. Suppose the Green functions of the problems (16), (14) and (18) exist and are denoted by $G(x,t)$ and $\widehat{G}(x,t)$, respectively. Then their solutions are represented in the forms

$$u(x) = \int_0^1 G(x,t)\varphi(t)dt, \quad v(x) = \int_0^1 \widehat{G}(x,t)\varphi(t)dt. \tag{19}$$

From the differentiability property of Green functions we have

$$u'(x) = \int_0^1 G_1(x,t)\varphi(t)dt, \quad v'(x) = \int_0^1 \widehat{G}_1(x,t)\varphi(t)dt, \tag{20}$$

where $G_1(x,t) = G'_x(x,t)$ is a function continuous in the square $[0,1]^2$ and $\widehat{G}_1(x,t) = \widehat{G}'_x(x,t)$ is continuous in the square $[0,1]^2$ except for the line $t = x$. Further, suppose that for any $0 \le x \le 1$ there hold the estimates

$$\begin{aligned} &\int_0^1 |G(x,t)|dt \le M_0,\ \int_0^1 |G_1(x,t)|dt \le M_1, \\ &\int_0^1 |\widehat{G}(x,t)|dt \le M_2,\ \int_0^1 |\widehat{G}_1(x,t)|dt \le M_3, \end{aligned} \tag{21}$$

where M_0, M_1, M_2 and M_3 are some constants.
Also, denote

$$y(x) = u'(x), \quad z(x) = v'(x).$$

Then from (20) we have

$$y(x) = \int_0^1 G_1(x,t)\varphi(t)dt, \quad z(x) = \int_0^1 \widehat{G}_1(x,t)\varphi(t)dt.$$

Next, for each real number $M > 0$ introduce the domain

$$\begin{aligned} \mathcal{D}_M = \{(x,u,y,v,z)|\ 0 \le x \le 1,\ |u| \le M_0 M,\ |y| \le M_1 M, \\ |v| \le M_2 M,\ |z| \le M_3 M\}, \end{aligned}$$

and as usual, by $B[O,M]$ we denote

$$B[O,M] = \{\varphi \in C[0,1] |\ \|\varphi\| \le M\},$$

where

$$\|\varphi\| = \max_{0\le x\le 1} |\varphi(x)|.$$

Theorem 1 (Existence of solutions)**.** *Suppose that there exists a number $M > 0$ such that the function $f(x,u,y,v,z)$ is continuous and bounded by M in the domain $\mathcal{D}_M$, i.e.,*

$$|f(x,u,y,v,z)| \le M, \tag{22}$$

for any $(x,u,y,v,z) \in \mathcal{D}_M$.

Then, the problem (13)-(14) *has a solution $u(x)$ satisfying $|u(x)| \le M_0M$, $|u'(x)| \le M_1M$, $|u''(x)| \le M_2M$, $|u'''(x)| \le M_3M$ for any $0 \le x \le 1$.*

Proof. By Proposition 1, the problem (13)-(14) is reduced to the operator equation (17). Therefore, the existence of solution of the problem will be proved if we show that this associated operator equation has a solution. For this purpose, we first show that the operator A defined by (15), (16) and (14) maps any closed ball $B[O,M]$ into itself.
Indeed, for any $\varphi \in B[O,M]$ from the representations of the solution $u(x)$ of the problem (13)-(14) and its derivatives $u'(x), u''(x) = v(x), u'''(x) = v'(x)$ by the formulas (19)-(20) and the estimates (21) we have

$$|u(x)| \le M_0M,\ |u'(x)| \le M_1M,$$
$$|u''(x)| \le M_2M,\ |u'''(x)| \le M_3M$$

for any $0 \le x \le 1$. Therefore, for any $0 \le x \le 1$ we have $(x, u(x), u'(x), u''(x), u'''(x)) \in \mathcal{D}_M$, and consequently, by the assumption (22)

$$|A\varphi(x)| = |f(x, u(x), u'(x), u''(x), u'''(x))| \le M.$$

It follows that $\|A\varphi\| \le M$ and thus, the operator A maps the ball $B[O,M]$ into itself.

Next, we prove that the operator A is compact one in the space $C[0,1]$.
According to [13, Sec. 31] (see APPENDIX) the integral operators (19),(20) which put each function $\varphi \in C[0,1]$ in correspondence to the functions u, u', u'', u''', respectively, are compact operators. Therefore, in view of the continuity of the function $f(x,u,y,v,z)$ it is easy to see that the operator A defined by (15) is a compact operator in the space $C[0,1]$. Thus, the operator A is a compact one

mapping the closed ball $B[O, M])$ into itself. By the Schauder Fixed Point Theorem [22], the operator equation (17) has a solution. The theorem is proved. ◀

Now suppose that one or more of the Green functions $G(x,t), \widehat{G}(x,t)$ and their derivatives $G_1(x,t), \widehat{G}_1(x,t)$ are of constant signs in the square $[0,1]^2$. In order to investigate the existence and properties of positive solutions of the problem (13)-(14) we introduce the notation

$$S_M = \{\varphi \in C[0,1] |\ 0 \le \varphi \le M\}.$$

Theorem 2 (Existence of positive solutions)**.** *Suppose we are given the Green function $G(x,t) \ge 0$ in the square $Q = [0,1]^2$ and there exists a number $M > 0$ such that the function $f(x,u,y,v,z)$ is continuous and*

$$0 \le f(x,u,y,v,z) \le M$$

for any $(x,u,y,v,z) \in \mathcal{D}_M^{u^+}$, where

$$\begin{aligned} \mathcal{D}_M^{u^+} = \{(x,u,y,v,z) |\ & 0 \le x \le 1,\ 0 \le u \le M_0 M, \\ & |y| \le M_1 M,\ |v| \le M_2 M,\ |z| \le M_3 M\}. \end{aligned}$$

Besides, suppose that $f(x,0,0,0,0) \not\equiv 0$. Then, the problem (13)-(14) *has a positive solution $u(x)$ satisfying*

$$\begin{gathered} 0 \le u(x) \le M_0 M,\ |u'(x)| \le M_1 M, \\ |u''(x)| \le M_2 M,\ |u'''(x)| \le M_3 M \end{gathered}$$

for $0 \le x \le 1$.

Proof. Similarly to the proof of Theorem 1, where instead of $\mathcal{D}_M$ and $B[O,M]$ there stand $\mathcal{D}_M^{u^+}$ and S_M, we conclude that the problem (13)-(14) has a nonnegative solution. Due to the condition $f(x,0,0,0,0) \not\equiv 0$, this solution must be positive. ◀

In a similar way it is easy to prove the following results.

Theorem 3 (Existence of positive monotone solutions)**.** *Suppose that $G(x,t) \ge 0$, $G_1(x,t) \ge 0$ ($G_1(x,t) \le 0$) in the square $Q = [0,1]^2$. Further, suppose that there exists a number $M > 0$ such that the function $f(x,u,y,v,z)$ is continuous and*

$$0 \le f(x,u,y,v,z) \le M$$

for any $(x,u,y,v,z) \in \mathcal{D}_M^{u^+y^+}$ $(\mathcal{D}_M^{u^+y^-})$, *where*

$$\begin{aligned}\mathcal{D}_M^{u^+y^+} &= \{(x,u,y,v,z)|\ 0 \le x \le 1,\ 0 \le u \le M_0M,\\ &\quad 0 \le y \le M_1M,\ |v| \le M_2M,\ |z| \le M_3M\},\end{aligned}$$

$$\begin{aligned}\mathcal{D}_M^{u^+y^-} &= \{(x,u,y,v,z)|\ 0 \le x \le 1,\ 0 \le u \le M_0M,\\ &\quad -M_1M \le y \le 0,\ |v| \le M_2M,\ |z| \le M_3M\}.\end{aligned}$$

Besides, suppose that $f(x,0,0,0,0) \not\equiv 0$. *Then, the problem* (13)-(14) *has a positive, increasing (decreasing) solution* $u(x)$ *satisfying*

$$\begin{aligned}0 \le u(x) \le M_0M,\ 0 \le u'(x) \le M_1M,\ (-M_1M \le u'(x) \le 0)\\ |u''(x)| \le M_2M,\ |u'''(x)| \le M_3M\end{aligned}$$

for $0 \le x \le 1$.

Theorem 4 (Existence of positive convex solutions)**.** *Suppose that* $G(x,t) \ge 0$, $\widehat{G}(x,t) \ge 0$ $(\widehat{G}(x,t) \le 0)$ *in the square* $Q = [0,1]^2$. *Further, suppose that there exists a number* $M > 0$ *such that the function* $f(x,u,y,v,z)$ *is continuous and*

$$0 \le f(x,u,y,v,z) \le M$$

for any $(x,u,y,v,z) \in \mathcal{D}_M^{u^+v^+}$ $(\mathcal{D}_M^{u^+v^-})$, *where*

$$\begin{aligned}\mathcal{D}_M^{u^+v^+} &= \{(x,u,y,v,z)|\ 0 \le x \le 1,\ 0 \le u \le M_0M,\\ &\quad |y| \le M_1M,\ 0 \le v \le M_2M,\ |z| \le M_3M\},\end{aligned}$$

$$\begin{aligned}\mathcal{D}_M^{u^+v^-} &= \{(x,u,y,v,z)|\ 0 \le x \le 1,\ 0 \le u \le M_0M,\\ &\quad |y| \le M_1M,\ -M_2M \le v \le 0,\ |z| \le M_3M\}.\end{aligned}$$

Besides, suppose that $f(x,0,0,0,0) \not\equiv 0$. *Then, the problem* (13)-(14) *has a positive, convex (concave) solution* $u(x)$ *satisfying*

$$\begin{aligned}0 \le u(x) \le M_0M,\ |u'(x)| \le M_1M,\\ 0 \le u''(x) \le M_2M\,(-M_2M \le u''(x) \le 0),\ |u'''(x)| \le M_3M\end{aligned}$$

for $0 \le x \le 1$.

Theorem 5 (Uniqueness of solution)**.** *Suppose that the function* $f(x,u,y,v,z)$ *satisfies the Lipschitz condition in variables* u,y,v,z. *Namely, there exist numbers* $c_0,c_1,c_2,c_3 \ge 0$ *such that*

$$|f(x,u_2,y_2,v_2,z_2) - f(x,u_1,y_1,v_1,z_1)| \leq c_0|u_2-u_1| + c_1|y_2-y_1| + c_2|v_2-v_1| + c_3|z_2-z_1| \quad (23)$$

for any $(x,u_i,y_i,v_i,z_i) \in [0,1]\times\mathbb{R}^4$ $(i=1,2)$ *and*

$$q := c_0M_0 + c_1M_1 + c_2M_2 + c_3M_3 < 1. \quad (24)$$

Then the solution of the problem (13)-(14) *is unique if it exists.*

Proof. Suppose the problem has two solutions $u_1(x)$ and $u_2(x)$. Set

$$v_i = u_i'',\ y_i = u_i',\ z_i = v_i',\ \varphi_i = f(x,u_i,y_i,v_i,z_i)\ (i=1,2).$$

Then, using the representations of the types (19), (20) for u_i, y_i, v_i, z_i and the estimates (21) we obtain

$$\begin{aligned} \|u_2-u_1\| &\leq M_0\|\varphi_2-\varphi_1\|, \quad \|y_2-y_1\| \leq M_1\|\varphi_2-\varphi_1\|, \\ \|v_2-v_1\| &\leq M_2\|\varphi_2-\varphi_1\|, \quad \|z_2-z_1\| \leq M_3\|\varphi_2-\varphi_1\|. \end{aligned} \quad (25)$$

From (23) and the above estimates we have

$$\begin{aligned} |\varphi_2-\varphi_1| &= |f(x,u_2,y_2,v_2,z_2) - f(x,u_1,y_1,v_1,z_1)| \\ &\leq c_0|u_2-u_1| + c_1|y_2-y_1| + c_2|v_2-v_1| + c_3|z_2-z_1| \\ &\leq (c_0M_0 + c_1M_1 + c_2M_2 + c_3M_3)\|\varphi_2-\varphi_1\|. \end{aligned}$$

It follows that

$$\|\varphi_2-\varphi_1\| \leq q\|\varphi_2-\varphi_1\|$$

with q defined by (24). Since $q<1$, the above inequality occurs only in the case $\varphi_2=\varphi_1$. This implies $u_2=u_1$ due to (25). The theorem is proved. ◄

Theorem 6 (Existence and uniqueness of solution)**.** *Assume that there exist numbers* $M, c_0, c_1, c_2, c_3 \geq 0$ *such that*

$$|f(x,u,y,v,z)| \leq M, \quad (26)$$

$$|f(x,u_2,y_2,v_2,z_2,) - f(x,u_1,y_1,v_1,z_1)| \leq c_0|u_2-u_1| + c_1|y_2-y_1| + c_2|v_2-v_1| + c_3|z_2-z_1| \quad (27)$$

for any $(x,u,y,v,z), (x,u_i,y_i,v_i,z_i) \in \mathcal{D}_M$ $(i=1,2)$ *and*

$$q := c_0M_0 + c_1M_1 + c_2M_2 + c_3M_3 < 1.$$

Then, the problem (13)-(14) *has a unique solution* $u(x)$ *such that* $|u(x)| \leq M_0M$, $|u'(x)| \leq M_1M$, $|u''(x)| \leq M_2M$, $|u'''(x)| \leq M_3M$ *for any* $0 \leq x \leq 1$.

Proof. Under the assumption (26), as proven in Theorem 1, the operator A, defined by (15), maps the closed ball $B[O,M]$ into itself. Besides, for $\varphi_1, \varphi_2 \in B[O,M]$, as shown in the same theorem, $(x, u_i(x), u_i'(x), u_i''(x),\ u_i'''(x)) \in \mathcal{D}_M$, where $u_i(x)$ is the solution of the problem (16), (14) with $\varphi = \varphi_i$ $(i = 1, 2)$. Therefore, using the Lipschitz condition (27) in $\mathcal{D}_M$, as in Theorem 5, we obtain

$$|(A\varphi_2)(x) - (A\varphi_1)(x)| = |f(x, u_2, y_2, v_2, z_2) - f(x, u_1, y_1, v_1, z_1)| \le q\|\varphi_2 - \varphi_1\|$$

for all $x \in [0, 1]$. This implies that $\|A\varphi_2 - A\varphi_1\| \le q\|\varphi_2 - \varphi_1\|$. So, with the assumption $q < 1$, A is a contraction mapping in $B[O,M]$. By the contraction principle, the operator A has a unique fixed point in $B[O,M]$, which corresponds to the unique solution $u(x)$ of the problem (13)-(14).
The estimations for $u(x)$ and its derivatives are obtained as in Theorem 1. Thus, Theorem 6 is proved. ◀

Remark 1. *In Theorem 5 the Lipschitz condition is required to be satisfied in* $[0, 1] \times \mathbb{R}^4$*, while in Theorem 6 under the condition* (26) *it is required only in* $\mathcal{D}_M$*.*

Combining Theorems 2, 3 and 4 with Theorem 5, we obtain the following theorem.

Theorem 7 (Existence of unique solution and its properties)**.** *Suppose that the Lipshitz condition* (23)-(24) *is satisfied in corresponding domains together with the conditions of Theorem 2/ Theorem 3/Theorem 4. Then the problem* (13)-(14) *has a unique positive solution/ a unique positive monotone solution/ a unique positive convex (concave) solution, which satisfies the corresponding estimates as in Theorems 2, 3 and 4.*

3. Some particular cases

3.1. The problem (1), (2)

Consider the equation (1) with the boundary conditions (2). In [14], the existence of a positive solution of this problem was established under many conditions posed on the positive function $f(x, u, y, v, z)$, including a growth condition on u, y, v, z at infinity and a Nagumo-type condition on v and z. In 2017, Zhou [23] also studied the problem (1)-(2) by using the method of order reduction and the fixed point index. He established the existence of positive solutions under some conditions which are difficult to verify. Maybe, this is the reason that no examples are shown for illustration of theoretical results. Also in 2017, Wei and Li in [24] using the lower and upper solutions method and a new maximum principle established the existence and uniqueness of positive solution. In a recent work

[4], freeing all growth conditions but requiring the Lipschitz condition only in a bounded domain we established the existence and uniqueness of a nonnegative solution.

As was shown in [4]

$$G(x,t) = \begin{cases} -\dfrac{t^3}{6} + \dfrac{t^2 x}{2}, & 0 \le t \le x \le 1 \\ -\dfrac{x^3}{6} + \dfrac{x^2 t}{2}, & 0 \le x \le t \le 1. \end{cases}$$

$$\widehat{G}(x,t) = \begin{cases} 0, & 0 \le t \le x \le 1, \\ t - x, & 0 \le x \le t \le 1. \end{cases}$$

$$G_1(x,t) = G'_x(x,t),\ \widehat{G}_1(x,t) = \widehat{G}'_x(x,t),$$

$$M_0 = \frac{1}{8},\ M_1 = \frac{1}{6},\ M_2 = \frac{1}{2},\ M_3 = 1.$$

It is easy to verify that $G(x,t), \widehat{G}(x,t), G_1(x,t) \ge 0$ and $\widehat{G_1}(x,t) \le 0$ for $0 \le x,t \le 1$.

Example 1. *(Example 4.1 in [4]) Consider the problem*

$$\begin{cases} u^{(4)}(x) = -\dfrac{3u'u'''}{1152} + \dfrac{(u'')^2}{576} + \dfrac{x}{4} + \dfrac{95}{4}, & 0 < x < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases}$$

It is easy to verify that for this problem all conditions of Theorem 7 are satisfied with $M = 25$. Therefore, the problem has a unique positive, increasing and convex solution. This solution is $u(x) = x^4 - 4x^3 + 6x^2$ with the above properties.

Example 2. *Consider the boundary value problem*

$$\begin{cases} u^{(4)}(x) = u^2 + |u'|^{\frac{1}{2}} + |u''|^{\frac{1}{2}} + |u'''|^{\frac{1}{2}} + 1, & 0 < x < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases}$$

In this example $f(x,u,y,v,z) = u^2 + |y|^{\frac{1}{2}} + |v|^{\frac{1}{2}} + |z|^{\frac{1}{2}} + 1$. It is possible to verify that for $M = 9$ all the conditions of Theorem 2 are satisfied. Therefore, the problem has a positive solution. Note that neither the condition $F1$ of [14, Theorem 3.1] nor the condition $F4$ of [14, Theorem 3.2] is satisfied, so the results of Li [14] cannot ensure the existence of positive solutions.

Example 3. *(Example 1 in [24]) Consider the boundary value problem*

$$\begin{cases} u^{(4)}(x) = \frac{1}{3}\sin x.u(x) + \frac{1}{3}\cos x.u'(x) + \frac{1}{2}e^x, & 0 < x < 1, \\ u(0) = u'(0) = u''(1) = u'''(1) = 0. \end{cases}$$

In [24] the authors obtained the result that the problem has a unique solution $u(x)$ *which satisfies* $0 \le u(x) \le e^x,\ 0 \le u'(x) \le e^x$.

It is easy to verify that for $M = \dfrac{36}{65} = 1.5055$ *all the conditions of Theorem 3 and Theorem 6 are satisfied with* $q < \frac{7}{72}$. *Therefore, the problem has a unique positive increasing solution satisfying* $0 \le u(x) \le \frac{M}{8} = 0.1882,\ 0 \le u'(x) \le \frac{M}{6} = 0.2509$. *Clearly, this result is much better than that in [24].*

3.2. The problem (1), (3)

Now we consider the equation (1) with the boundary conditions (3). The existence of a solution for this problem was studied by Li and Liang in [15] by the Fourier analysis method and Leray-Schauder fixed point theorem. A growth condition and the Lipschitz condition on the function $f(x,u,y,v,z)$ are imposed. Very recently, in [5], following the method used for the equation $u^{(4)}(x) = f(x, u(x), u''(x))$, we obtained the results for existence and uniqueness of a solution and the convergence of an iterative method. But in that work, disusing the first representation in (19), we obtained the estimate $\|u\| \le \frac{1}{64}\|\varphi\|$. Here, applying the general methodology presented in the previous section to the problem (1), (3), we obtain a somewhat better estimate: $\|u\| \le \frac{5}{384}\|\varphi\|$.

For this problem

$$G(x,t) = \frac{1}{6}\begin{cases} x(t-1)(t^2+x^2-2t), & 0 \le x \le t \le 1, \\ t(x-1)(t^2+x^2-2x), & 0 \le t \le x \le 1, \end{cases}$$

$$\widehat{G}(x,t) = \begin{cases} x(t-1), & 0 \le x \le t \le 1, \\ t(x-1), & 0 \le t \le x \le 1, \end{cases}$$

$$G_1(x,t) = G'_x(x,t),\ \widehat{G}_1(x,t) = \widehat{G}'_x(x,t).$$

It is easy to verify that $G(x,t) \ge 0$ and $\widehat{G}(x,t) \le 0$ for $0 \le x, t \le 1$ and

$$M_0 = \frac{5}{384},\ M_1 = \frac{1}{24},\ M_2 = \frac{1}{8},\ M_3 = \frac{1}{2}.$$

Therefore, with a suitable selection of $M > 0$ Theorem 4 guarantees the existence of a positive concave solution of the problem (1), (3) without the Lipschitz condition and any growth conditions on f at infinity.

Example 4. *Consider the problem (see [16])*

$$u^{(4)} = u^m - \frac{1}{\pi^{10}}(u'')^5 + \sin \pi x, \quad m \geq 1$$

In this example $f(x,u,y,v,z) = u^m - \dfrac{1}{\pi^{10}}v^5 + \sin \pi x$. *In order to satisfy the conditions of Theorem 4 for all* $m \geq 1$, *we can take* $M = 1.1$. *Hence, we conclude that the problem has a unique positive concave solution. It should be noted that in [16] the authors only conclude that this example has at least one solution, with no judgement about its uniqueness.*

Example 5. *Consider the problem (see [5, Example 4.6])*

$$\begin{cases} u^{(4)}(x) = \pi|u|^{\frac{1}{2}} + \pi|u'|^{\frac{1}{2}} + |u''|^{\frac{1}{2}} + |u'''|^{\frac{1}{2}} + 1, \quad 0 < x < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

In this example $f(x,u,y,v,z) = \pi|u|^{\frac{1}{2}} + \pi|y|^{\frac{1}{2}} + |v|^{\frac{1}{2}} + |z|^{\frac{1}{2}} + 1$. *It is easy to verify that for* $M = 2.5$ *all the conditions of Theorem 4 are satisfied. Therefore, by this theorem the problem has a positive concave solution. Note that in [5] we could not conclude the existence of solutions, but by an iterative method we found a positive, concave solution. Besides, the results of [16] do not apply to this problem.*

Example 6. *Consider the problem (see [17, Remark 3.3]*

$$\begin{cases} u^{(4)}(x) = -3u + u^3 + 4u'' + \sin x, \quad 0 < x < 1, \\ u(0) = u(1) = u''(0) = u''(1) = 0. \end{cases}$$

In [17] by the method of lower and upper solutions the authors showed that the problem has a solution u *satisfying* $-1 \leq u(x) \leq 1,\ x \in [0,1]$. *Below, using the theoretical results in the previous section we shall obtain better results.*

Indeed, for this function we have $f(x,u,y,v,z) = -3u + u^3 + 4v + \sin x$. *It is easy to verify that for* $M = 2.5$ *all the conditions of Theorem 6 are satisfied. Therefore, the problem has a unique solution satisfying the estimates*

$$|u(x)| \leq 0.0326,\ |u'(x)| \leq 0.1042,\ |u''(x)| \leq 0.3125,\ |u'''(x)| \leq 1.25.$$

Clearly, these results are better than the result in [17].

3.3. The problem (1), (4)

Consider now the equation (1) with the boundary conditions (4).
In [12], by using the upper and lower solutions method, the authors established

the existence and uniqueness of solution of this problem. In this work, the nonlinear function $f(x,u,y,v,z)$ was assumed to satisfy a Nagumo-type condition with respect to α'', β'' (where α, β are the upper and lower solutions of the problem such that $\alpha'' \le \beta''$) and to be decreasing in x, y and strictly increasing in z.

For this problem we have (see [6])

$$G(x,t) = \frac{1}{6\rho}\begin{cases} x(ac(1-t)(3t-x^2)+ad(6t-x^2-3t^2) \\ +bc(1-t)(3+3t-3x)+bd(6-3x)), \quad 0 \le x \le t \le 1, \\ ac(3xt-3x^2t+x^3t-t^3)+ad(6xt-3x^2t-t^3) \\ +bc(3x-3x^2+x^3-t^3)+bd(6x-3x^2), \quad 0 \le t \le x \le 1, \end{cases}$$

$$\widehat{G}(x,t) = \frac{1}{\rho}\begin{cases} (ct-c-d)(b+ax), & 0 \le x \le t \le 1, \\ (b+at)(cx-c-d), & 0 \le t \le x \le 1, \end{cases} \tag{28}$$

$$G_1(x,t) = G'_x(x,t) = \frac{1}{2\rho}\begin{cases} ac(1-t)(t-x^2)+ad(2t-x^2-t^2) \\ +bc(1-t)(1+t-2x)+bd(2-2x), \\ \qquad\qquad 0 \le x \le t \le 1, \\ act(1-x)^2+ad(2t-2tx)+bc(1-x)^2 \\ +bd(2-2x), \quad 0 \le t \le x \le 1, \end{cases} \tag{29}$$

$$\widehat{G}_1(x,t) = \widehat{G}'_x(x,t) = \frac{1}{\rho}\begin{cases} a(ct-c-d), & 0 \le x \le t \le 1, \\ c(b+at), & 0 \le t \le x \le 1, \end{cases} \tag{30}$$

It is easy to verify that $G(x,t), G_1(x,t) \ge 0$ and $\widehat{G}(x,t) \le 0$ for $0 \le x,t \le 1$ and

$$M_0 = \frac{1}{24} + \frac{2ad+bc+6bd}{12\rho}, \quad M_1 = \frac{1}{12} + \frac{ad+bc+4bd}{4\rho}, \tag{31}$$

$$M_2 = \frac{1}{2}\Big(\frac{a(d+c/2)}{\rho}\Big)^2 + \frac{b(d+c/2)}{\rho}, \quad M_3 = \frac{1}{\rho}\Big(\frac{ac}{2} + \max(ad, bc)\Big). \tag{32}$$

Therefore, if a suitable $M > 0$ is selected, the above Theorems 3, 4 guarantee the existence of a positive, increasing and concave solution of the problem (1), (4) without the Lipschitz condition and any growth conditions on f at infinity.

Example 7. *Consider the problem (Example 2 in [6])*

$$\begin{cases} u^{(4)}(x) = \dfrac{u^2(x)}{4} + \dfrac{u'(x)}{10} - u''(x) + \dfrac{\sin u'''(x)}{10} + \dfrac{x}{10} + 0.1, \quad 0 < x < 1, \\ u(0) = 0, \quad u'(1) = 0, \quad 2u''(0) - u'''(0) = 0, \quad u''(1) + u'''(1) = 0. \end{cases}$$

In this example $a = 2,\ b = 1,\ c = 1,\ d = 1$ *and*

$$f(x,u,y,v,z) = -\frac{u^2}{4} + \frac{y}{10} - v + \frac{\sin z}{10} + \frac{x}{10} + 0.1.$$

It is possible to verify that with $M = 2$ all the conditions of Theorem 7 are satisfied. Therefore, the problem has a unique positive, increasing and concave solution. The numerical solution found by an iterative method in [6] possesses the above properties.

Remark 2. *In [20], the existence of a solution of the problems for the equation* (1) *with one of the boundary conditions* (5)-(12) *were studied by Minhós, Gyulov and Santos by using a a priori estimation, lower and upper solutions method and degree theory. In that work, many conditions of the nonlinear function $f(x,u,y,v,z)$ were required, including a growth condition v, z at infinity and a Nagumo-type condition on z.*

3.4. The problems (1), (5) and (1), (6)

The problem (1), (5) and the problem (1), (6) are particular cases of the problem (1), (4) when $b = c = 0$ and $a = d = 0$.

3.5. The problems (1) with one of the boundary conditions (7)-(12)

Consider the more general problems:

$$\begin{cases} u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \quad 0 < x < 1, \\ u(1) = 0, u'(0) = 0, au''(0) - bu'''(0) = 0, cu''(1) + du'''(1) = 0, \end{cases} \tag{33}$$

$$\begin{cases} u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \quad 0 < x < 1, \\ u(0) = 0, u'(0) = 0, au''(0) - bu'''(0) = 0, cu''(1) + du'''(1) = 0, \end{cases} \tag{34}$$

$$\begin{cases} u^{(4)}(x) = f(x, u(x), u'(x), u''(x), u'''(x)), \quad 0 < x < 1, \\ u(1) = 0, u'(1) = 0, au''(0) - bu'''(0) = 0, cu''(1) + du'''(1) = 0, \end{cases} \tag{35}$$

where $a, b, c, d \geq 0, \rho := ad + bc + ac > 0$.
We will apply the general methodology presented in the previous section to the above problems. To find the Green functions of the above boundary value problems, we follow the method in [18]. As a result, for the problem (33) we obtain

$$G(x,t) = \frac{1}{6\rho} \begin{cases} ac(1-t)(2t - t^2 - x^3) + ad(t^3 - x^3 + 3t - 3t^2) \\ +bc(1-t)(2 + 2t - t^2 - 3x^2) + bd(3 - 3x^2), \quad 0 \leq x \leq t \leq 1, \\ (1-x)\Big(ac(2t + 2xt - 3t^2 - x^2 t) + ad(3t + 3xt - 3t^2) \\ +bc(2 + 2x - 3t^2 - x^2) + bd(3 + 3x)\Big), \quad 0 \leq t \leq x \leq 1, \end{cases}$$

$$G_1(x,t) = G'_x(x,t) = \frac{1}{2\rho} \begin{cases} acx^2(t-1) - adx^2 + 2bcx(t-1) - 2bdx, \\ \qquad\qquad 0 \leq x \leq t \leq 1, \\ ac(t^2 + x^2 t - 2xt) + ad(t^2 - 2xt) \\ +bc(x^2 + t^2 - 2x) - 2bdx, \quad 0 \leq t \leq x \leq 1. \end{cases} \tag{36}$$

The functions $\widehat{G}(x,t), \widehat{G}_1(x,t)$ are defined by (28), (30), respectively,

$$M_0 = \frac{1}{24} + \frac{ad+2bc+6bd}{12\rho},$$

and M_1, M_2, M_3 are given in (31), (32), respectively.

For the problem (34) we have

$$G(x,t) = \frac{1}{6\rho}\begin{cases} x^2\Big(acx(t-1) - adx + 3bc(t-1) - 3bd\Big), \quad 0 \le x \le t \le 1, \\ ac(3xt^2 + x^3t - t^3 - 3x^2t) + ad(3xt^2 - 3x^2t - t^3) \\ +bc(3xt^2 - t^3 - 3x^2 + x^3) - 3bdx^2, \quad 0 \le t \le x \le 1, \end{cases}$$

$G_1(x,t), \widehat{G}(x,t), \widehat{G}_1(x,t)$ are defined by (36), (28), (30), respectively,

$$M_0 = \frac{1}{24} + \frac{ad+2bc+6bd}{12\rho},$$

and M_1, M_2, M_3 are given by (31), (32), respectively.

For the problem (35),

$$G(x,t) = \frac{1}{6\rho}\begin{cases} ac(t-1)(t^2+t+x^3-3xt) + ad(t^3-3xt^2-3t+6xt-x^3) \\ \qquad +bc(t-1)(t^2+t+1+3x^2-3x-3xt) - 3bd(x-1)^2, \\ \qquad\qquad 0 \le x \le t \le 1, \\ (x-1)^2\Big(act(x-1) - 3adt + bc(x-1) - 3bd\Big), \\ \qquad\qquad 0 \le t \le x \le 1, \end{cases}$$

$G_1(x,t), \widehat{G}(x,t), \widehat{G}_1(x,t)$ are defined in (29), (28), (30), respectively,

$$M_0 = \frac{1}{24} + \frac{2ad+bc+6bd}{12\rho}$$

and M_1, M_2, M_3 are defined in (31), (32), respectively.

It is easy to see that without the Lipschitz condition and any growth conditions on f at infinity the above Theorem 1 can guarantee the existence of a solution of the problem (33)-(35).

Note that the problems (1), (7); (1), (9); (1), (11) are particular cases of the problems (33), (34), (35), respectively (when $b = c = 0$) and the problems (1), (8); (1), (10); (1), (12) are also particular cases of the problems (33), (34), (35), respectively (when $a = d = 0$). Therefore, following the method used for the problems (33)-(35), we obtain also the existence of a solution of the problems (1) with one of the boundary conditions (7)-(12).

4. Concluding remarks

In this paper we have proposed an efficient unified method for investigating the existence and properties of solutions such as uniqueness, positivity, monotony and convexity for boundary value problems of fully fourth order equation. Many of the considered problems were investigated by other authors using different methods. It should be emphasized that those authors obtained the existence results under many strong assumptions on the nonlinear term $f(x, u, u', u'', u''')$ including the Nagumo-type conditions and the growth at infinity. These assumptions, in general, are not easy to verify. In addition, the proof of the existence of solutions is often very complicated. Differently from those authors, to study the boundary value problems for fully fourth order equation we reduced them to operator equations for the nonlinear terms, but not for the sought functions. Due to this approach, we have established the existence, uniqueness, positivity, monotony and convexity of solutions of the boundary value problems under some easily verified conditions in bounded domains. The proof of the results, as seen from Section 2, is simple. Many examples illustrated the effectiveness of the proposed method.

It should be noted that in the considered problems *the nonlinear terms are local.* Very recently, in [7], we also successfully applied the above method to a boundary value problem for nonlinear nonlocal fourth order equation, namely, the Kirchhoff type equation. The approach of reducing BVPs for nonlinear ODE and PDE to operator equations for nonlinear terms is also used by ourselves to treat a sixth order nonlinear BVP in [8] and nonlocal biharmonic equation in [9].

The proposed method can be applied to some other problems for ordinary and partial differential equations of higher order, including even and odd orders. This is the direction of our research in the future. Besides, we will apply this method to the systems of coupled beam equations with various boundary conditions including the case of simply supported ends considered in [19].

APPENDIX

In the space $C[a, b]$, consider the operator $y = Ax$ defined by the formula

$$y(t) = \int_a^b K(t, s)x(s)ds.$$

Theorem 8. *(see [13, Sec. 31]) The above formula defines a compact operator in the space $C[a, b]$ if the function $K(t, s)$ is bounded on the square $a \leq t \leq b, a \leq s \leq b$ and all points of discontinuity of the function $K(s, t)$ lie on a finite number* *of curves*

$$s = \varphi_k(t), \quad k = 1, 2...., n,$$

where $\varphi_k(t)$ *are continuous functions.*

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Quang A. Dang
Centre for Informatics and Computing, VAST, 18 Hoang Quoc Viet, Cau Giay, Hanoi, Vietnam
E-mail: dangquanga@cic.vast.vn

Thanh Huong Nguyen
College of Sciences, Thai Nguyen University, Thai Nguyen, Vietnam
E-mail: nguyenthanhhuong2806@gmail.com

Ha Hai Truong
College of Information and Telecommunication Technology, Thai Nguyen University, Thai Nguyen, Vietnam
E-mail: haininhtn@gmail.com

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