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The General Solution of the Homogeneous Riemann Problem in Weighted Smirnov Classes with General Weight

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Abstract. Homogeneous Riemann problem of the theory of analytic functions with a piecewise continuous coefficient in weighted Smirnov classes with a general weight is considered. The conditions on the coefficient of the problem and on the weight function are found, which ensure the construction of a general solution for the homogeneous problem in the corresponding weighted Smirnov classes. Special cases of weight function are considered.
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1. Introduction

Theory of Riemann problems has a long history. These problems probably date back to B.Riemann's [1]. Later D.Hilbert [2,3] also considered them and stated a problem which is now referred to as Riemann-Hilbert problem. In the context of applications to some problems of mechanics and mathematical physics, this field has been significantly developed over the years by well-known mathematicians (see, e.g., [4,5]) and the theory of these problems has been well covered in the literature [6-13]. Note that the methods of this theory are also used in other fields of mathematics such as approximation theory, spectral theory of differential operators, etc. The idea of using Riemann-Hilbert problem in the study of approximation properties of perturbed trigonometric systems belongs to A.V.Bitsadze [26]. This method has been successfully used by S.M.Ponomarev [27,28], E.I.Moiseev [29,30] to establish the basicity of linear phase trigonometric systems for Lebesgue spaces. Further development of this method, used in establishing basis properties of special function systems in Lebesgue spaces, is due to B.T.Bilalov [31-36]. This method allowed B.T.Bilalov to find Riesz basicity criterion for the well-known Kostyuchenko system (see [33,36]) in the space $L_2(0,\pi)$.

The Riemann-Hilbert problems are still of great interest. As the harmonic analysis develops further and new function spaces arise, new statements of Riemann problem appear. For example, in the context of applications to some problems of mechanics and pure mathematics, since recently there arose great interest in the non-standard spaces of functions such as Lebesgue space with variable summability index, Morrey space, grand Lebesgue space, etc. (see, e.g., [14-16]). Various issues of mathematical analysis (such as boundedness of singular integral operators, Riesz potentials, direct and inverse problems of approximation theory with respect to Faber polynomials [22], etc.) are being studied in such spaces. Riemann-Hilbert problems also began to be studied in these spaces in different statements (see, e.g., [17-21]), and many relevant issues still remain unsolved.

Results relevant to the subject of this work can be found in [17-19]. In [18], Riemann-Hilbert boundary value problem has been considered in weighted Smirnov class in a simply connected bounded domain with piecewise smooth boundary, where the weight function is analytically extended inside the domain and has a unique degeneration point on the boundary. In [17], the same problem has been considered in weighted Smirnov classes with the variable summability index. In [19], Riemann problem has been considered in weighted Smirnov space in a domain with piecewise smooth boundary, where the weight has a power form and satisfies the Muckenhoupt condition. Note that the weighted Smirnov classes of analytic functions, where the solutions to boundary value problems are sought, are determined in the mentioned works (and in this work, too) in different ways. Moreover, in [17; 18], the authors consider so-called Riemann-Hilbert problems

$$Re\left[\left(a(\tau)+ib(\tau)\right)\Phi^{+}(\tau)\right]=f(\tau),\tau\in\Gamma,$$

which can be reduced to Riemann problem

$$\Phi^{+}(\tau)+G(\tau)\Phi^{-}(\tau)=f(\tau),\tau\in\Gamma,$$

by means of conformal mapping (see, e.g., [7;12]). Besides, in this work we don't impose Muckenhoupt type condition on the weight function in case of homogeneous problem. Note that the techniques used in this work are different from those used in above-mentioned works.

In this paper, homogeneous Riemann problem of the theory of analytic functions with a piecewise continuous coefficient in weighted Smirnov classes with a general weight is considered. The conditions on the coefficient of the problem and on the weight function are found, which ensure the construction of a general solution of the homogeneous problem in the corresponding weighted Smirnov classes. Special cases of weight function are considered.

2. Necessary information

In this section, we state some notations and facts to be used to obtain our results. By $O_r(z_0)$ we denote a circle of radius r centered at z_0 on the complex plane C, i.e. $O_r(z_0) \equiv \{z \in C : |z - z_0| < r\}$. $|M|$ means the Lebesgue (linear) measure of the set $M \subset \Gamma$, where $\Gamma \subset C$ is some rectifiable curve. $Z-$is the set of integers. By $k \in n : m$ we denote $k \in \{n, n+1, ..., m\}$. $[x]$ means the integer part of the number x. Notation $f(x) \sim g(x)\ , x \in M$, means $\exists \delta > 0 : 0 < \delta \leq \left|\frac{f(x)}{g(x)}\right| \leq \delta^{-1}$, $\forall x \in M$.

Let's give a definition for Carleson curve.

Definition 1. *Jordan rectifiable curve Γ on the complex plane is called Carleson curve or regular curve if*

$$\sup_{z \in \Gamma} |\Gamma \cap O_r(z)| \leq cr\ ,\ \ \forall r > 0\,,$$

where c is a constant independent of r.

For more information about this concept see, e.g., [23-25].

Let Γ be some Jordan rectifiable curve and $\omega(\cdot)$ be a weight function on Γ, i.e. $\omega(\xi) > 0$ for a.e. $\xi \in \Gamma$.

Definition 2. *We will say that the weight function $\omega : \Gamma \to R^+ = (0, +\infty)$ belongs to the Muckenhoupt class $A_p(\Gamma)$ $(p > 1)$, if*

$$\sup_{z\in\Gamma}\sup_{r>0} \left(\frac{1}{r}\int_{\Gamma \cap O_r(z)} \omega(\xi)\ |d\xi|\right) \left(\frac{1}{r}\int_{\Gamma \cap O_r(z)} |\omega(\xi)|^{-\frac{1}{p-1}}\ |d\xi|\right)^{p-1} < +\infty.$$

We will need some facts about the weights $\omega(\cdot)$, which satisfy the Muckenhoupt condition $A_p(\Gamma)$, $1 \leq p \leq +\infty$, on the rectifiable curve Γ. The following statement is true:

Statement 1. *i) If $\omega \in A_p(\Gamma)$, $1 \leq p < +\infty$, then $\omega \in A_q(\Gamma)$, for $q > p$; ii) $\omega \in A_p(\Gamma)$, $1 < p < +\infty$, if and only if $\omega^{-\frac{1}{p-1}} \in A_{p'}(\Gamma)$, $\frac{1}{p} + \frac{1}{p'} = 1$; iii) if $\omega \in A_p(\Gamma)$, $1 < p < +\infty$, then $\omega \in A_q(\Gamma)$, for some $q : 1 < q < p$.*

For more information about this statement see, e.g., [24].

We will also use the following statement by R.R. Coifman, C. Fefferman [23].

Statement 2. *If the function $\omega(\cdot) > 0$ satisfies the Muckenhoupt condition $A_p(\Gamma)$, $1 < p < +\infty$, then for sufficiently small $\delta > 0$ the "inverse Hölder inequality"*

$$\left(\frac{1}{r}\int_{\Gamma\cap O_r(z)} |\omega(\xi)|^{1+\delta}\,|d\xi|\right)^{\frac{1}{1+\delta}} \le c\left(\frac{1}{r}\int_{\Gamma\cap O_r(z)} \omega(\xi)\,|d\xi|\right),\ \forall r > 0,\ \forall z \in \Gamma,$$

holds, where $c = c(\delta)$ is a constant independent of r and $z \in \Gamma$.

As usual, by $L_p(\Gamma;\omega)$ we denote the weighted Lebesgue space of functions endowed with the norm $\|\cdot\|_{p,\omega}$:

$$\|f\|_{L_p(\Gamma;\omega)} = \left(\int_\Gamma |f(\xi)|^p\,\omega(\xi)\,|d\xi|\right)^{\frac{1}{p}}.$$

3. General assumptions. Weighted Smirnov classes

Let $G(\xi) = |G(\xi)|\,e^{i\theta(\xi)}$ be complex-valued functions on the curve Γ. We make the following basic assumptions on the coefficient $G(\cdot)$ of the considered boundary value problem and Γ:

(i) $|G(\cdot)|^{\pm 1} \in L_\infty(\Gamma)$;

(ii) $\theta(\cdot)$ *is piecewise continuous on* Γ, *and* $\left\{\xi_k,\, k = \overline{1,r}\right\} \subset \Gamma$ *are discontinuity points of the function* $\theta(\cdot)$:

We impose the following condition on the curve Γ.

(iii) Γ *is either Lyapunov or Radon curve (i.e. it is a limited rotation curve) with no cusps. Direction along* Γ *will be considered as positive, i.e. when moving along this direction the domain* D *stays on the left side. Let* $a \in \Gamma$ *be an initial (and also a final) point of the curve* Γ. *We will assume that* $\xi \in \Gamma$ *follows the point* $\tau \in \Gamma$, *i.e.* $\tau \prec \xi$, *if* ξ *follows* τ *when moving along a positive direction on* $\Gamma\backslash a$, *where* $a \in \Gamma$ *represents two stuck points* $a^+ = a^-$, *with* a^+ *a beginning, and* a^- *an end of the curve* Γ.

So, without loss of generality, we will assume that $a^+ \prec \xi_1 \prec \ldots \prec \xi_r \prec b = a^-$. Denote one-sided limits $\lim_{\substack{\xi\to\xi_0\pm 0\\ \xi\in\Gamma}} g(\xi)$ of the function $g(\xi)$ at the point $\xi_0 \in \Gamma$ generated by this order by $g(\xi_0 \pm 0)$, respectively. The jumps of the function $\theta(\xi)$ at the points ξ_k, $k = \overline{1,r}$, are denoted by h_k: $h_k = \theta(\xi_k + 0) - \theta(\xi_k - 0)$, $k = \overline{1,r}$.

Let $D^+ \subset C$ be a bounded domain with the boundary $\Gamma = \partial D^+$, which satisfies the condition iii). Denote by $E_p(D^+)$, $1 < p < \infty$, a Smirnov Banach space of analytic functions in D^+ with the norm $\|\cdot\|_{E_p(D^+)}$:

$$\|f\|_{E_p(D^+)} =: \left\|f^+\right\|_{L_p(\Gamma)}, \forall f \in E_p\left(D^+\right), \tag{1}$$

where $f^+ = f/_\Gamma$ are non-tangential boundary values of the function f on Γ.

Based on the norm (1), we define the weighted Smirnov class. Let $\rho \in L_1(\Gamma)$ be some weight function. Define weighted Smirnov class $E_{p,\rho}(D^+)$:

$$E_{p,\rho}\left(D^+\right) \equiv \left\{f \in E_1\left(D^+\right) : \left\|f^+\right\|_{L_{p,\rho}(\Gamma)} < +\infty\right\},$$

and let

$$\|f\|_{E_{p,\rho}(D^+)} = \left\|f^+\right\|_{L_{p,\rho}(\Gamma)}. \tag{2}$$

Similarly we define the Smirnov classes in unbounded domain. Let $D^- \subset C$ be an unbounded domain containing infinitely remote point (∞). Denote by ${}_mE_1(D^-)$ a class of functions from $E_1(D^-)$ which are analytic in D^- and have an order $k \leq m$ at infinity, i.e. the function $f \in E_1(D^-)$ has a Laurent decomposition $f(z) = \sum_{k=-\infty}^{m} a_k z^k$ in the vicinity of the infinitely remote point $z = \infty$, where m is some integer.

For a given weight function $\rho \in L_1(\Gamma)$, the weighted class ${}_mE_{p,\rho}(D^-)$ is defined as follows:

$${}_mE_{p,\rho}\left(D^-\right) \equiv \left\{f \in_m E_1\left(D^-\right) : \left\|f^-\right\|_{L_{p,\rho}(\Gamma)}\right\},$$

with

$$\|f\|_{{}_mE_{p,\rho}(D^-)} = \left\|f^-\right\|_{L_{p,\rho}(\Gamma)}, \tag{3}$$

where f^- are non-tangential boundary values of the function f on Γ.

The following lemma is true.

Lemma 1. *Let $\rho^{-\frac{q}{p}} \in L_1(\Gamma)$. Then the classes $E_{p,\rho}(D^+)$ and ${}_mE_{p,\rho}(D^-)$ are Banach spaces with respect to the norms* (2) *and* (3)*, respectively, where $1 \leq p < +\infty$, $\frac{1}{p} + \frac{1}{q} = 1$.*

4. The general solution of the homogeneous problem

Consider the following homogeneous Riemann problem in weighted Smirnov classes

$$F^+(\xi) + G(\xi) F^-(\xi) = 0,, \text{ a.e. } \ \xi \in \Gamma. \tag{4}$$

By the solution of the problem (4) we mean a pair of analytic functions

$$\left(F^{+};F^{-}\right) \in E_{p,\rho}\left(D^{+}\right) \times_{m} E_{p,\rho}\left(D^{-}\right),$$

whose non-tangential boundary values $F^{\pm}(\xi)$ satisfy the equality (4) a.e. on Γ. In weightless case, this problem has been well enough studied in the monograph by I.I.Danilyuk [8].

In construction of general solution to homogeneous Riemann problem, an important role is played by the following lemma of uniqueness of the solution for a simplest homogeneous problem.

Lemma C [8]. *Assume that D^{+} is an arbitrary domain bounded by the rectifiable curve Γ. Homogeneous problem*

$$\Phi^{+}(\xi) - \Phi^{-}(\xi) = 0, \xi \in \Gamma,$$

in a class of functions $\Phi(\cdot)$, belonging to Smirnov classes $E_1\left(D^{\pm}\right)$ $\left(D^{-} = C \backslash \bar{D}^{+}\right)$ and having a finite order k at infinity, admits only trivial solutions in the form of polynomials, whose degree does not exceed k.

Let S be a length of the curve Γ and $z = z(s)$, $0 \leq s \leq S$, be a parametric representation of Γ with respect to the length of the arc s. Rewrite the problem (4) as follows

$$F^{+}[z(s)] + G(z(s)) F^{-}[z(s)] = 0, \text{ a.e. } s \in [0, S], \tag{5}$$

Let $\Omega(s) \equiv \theta(z(s))$, $0 \leq s \leq S$, and suppose

$$h_k = \Omega(s_k + 0) - \Omega(s_k - 0), k = \overline{1,r}; h_0 = \Omega(+0) - \Omega(S - 0),$$

where $\xi_k = z(s_k)$, $0 < s_k < S$, $a = z(0) = z(S)$, are discontinuity points of the argument $\Omega(\cdot)$. Consider the following piecewise holomorphic functions

$$Z_{(1)}(z) = \exp\left\{\frac{1}{2\pi i}\int_{\Gamma} \ln\left|G(z(s))\right| \frac{dz(s)}{z(s) - z}\right\},$$

$$\tilde{Z}_{\theta}(z) = \exp\left\{\frac{1}{2\pi}\int_{\Gamma} \Omega(s) \frac{dz(s)}{z(s) - z}\right\} = \exp\left\{\frac{1}{2\pi}\int_{\Gamma} \theta(z(s)) \frac{dz(s)}{z(s) - z}\right\}, z \notin \Gamma.$$

As the argument $\theta(\cdot)$ is defined ambiguously, it is clear that the value of the function $\tilde{Z}_{\theta}(\cdot)$ depends on the chosen argument. Denote the product of these functions by

$$Z_{\theta} : Z_{\theta}(z) \equiv Z_{(1)}(z) \tilde{Z}_{\theta}(z), z \in C \backslash \Gamma.$$

Hereinafter, the function $Z_{\theta}(\cdot)$ will be called a canonical solution of the problem (4) *corresponding to the argument $\theta(\cdot)$.*

The following lemma is true for the first multiplier $Z_{(1)}(z)$.

Lemma 2. *[8] Let the conditions i)-iii) be satisfied for the coefficient $G(\cdot)$ and the curve Γ. Then the functions $Z_{(1)}(z)$; $Z_{(1)}^{-1}(z)$ are bounded in each of the domains $D^{\pm}$.*

To proceed further, we represent the function $\Omega(s)$ in the following form

$$\Omega(s) = \Omega_0(s) + \Omega_1(s) \ , \ 0 \le s \le S,$$

where $\Omega_0(s)$ is a continuous part, and $\Omega_1(s)$ is a jump function defined by

$$\Omega_1(0) = 0,$$

$$\Omega_1(s) = [\Omega(+0) - \Omega(0)] + \sum_{0<s_k<s} h_k + [\Omega(s) - \Omega(s-0)] \ , \ 0 < s < S.$$

Denote

$$h_0^{(0)} = \Omega_0(S) - \Omega_0(0) \ , \quad h_0^{(1)} = \Omega_1(+0) - \Omega_1(S-0) \, .$$

Let

$$Z_{(2)}(z) = \exp\left\{ \frac{1}{2\pi} \int_\Gamma \Omega_0(s) \frac{dz(s)}{z(s) - z} \right\},$$

and

$$Z_{(3)}(z) = \exp\left\{ \frac{1}{2\pi} \int_\Gamma \Omega_1(s) \frac{dz(s)}{z(s) - z} \right\}.$$

It was proved in [8] that the following inclusion is true

$$\tilde{Z}_{(2)}^{\pm 1}(s) = |z(s) - z(0)|^{\pm \frac{h_0^{(0)}}{2\pi}} \left| Z_{(2)}^{\pm}[z(s)] \right|^{\pm 1} \in L_q(\Gamma) \ , \ \forall q \in (0, +\infty) \, . \quad (6)$$

The modulus of boundary values of the function $Z_{(3)}(\cdot)$ can be represented as follows [8]:

$$\left| Z_{(3)}^{+}[z(\sigma)] \right| \equiv |z(0) - z(\sigma)|^{-\frac{h_0^{(1)}}{2\pi}} \prod_{0<s_k<S} |z(s_k) - z(\sigma)|^{-\frac{h_k}{2\pi}} \, , \quad (7)$$

which follows directly from the lemma below.

Lemma 3. *[8] Let the curve Γ satisfy the condition iii) and $\Omega_1(s)$ be an arbitrary jump function with the jumps $h_0^{(1)} = \Omega_1(+0) - \Omega_1(S-0)$ at the point $z(0)$. Then the modulus of boundary values of the function $Z_{(3)}(\cdot)$ can be represented by the formula (7) for a.e. $\sigma \in [0, S]$.*

Before stating our main result, let us introduce the following weight function

$$\sigma(s) \equiv |z(0) - z(s)|^{-\frac{h_0}{2\pi}} \prod_{0<s_k<S} |z(s_k) - z(s)|^{-\frac{h_k}{2\pi}}. \tag{8}$$

Let $\rho : \Gamma \to (0, +\infty)$ be some weight function. Assume that $\exists p_1; p_2 \in (1, +\infty)$, such that the following conditions hold

$$\int_0^S \sigma^{pp_1}(s)\, \rho^{p_1}(z(s))\, ds < +\infty, \tag{9}$$

$$\int_0^S \sigma^{-qp_2}(s)\, \rho^{-\frac{q}{p}p_2}(z(s))\, ds < +\infty. \tag{10}$$

So, the following main theorem is true.

Theorem 1. *Let the conditions i)-iii) hold for complex-valued functions* $G(\cdot)$ *and the curve* Γ. *Assume that the conditions* (9) *and* (10) *are satisfied for the weight function* $\rho(\cdot)$. *Then the general solution of homogeneous problem* (4) *in the classes* $E_{p,\rho}(D^+) \times_m E_{p,\rho}(D^-)$ *has a representation*

$$F(z) = Z_\theta(z)\, P_m(z), \tag{11}$$

where $Z_\theta(\cdot)$ *is a canonical solution corresponding to the argument* $\theta(\cdot)$, *and* $P_m(\cdot)$ *is an arbitrary polynomial of degree* $k \le m$ *(for* $m \le -1$ *we assume* $P_m(z) \equiv 0$*).*

Proof. Introduce the following piecewise analytic function

$$\Phi(z) \equiv \frac{F(z)}{Z_\theta(z)}, \tag{12}$$

where $F(\cdot)$ is a solution of homogeneous problem (4) in the classes $E_{p,\rho}(D^+) \times {}_mE_{p,\rho}(D^-)$. It is not difficult to see that the following relation holds

$$\Phi^+(\tau) = \Phi^-(\tau), \quad \text{a.e. } \tau \in \Gamma.$$

Let's show that the function Φ satisfies all the conditions of Lemma C. So, $Z_\theta(z)$ has neither zeros nor poles when $z \notin \Gamma$. Therefore, the functions $\Phi(z)$ and $F(z)$ have the same order at infinity. By definition of solution, we have $F \in E_1(D^+)$. From the results of I.I.Danilyuk [8] (see, e.g., Lemma 16.5, p. 148) it follows that the function $Z_\theta(z)$ belongs to the classes $E_\delta(D^\pm)$ for sufficiently small $\delta > 0$, if conditions i)-iii) hold. Then from (12) we obtain that the function $\Phi(z)$ belongs to the classes $E_\mu(D^\pm)$ for sufficiently small $\mu > 0$. Thus, it follows from the Smirnov theorem [8] that if $\Phi^+ \in L_1(\Gamma)$, then clearly $\Phi \in E_1(D^+)$. As $\Phi^+(\tau) = \Phi^-(\tau)$

a.e. $\tau \in \Gamma$, then it suffices to prove that $\Phi^{-}(\tau)$ belongs to the space $L_1(\Gamma)$. We have

$$\left|\Phi^{-}(\tau)\right| = \left|F^{-}(\tau)\right| \left|Z_{\theta}^{-}(\tau)\right|^{-1}, \text{ a.e. } \tau \in \Gamma.$$

By definition of solution, we have the inclusion $|F^{-}| \in L_{p,\rho}(\Gamma)$. Therefore, if $\left|Z_{\theta}^{-}\right|^{-1} \in L_{q,\tilde{\rho}}(\Gamma)$, then $\Phi^{-} \in L_1(\Gamma)$, where $\tilde{\rho}(\cdot) = \rho^{-\frac{q}{p}}(\cdot)$, which follows directly from Hölder's inequality

$$\int_{\Gamma} \left|\Phi^{-}(\tau)\right| |d\tau| \leq \left(\int_{\Gamma} \left|F^{-}(\tau)\right|^{p} \rho(\tau) |d\tau|\right)^{\frac{1}{p}} \left(\int_{\Gamma} \left|Z_{\theta}^{-}(\tau)\right|^{-q} \rho^{-\frac{q}{p}}(\tau) |d\tau|\right)^{\frac{1}{q}}.$$

Then, by Smirnov theorem [8] we obtain that the function $\Phi(z)$ belongs to the class $E_1(D^{\pm})$. As $\Phi^{+}(\tau) = \Phi^{-}(\tau)$ a.e. $\tau \in \Gamma$, it follows from the uniqueness theorem (i.e. from Lemma C) that $\Phi(z)$ is a polynomial of degree $k \leq m$, i.e. $\Phi(z) \equiv P_m(z)$, where $P_m(z)$ is a polynomial of degree $k \leq m$. So we obtain the representation

$$F(z) = Z_{\theta}(z) P_m(z).$$

Now we have to find out whether the function $F(z)$ belongs to the required classes. It is absolutely clear that if $Z_{\theta}^{+}(\cdot) \in L_{p,\rho}(\Gamma)$, then $F^{+}(\cdot) \in L_{p,\rho}(\Gamma)$. As a result, by definition of weighted Smirnov classes, we obtain that $F(\cdot) \in E_{p,\rho}(D^{+}) \times_m E_{p,\rho}(D^{-})$.

The modulus of boundary values $\left|Z_{\theta}^{+}(z(s))\right|$ can be represented as follows

$$\left|Z_{\theta}^{+}(z(s))\right| \equiv \left|Z_{(1)}^{+}(z(s))\right| \left|\tilde{Z}_{(2)}(s)\right| \sigma(s).$$

From Lemma 2 we have

$$\left|Z_{\theta}^{+}(z(s))\right| \sim \sigma(s) \left|\tilde{Z}_{(2)}(s)\right|, s \in [0, S]. \tag{13}$$

Applying Hölder's inequality we get

$$\int_{\Gamma} \left|Z_{\theta}^{+}(z(s))\right| |dz(s)| \sim \left(\int_{\Gamma} |\sigma(s)|^{p_0} |dz(s)|\right)^{\frac{1}{p_0}} \left(\int_{\Gamma} \left|\tilde{Z}_2(s)\right|^{p_0'} |dz(s)|\right)^{\frac{1}{p_0'}}, \tag{14}$$

where $\frac{1}{p_0} + \frac{1}{p_0'} = 1$. As is known (see, e.g., I.I.Danilyuk [8]),

$$\left|\frac{dz}{ds}\right| = 1, \quad \text{a.e. } \in [0, S],$$

moreover, as the curve Γ has no cusp, we have $\exists \delta_0; k_0 > 0:$

$$k_0 |s - \sigma| \leq |z(s) - z(\sigma)| \leq |s - \sigma|, \forall s, \sigma : |s - \sigma| \leq \delta_0. \tag{15}$$

Taking into account these relations and using the inclusion (6), from (14) we obtain $Z_\theta^+ \in L_1(\Gamma)$, and, as a result, $F^+ \in L_1(\Gamma)$. Then it follows from Smirnov theorem that the function $F(z)$ belongs to Smirnov class $E_1(D^+)$. It is absolutely clear that the boundary values $F^\pm$ of the function F on Γ satisfy (4). From the condition i) it follows that $|D|^{\pm 1} \in L_\infty$, therefore we clearly have

$$\left|Z_\theta^+(z(s))\right| \sim \left|Z_\theta^-(z(s))\right| \ , \ s \in (0, S) \, .$$

Using this relation, it is easy to conclude that $F^- \in L_1(\Gamma)$, and, as a result, $F \in_m E_1(D^-)$. Let's find the conditions under which the boundary values $F^\pm$ belong to the space $L_{p,\rho}(\Gamma)$. It is absolutely clear that if $Z_\theta^+ \in L_{p,\rho}(\Gamma)$, then $F^\pm \in L_{p,\rho}(\Gamma)$.

Then, considering (14), we obtain

$$\int_0^S \left|Z_\theta^+(z(s))\right|^p \rho(z(s))\, ds \leq$$

$$\leq M \left(\int_0^S \sigma^{pp_1}(s)\, \rho^{p_1}(z(s))\, ds\right)^{\frac{1}{p_1}} \left(\int_0^S \left|\tilde{Z}_{(2)}(s)\right|^{q_1} ds\right)^{\frac{1}{q_1}} ,$$

where M is some constant and $\frac{1}{p_1} + \frac{1}{q_1} = 1$.

Again, by paying attention to condition (10), from (13) we directly get the inclusion $\left|Z_\theta^-(\cdot)\right|^{-1} \in L_{p;\tilde{\rho}}$, where $\tilde{\rho} = \rho^{-\frac{q}{p}}$. Summing up the obtained results we finish the proof of theorem. ◄

This theorem has a following direct corollary.

Corollary 1. *Let all the conditions of Theorem 1 be satisfied. Then, if* $F(\infty) = 0$*, the problem* (4) *has only a trivial solution in the classes* $E_{p,\rho}(D^+) \times_m E_{p,\rho}(D^-)$*, i.e. when* $m \leq -1$*, the problem* (4) *has only a trivial solution in the classes* $E_{p,\rho}(D^+) \times_m E_{p,\rho}(D^-)$*.*

In fact, as $\exists \lim\limits_{z\to\infty} Z_\theta(z) \neq 0$, it is clear that the function $Z_\theta(\cdot)$ does not influence the order of the function $F(\cdot)$ at infinitely remote point. Then it follows from the representation $F(z) = Z_\theta(z)\, P_m(z)$ that for $m \leq -1$ the homogeneous problem has only a trivial solution.

Let's consider some special cases of the weight function $\rho(\cdot)$.

Example 1. *Let* $\rho(\cdot)$ *have the following power form*

$$\rho(s) = \rho(z(s)) = \prod_{k=0}^{m_0} |z(s) - z(t_k)|^{\alpha_k} , \qquad (16)$$

where $\{t_k\}_1^m \subset [0,S)$ *are different points, and* $\{\alpha_k\}_0^{m_0} \subset [0,S)$ *are some numbers. Denote the union of sets* $\{s_k\}_0^r$ *and* $\{t_k\}_0^{m_0}$ *by* $\{\sigma_k\}_0^l : \{\sigma_k\}_1^l \equiv \{s_k\}_0^r \bigcup \{t_k\}_0^{m_0}$. *Let* $\chi_A(\cdot)$ *be a characteristic function of the set A. Denote one-point sets* $\{\sigma_k\}$, $k = \overline{0,l}$, *by* $T_k : T_k \equiv \{\sigma_k\}$, $k = \overline{0,l}$. *Let*

$$\beta_k = -\frac{p}{2\pi}\sum_{i=0}^{r} h_i \chi_{T_k}(s_i) + \sum_{i=0}^{m_0} \alpha_i \chi_{T_k}(t_i), k = \overline{0,l}. \qquad (17)$$

Assume that the following inequalities hold

$$-1 < \beta_k < \frac{p}{q}, k = \overline{0,l}. \qquad (18)$$

It is not difficult to show that if the inequalities (18) hold, then so do the relations (9), (10) and, as a result, Theorem 1 is true. Then we have the following

Corollary 2. *Let the function* $G(\cdot)$ *and the curve* Γ *satisfy the conditions i)-iii), and the weight function* $\rho(\cdot)$ *have the form* (16). *Assume that the inequalities* (18) *hold, where* β_k*'s are defined by* (17). *Then the general solution of the problem* (4) *in the classes* $E_{p,\rho}(D^+) \times_m E_{p,\rho}(D^-)$ *has a representation* (11).

Example 2. *Consider* (16) *as a weight function* $\rho(\cdot)$ *again, but now assume that* $\{s_k\}_0^r \bigcap \{t_k\}_0^{m_0} = \emptyset$. *Then the following corollary is true.*

Corollary 3. *Let all the conditions of Corollary 2 be fulfilled and* $\{s_k\}_0^r \bigcap \{t_k\}_0^{m_0} = \emptyset$. *If the inequalities*

$$-\frac{1}{q} < \frac{h_k}{2\pi} < \frac{1}{p}, \quad k = \overline{0,r}; -1 < \alpha_i < \frac{q}{p}, \quad i = \overline{0,m_0},$$

hold, then the general solution of the problem (4) *in the classes* $E_{p,\rho}(D^+) \times_m E_{p,\rho}(D^-)$ *has a representation* (11).

Note that the weight (16) belongs to the Muckenhoupt class $A_p(\Gamma)$, $1 < p < +\infty$, if and only if the degeneration orders satisfy the inequalities

$$-1 < \alpha_i < \frac{q}{p}, \quad i = \overline{0,m_0}.$$

Remark 1. *It is easy to see that if the weight function* $\nu(s) = \sigma(s)\rho^{\frac{1}{p}}(z(s))$, $s \in [0,S]$, *belongs to the Muckenhoupt class* $A_p(0,S)$, *then the conditions* (9), (10) *are fulfilled.*

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References

[1] B. Riemann, *Grundlagen für eine allgemeine Theorie der. Functionen einer veränderlichen complexen Grösse*, Göttingen, 1851; Werke, Leipzig, 1876.

[2] D. Hilbert, *Über eine Anwendung der Integral der Integralgleichungen auf ein Problem der Fuktionentheorie*, Verhand 3 I. Math. Kong, Heidelberg, 1904.

[3] D. Hilbert, *Grundzüge einer allgemeinen Theorie der linearen Integralgleichungen*, Teubner, Leipzig and Berlin, 1912.

[4] T. Carleman, *Sur la theorie des equations integrals etses applications*, Verband des Int. math. Kongr.,Zurich, **1**, 1932.

[5] J. Plemelj, *Riemannsche Functionenscharen mit gegebener Monodromiargruppe*, Monatsh. Math. Phys., **19**, 1908, 211-2455.

[6] I. Privalov, *Boundary Properties of Analytic Functions*, Moscow, Gostekhizdat, 1950. (*in Russian*)

[7] N. Muskhelishvili, *Singular Integral Equations*, Noordhoff, Groningen–Leyden, 1953.

[8] I.I. Danilyuk, *Irregular boundary value problems on half plane*, Moscow, Nauka, 1975. (*in Russian*)

[9] I.N. Vekua, *New Methods of the Solutions of Elliptic Equations*, Moscow, Gostekhizdat, 1948. (*in Russian*)

[10] I.N. Vekua, *Generalized Analytic Functions*, Pergamon Press, Oxford, 1962.

[11] I.N. Vekua, *Systems of Singular Integral Equations*, Moscow, Nauka, 1970. (*in Russian*)

[12] F.D. Gakhov, *Boundary value problems*, Moscow, Nauka, 1977. (*in Russian*)

[13] G. Litvinchuk, *Boundary-Value Problems and Singular Integral Equations with Shift*, Moscow, Nauka, 1977. (*in Russian*)

[14] D.V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue spaces: Foundations and Harmonic Analysis*, Springer, 2013.

[15] D.R. Adams, *Morrey Spaces*, Springer, Switzerland, 2016.

[16] V.M. Kokolashvili, A. Meskhi, H. Rafeiro, S. Samko, *Integral operators in non-standart function spaces*, Birkhauser, 2010.

[17] V.M. Kokilashvili, *Boundary value problems of analytic and harmonic functions in a domain with piecewise smooth boundary in the frame of variable exponent Lebesgue spaces*, Operator Theory, **216**, 2011, 17-39.

[18] Z. Meshveliani, *The Riemann–Hilbert problem in weighted Smirnov classes of analytic functions*, Proc. Razmadze Math. Inst., **137**, 2005, 65-86.

[19] G. Manjavidze, N. Manjavidze, *Boundary value problems for analytic and generalized analytic functions*, J. Math. Sci., **160**, 2009, 745-821.

[20] T.I. Najafov, N.P. Nasibova, *On the Noetherness of the Riemann problem in a generalized weighted Hardy classes*, Azerb. J. of Math., **5(2)**, 2015, 109-139.

[21] B.T. Bilalov, T.B. Gasymov, A.A. Guliyeva, *On solvability of Riemann boundary value problem in Morrey-Hardy classes*, Turk. J. of Math., **40(5)**, 2016, 1085-1101.

[22] D.M. Israfilov, N.P. Tozman, *Approximation in Morrey-Smirnov classes*, Azerb. J. of Math., **1(1)**, 2011, 99-113.

[23] R.R. Coifman, C. Fefferman, *Weighted norm inequalities for maximal functions and singular integrals*, Studia Mathematica, **51(3)**, 1974, 241-249.

[24] E.M. Dynkin, *Methods of theory of singular integrals*, Itoginauki i texn., ser. matem. analiz, **15**, 1987, 197-292. (*in Russian*)

[25] G. David, *Operateur sintegraux singulars sur certain scourbes du plan complexe*, Ann. Sci. Ecole Norm. Sup., **17**, 1984, 157-189.

[26] A.V. Bitsadze, *On one system of functions*, UMN, **5(4(38)**, 1950, 150-151 (*in Russian*)

[27] S.M. Ponomarev, *On an eigenvalue problem*, Dokl. Akad. Nauk SSSR, **249(5)**, 1979, 1068–1070 (*in Russian*)

[28] S.M. Ponomarev, *On the theory of boundary value problems for mixed type equations in three-dimensional domains*, DAN SSSR, **246(6)**, 1979, 1303-1304. (*in Russian*)

[29] E.I. Moiseev, *On basicity of sine and cosine systems*, DAN SSSR, **275(4)**, 1984, 794-798. (*in Russian*)

[30] E.I. Moiseev, *On basicity of one sine system*, Dif. uravneniya, **23(1)**, 1987, 177-179. (*in Russian*)

[31] B.T. Bilalov, *Basicity of some exponential, sine and cosine systems*, Dif. uravneniya, **26(1)**, 1990, 10-16. (*in Russian*)

[32] B.T. Bilalov, *Basis properties of some exponential, sine and cosine systems*, Sibirski M. Jurnal, **45(2)**, 2004, 264-273. (*in Russian*)

[33] B.T. Bilalov, *On basicity of the system and exponential shift systems*, Dokl. RAN, **345(2)**, 1995, 644-647. (*in Russian*)

[34] B.T. Bilalov, *Basis properties of power systems in* L_p, Sibirski M. Jurnal, **47(1)**, 2006, 1-12. (*in Russian*)

[35] B.T. Bilalov, *Exponential shift system and the Kostyuchenko problem*, Sibirski M. Jurnal, **50(2)**, 2009, 279-288. (*in Russian*)

[36] B.T. Bilalov, *On solution of the Kostyuchenko problem*, Siberian Math. Journal, **53(3)**, 2012, 509-526.

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