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Multiparameter Inverse Spectral Problems in the Oscillation Model of an Orthotropic Plate

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Abstract. In this paper, the problems of constructing a mathematical model for small linear oscillations of an orthotropic plate carrying concentrated masses in a finite set of points connected with a fixed base by elastic struts (springs) with certain stiffness coefficients, as well as the direct and inverse spectral problems for such a model in a new formulation are considered.

Key Words and Phrases: mathematical model, oscillation, orthotropic plate, direct spectral problem, inverse spectral problem.

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1. Introduction

In this paper, the mathematical models describing vibrations of elastic plates and shells are studied. Plates are equipped with additional fasteners and contain concentrated masses at some points. At the same time, structural elements (masses, fastenings) have exact and relatively small dimensions compared with the plate. Similar designs are common in various fields of mechanics and engineering.

The frequencies and modes of oscillations significantly depend on the elasticity coefficients and point masses of fasteners, which entails the possibility of controlling these characteristics using the available fastener parameters. In addition to the control problems for such systems, the problems of diagnosing (or identifying) the parameters of the connections from known data (the finite set of eigenvalues of oscillations) are relevant.

In this paper, a different approach to the study of such structures, based on the theory of multiparameter inverse spectral problems (briefly MPISP) are

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proposed. For the MPISP, it is appropriate to apply the methods and approaches developed in [1, 2, 8], which differ significantly from the methods of classical inverse spectral problems.

Let us directly proceed to the mathematical model of the structure. Let an orthotropic plate Ω be given carrying the concentrated masses m_j at certain points $M_j = M_j(x_j, y_j)$, which are connected to the fixed base by elastic stiffness springs k_j. We will assume the validity of the Kirchhoff-Love hypotheses with the boundary conditions of any of three possible types (free edge, hinge bearing, pinching). The interaction of point masses with elastic columns (springs) is well managed to be modeled using Dirac delta functions.

It is known that within the assumptions on the model, the natural oscillations of the plate are described by a biharmonic operator. In our case, this operator has distribution coefficients of δ -like form. Such operators were studied, for example, in [7, 3].

Below it will be shown that the form of the plate's natural oscillations is described by an operator pencil. A direct problem is to investigate the nature of the spectrum of this operator pencil. The inverse spectral problem consists in determining the coefficients of spring stiffness and the values of the concentrated masses from the known finite set of natural frequencies of oscillations.

2. Construction of a mathematical model

We assume that at rest the form of the considered plate is a simply connected bounded domain $\Omega \subset \mathbf{R}^2$ with a smooth boundary $\Gamma = \partial\Omega$. $M_j = (x_j, y_j) \in \Omega$ are the points where elastic supports with elasticity coefficients k_j and point masses m_j, $j = \overline{1, n}$ are found.

Let the function $W(M, t)$, where $M = M(x, y) \in \Omega$, be the magnitude of the deviation from the equilibrium state, and either the free edge condition, or the hinged support condition, or the pinch condition be satisfied on the boundary. Without loss of generality, we shall further assume that for $W(M, t)$ the conditions of "pinching" are fulfilled: $W|_\Gamma = 0$, $\frac{\partial W}{\partial n}|_\Gamma = 0$. Denote these conditions by $l(W)|_\Gamma = 0$.

The plate thickness is assumed to be h, $h << 1$, density is denoted by ρ, Young's modulus by E and Poisson's ratio by σ . The potential energy of the plate, together with the energy of the elastic supports, has the following form:

$$\Pi = D \int\limits_{\Omega} \left((W_{xx} + W_{yy})^2 - 2(1-\sigma)(W_{xx}W_{yy} - W_{xy}^2) \right) dxdy +$$

$$+\frac{1}{2}\sum_{j=1}^{n} k_j W(x_j, y_j)^2, \tag{1}$$

where $D = \frac{Eh^2}{12(1-\sigma^2)}$ is a cylindrical rigidity of the plate. The kinetic energy of the entire plate, taking into account the added masses, has the following form:

$$T = \frac{\rho h}{2}\int\limits_{\Omega} |W_t^2| dxdy + \frac{1}{2}\sum_{j=1}^{n} m_j |W_t(x_j, y_j, t)|^2. \tag{2}$$

In this case, the action functional has the form

$$S = \int\limits_{t_1}^{t_2} (T - \Pi) dr \tag{3}$$

Considering S on the set of harmonic oscillations of the form

$$W(x, y, t) = \mathbf{e}^{i\omega t} u(x, y),$$

we obtain a quadratic functional

$$\widehat{S}(u, u) = D\int\limits_{\Omega} \left(|\Delta u|^2 - 2(1-\sigma)(u_{xx}u_{yy} - u_{xy}^2)\right) dxdy -$$

$$-\frac{\omega^2 \rho h}{2}\int\limits_{\Omega} |u^2| dxdy + \frac{1}{2}\sum_{j=1}^{n} (k_j - \omega^2 m_j)|u(x_j, y_j)|^2. \tag{4}$$

Note that the problem of finding the natural oscillations of a system is equivalent to the search for the extremals of quadratic form $\widehat{S}(u, u)$, see e.g. [6].

Now, consider the problem in dimensionless quantities, and also introduce the divergent-gradient operator

$$\Delta_0 u = div(a\bigtriangledown)u,$$

where

$$\mathbf{a} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \quad \mathbf{a}^* = \mathbf{a} > 0.$$

Let's pass from (4) to the quadratic form in dimensionless quantities

$$\widehat{Q}(u, u, \lambda) = \int\limits_{\Omega} |\Delta_0 u|^2 dxdy + \sum_{j=1}^{n} k_j |u(x_j, y_j)|^2 -$$

$$-\lambda \left(\int\limits_{\Omega} |u|^2 dxdy + \sum_{j=1}^{n} m_j |u(x_j, y_j)|^2 \right). \tag{5}$$

As the domain of the quadratic form $\widehat{Q}(u,u)$, it is natural to consider the space

$$D_{\widehat{Q}} = \{u \in W_2^2(\Omega) | l(u)|_{\partial\Omega=0}\}.$$

Let us introduce the Hilbert space H_1 with the norm

$$\| u \|_{H_1}^2 = \int\limits_{\Omega} (|\Delta_0 u|^2 + |u|^2) dxdy.$$

Then it is obvious that the norms $\| \cdot \|_{H_1}$ and $\| \cdot \|_{W_2^2}$ are equivalent.

Thus, the problem of finding the natural vibrations of a plate is reduced to finding extremals (eigenfunctions) of the quadratic functional $\widehat{Q}(u,u,\lambda)$ in the class of functions $D_{\widehat{Q}}$.

Let $Q(u,v,\lambda)$ be the sesquilinear form associated with the quadratic form $\widehat{S}$. Then the extremal u_0 of the quadratic form $\widehat{Q}(u,u,\lambda)$ satisfies the equation

$$Q(u_0, v, \lambda) = 0, \quad \forall v \in D_{\widehat{Q}}.$$

Following this scheme, from Sobolev embedding theorems we obtain the following estimate for the quadratic form $\widehat{Q}(u,u,\lambda)$:

$$\widehat{Q}(u,u,\lambda)| \leq C \| u \|_{H_1}^2, \quad C = const,$$

which, in turn, implies the following representation for the sesquilinear form $Q(u,v,\lambda)$ associated with $\widehat{Q}(u,u,\lambda)$ for all $u, v \in D_{\widehat{Q}}$:

$$Q(u,v,\lambda) = (L(\lambda)u, v)_{H_1}, \tag{6}$$

where $L(\lambda) : H_1(\Omega) \to H_1(\Omega)$ is some bounded operator. Let us find the form of this operator.

Theorem 1. *The operator $L(\lambda) : H_1(\Omega) \to H_1(\Omega)$, associated with the quadratic form $\widehat{Q}(u,u,\lambda)$, has the form*

$$L(\lambda)u = u - (\lambda+1)(\Delta_0^2 + I)^{-1}u + \sum_{j=1}^{n} (k_j - \lambda m_j) g_0(M; M; -1) u(M_j),$$

where $g_0(M;S;-1)$ is the Green's function of the boundary value problem

$$\begin{cases} (\Delta_0^2 + I)u = f(u), \quad M \in \Omega, \\ \qquad l(u)|_{\partial\Omega} = 0. \end{cases}$$

Proof. Let us consider the sesquilinear forms bounded on H_1 and their representations

$$\int_\Omega \Delta_0 u \overline{\Delta_0 v} dxdy = (A_1 u, v)_{H_1},$$

$$\int_\Omega u\bar{v} dxdy = (A_2 u, v)_{H_1},$$

$$u(M_*)\overline{v(M_*)} = (A_3(M_*)u, v)_{H_1},$$

and find the corresponding operator A_k, $k = 1, 2, 3$ for each form. Since all operators $A_k : H_1 \to H_1$ are bounded, then it suffices to consider them on functions of the space $\mathbf{C}_0^\infty(\Omega)$.

Note that

$$\int_\Omega \Delta_0 u \overline{\Delta_0 v} dxdy = (\Delta_0 u, \Delta_0 v)_{L_2(\Omega)} = (\Delta_0^2 u, v)_{L_2(\Omega)}.$$

On the other hand, for the same functions $u, v \in \mathbf{C}_0^\infty(\Omega)$ we have

$$(A_1 u, v)_{H_1} = (\Delta_0(A_1 u), \Delta_0 v)_{L_2(\Omega)} + (A_1 u, v)_{L_2(\Omega)} = ((\Delta_0^2 + I)(A_1 u), v)_{L_2(\Omega)},$$

which implies the validity of the following identity:

$$((\Delta_0^2 + I)(A_1 u), v)_{L_2(\Omega)} = (\Delta_0^2 u, v)_{L_2(\Omega)}.$$

Thus, the required operator has the representation

$$A_1 u = (\Delta_0^2 + I)^{-1}\Delta_0^2 u. \tag{7}$$

In view of the boundedness of the operator $A_1 : H_1 \to H_1$, its domain is the space H_1.

Reasoning similarly, from the identity

$$\int_\Omega u\bar{v} dxdy \equiv (A_2 u, v)_{H_1(\Omega)},$$

we obtain the representation

$$A_2 u = (\Delta_0^2 + I)^{-1} u. \tag{8}$$

Consider the third identity for an arbitrary point $M_* = M(x_*, y_*) \in \Omega$. The following lemma is true.

Lemma 1. *For* $\forall M_* \in \Omega,\ u \in H_1(\Omega)$

$$A_3(M_*)u = g_0(M, M_*, -1)u(M_*). \tag{9}$$

Proof. It is well known that

$$g_0(M, M_*, -1) = C_1\rho^2(M, M_*)\ln\rho(M, M_*) + \xi(M), \quad C_1 = const,$$

where $\xi(M)$ is such that

$$(\Delta_0^2 + I)\xi(M) = 0.$$

Consequently, $g_0(M, M_*, -1) \in H_1(\Omega)$.

From the embedding theorem for Sobolev spaces we obtain the estimate

$$|u(M_*)| \leq \| u \|_{C(\overline{\Omega})} \leq b \| u \|_{H_1(\Omega)},$$

where $b > 0$ is a constant which does not depend on the choice of the function $u \in H_1(\Omega)$. Then $A_3(M_*)$ is a bounded one-dimensional operator. It remains to show that for all $u, v \in \mathbf{C}_0^\infty(\Omega)$ the identity

$$u(M_*)\overline{v(M_*)} \equiv (A_3(M_*)u, v)_{H_1}$$

holds. We have

$$(A_3(M_*)u, v)_{H_1} = (\Delta_0(A_3(M_*)u), \Delta_0 v)_{L_2(\Omega)} +$$

$$+(A_3(M_*)u, v)_{L_2(\Omega)} = (A_3(M_*)u, (\Delta_0^2 + I)v)_{L_2(\Omega)}.$$

It is obvious that the last scalar product can be represented in the form

$$u(M_*)\int_\Omega g_0(M_*; \xi, \eta; -1)(\Delta_0^2 + 1)\overline{v(\xi, \eta)}d\xi d\eta =$$

$$= u(M_*)(\Delta_0^2 + I)^{-1}(\Delta_0^2 + I)v|_{M=M_*} = u(M_*)\overline{v(M_*)},$$

which proves the validity of the representation for the operator $A_3(M_*)$.
The lemma is proved. ◄

From formulas (7) - (9) we obtain the following representation for $L(\lambda)$:

$$Q(u, v, \lambda) = (L(\lambda)u, v) = (A_1 u, v)_{H_1} - \lambda(A_2 u, v)_{H_1} +$$

$$+\sum_{j=1}^{n}(k_j - \lambda m_j)(A_3(M_j)u, v)_{H_1}, \quad \forall u, v \in D_{\widehat{Q}},$$

from which the assertion of the theorem follows. ◂

Note that the constructed operator $L(\lambda) : H_1 \to H_1$ has an obvious connection with the biharmonic differential operator pencil with δ-like coefficients.

Indeed, we have the identity

$$\left((\Delta_0^2 + I)L(\lambda)u, v\right)_{L_2(\Omega)} = (\Delta_0 u, \Delta_0 v)_{L_2(\Omega)} - \lambda(u, v)_{L_2(\Omega)} +$$

$$+ \sum_{j=1}^{n} (k_j - \lambda m_j)\delta_j(M - M_j)u(M_j)\overline{v(M_j)}, \ \ \forall u, v \in D_{\widehat{Q}}. \tag{10}$$

The closure of the right-hand side of (10) defines a biharmonic operator with δ-like potential

$$L_\delta u = (\Delta_0^2 - \lambda I)u + \sum_{j=1}^{n} (k_j - \lambda m_j)\delta_j(M - M_j)u(M)$$

with domain $D_{L_\delta} \subset D_{\widehat{Q}}$.

Questions of the correct determination of similar operators with singular coefficients were studied, for example, in papers [4**?**].

3. Direct spectral problem

Now we proceed to the study of the spectral problem and its properties:

$$L(\lambda)u = 0, \quad u \in H_1, \quad \lambda \in \mathbb{C}. \tag{11}$$

The solutions of the spectral problem (11) are natural frequencies of oscillations and forms of natural oscillations. First, we show that the spectrum $L(\lambda)$ is discrete with a condensation point at infinity.

Denote

$$D_0(\alpha)u = (\Delta_0^2 + \alpha I)^{-1}, \quad u \in H_1,$$

$$P_j(\alpha)u = g_0(M, M_j, -\alpha)u(M_j), \ \ j = \overline{1, n}, \quad \mu(\alpha) = \lambda + \alpha.$$

Let us note that if $-\alpha$ is not an eigenvalue of the operator Δ_0^2, then the operator $(\Delta_0^2 + \alpha I)^{-1}(\Delta_0^2 + I)$ is bounded and boundedly invertible in the space H_1.

In this case it is evident that the equations $L(\lambda)u = 0$ and $(\Delta_0^2 + \alpha I)^{-1}(\Delta_0^2 + I)L(\lambda)u = 0$ are equivalent. Let us transform the last equation:

$$u(M) = \mu(\alpha)\left(D_0(\alpha) + \sum_{j=1}^{n} k_j P_j(\alpha)\right)u -$$

$$-\sum_{j=1}^{n}(k_j+m_j)P_j(\alpha)u. \tag{12}$$

In this equation, the operator

$$D_1(\alpha)=D_0(\alpha)+\sum_{j=1}^{n}m_jP_j(\alpha):H_1\to H_1$$

is compact, and the operator

$$D_2(\alpha)=\sum_{j=1}^{n}(k_j+m_j)P_j(\alpha):H_1\to H_1$$

is bounded and finite-dimensional.

Note that since $\| D_2(\alpha) \|\to 0$, for $\alpha\to+\infty$ and $\alpha\in\mathbb{R}$, then without loss of generality we can assume that the operator $(I+D_2(\alpha))^{-1}:H_1\to H_1$ is bounded for any set of fixed parameters $k_j\in\mathbb{C}, m_j\in\mathbb{C}$, $j=\overline{1,n}$. Consequently, equation (12) can be rewritten in the form

$$u(M)=\mu(\alpha)\left(I+D_2(\alpha)\right)^{-1}D_1(\alpha)u. \tag{13}$$

Let us show that the spectrum of the operator $(I+D_2(\alpha))^{-1}D_1(\alpha)$ consists of an infinite series of eigenvalues $\sigma_k=1/\mu_k(\alpha)$ with a single condensation point at the zero point.

The operator $(I+D_2(\alpha))^{-1}D_1(\alpha):H_1\to H_1$ is compact, hence its spectrum consists only of eigenvalues of finite multiplicity except possibly a zero point.

It is easy to show that the dimension of the kernel of the operator $(I+D_2(\alpha))^{-1}D_1(\alpha)$ can not be greater than $n+1$, and its image is an infinite-dimensional subspace of $W_2^2(\Omega)$.

Thus, the following theorem is true.

Theorem 2. *The spectrum of the operator pencil*

$$L(\lambda)u=u-(\lambda+1)\left(I+\Delta_0^2\right)^{-1}u+\sum_{j=1}^{n}(k_j-\lambda m_j)g_0(M,M_j,-1)u(M_j)$$

for an arbitrary set of parameters $\overrightarrow{k}=(k_1,..,k_n)\subset\mathbb{C}^n,\overrightarrow{m}=(m_1,..,m_n)\subset\mathbb{C}^n$ *is discrete with a single condensation point at infinity.*

Let λ be a regular point of the unperturbed operator $L_0(\lambda) : H_1 \to H_1$, where

$$L_0(\lambda)u = u - (\lambda + 1)(I + \Delta_0^2)^{-1}u. \tag{14}$$

Then the operator $(\Delta_0^2 - \lambda I)^{-1}(I + \Delta_0^2)$ is boundedly invertible in H_1 and the spectral problems $L(\lambda)u = 0$ and $\widehat{L}(\lambda)u = (\Delta_0^2 - \lambda I)^{-1}(I + \Delta_0^2)L(\lambda)u = 0$ are equivalent. It is clear that the equation $\widehat{L}(\lambda)u = 0$ can be written in the form

$$\widehat{L}(\lambda)u = u(M, \lambda) - \sum_{j=1}^{n}(\lambda m_j - k_j)g_0(M, M_j, \lambda)u(M_j, \lambda) = 0. \tag{15}$$

Considering the equation (15), let us show that only at the points $M = M_j, j = \overline{1, n}$, it is equivalent to a finite-dimensional system. For this purpose, let us write the equation (15) for a series of points $M = M_j, j = \overline{1, n}$:

$$\begin{cases} u(M_1, \lambda) = \sum_{j=1}^{n}(\lambda m_j - k_j)g_0(M_1, M_j, \lambda)u(M_j, \lambda) \\ u(M_2, \lambda) = \sum_{j=1}^{n}(\lambda m_j - k_j)g_0(M_2, M_j, \lambda)u(M_j, \lambda) \\ \qquad\qquad\qquad\qquad \ldots\ldots \\ u(M_n, \lambda) = \sum_{j=1}^{n}(\lambda m_j - k_j)g_0(M_n, M_j, \lambda)u(M_j, \lambda) \end{cases}$$

Let

$$G_0(\lambda) = \begin{pmatrix} g_0(M_1, M_1, \lambda) & \ldots & g_0(M_1, M_n, \lambda) \\ g_0(M_2, M_1, \lambda) & \ldots & g_0(M_2, M_n, \lambda) \\ \ldots & \ldots & \ldots \\ g_0(M_n, M_1, \lambda) & \ldots & g_0(M_n, M_n, \lambda) \end{pmatrix},$$

$A = diag\{k_1, k_2, ..., k_n\}$, $B = diag\{m_1, m_2, ..., m_n\}$ are diagonal matrices and $\vec{u}(\lambda) = (u(M_1, \lambda), u(M_2, \lambda), ..., u(M_n, \lambda))^T$. Then the series of obtained equalities can be written in the following matrix form:

$$(I + G_0(\lambda)(A - \lambda B))\, \vec{u}(\lambda) = 0. \tag{16}$$

For the solvability of system (16) it is necessary and sufficient that

$$f(\lambda) = \det\left(I + G_0(\lambda)(A - \lambda B)\right) = 0. \tag{17}$$

Let us note that the meromorphic function $f(\lambda)$ with poles at the points $\lambda \in \sigma(L_0)$ is the spectrum of the (discrete) operator L_0.

Let λ_i^* be a zero of the function $f(\lambda)$. Then it follows from (15) that the eigenfunctions of the operator $L(\lambda)$ admit a representation

$$u(M, \lambda_i^*) = \sum_{j=1}^{n}(\lambda_i^* m_j - k_j)g_0(M, M_j, \lambda_i^*)u(M_j, \lambda_i^*). \tag{18}$$

Thus, the following statement is true.

Theorem 3. *Each eigenvalue of the pencil* $L(\lambda)$ *for any set of parameters* $\overrightarrow{k} = (k_1, .., k_n) \subset \mathbb{C}^n, \overrightarrow{m} = (m_1, .., m_n) \subset \mathbb{C}^n$ *is either a zero of* $f(\lambda)$*, or is an eigenvalue of the pencil* $L_0(\lambda)$*.*

4. The inverse spectral problem and its properties

Consider the inverse spectral problem for the operator pencil $L(\lambda)$, which linearly depends on the parameters $\vec{k} = (k_1, ..k_n), \ \ \vec{m} = (m_1, .., m_n)$:

$$L(\lambda)u = L(\lambda, \vec{k}, \vec{m})u =$$

$$= u - (\lambda + 1)\left(I + \Delta_0^2)\right)^{-1} u + \sum_{j=1}^{n}(k_j - \lambda m_j) g_0(M, M_j, -1)u(M_j).$$

The eigenvalues of the pencil $L(\lambda, \vec{k}, \vec{m})$ also depend on the parameters $\vec{k} = (k_1, ..k_n), \ \ \vec{m} = (m_1, .., m_n)$, and this dependence is analytic, with the exception of singular points that form a set of measure zero.

Let us formulate a multiparameter inverse spectral problem (MPISP) for the operator pencil $L(\lambda, \vec{k}, \vec{m})$.

MPISP *It is required to choose such values* m_j *of point masses and stiffness coefficients of springs* k_j*, that* $2n$ *of eigenvalues of the operator pencil* $L(\lambda, \vec{k}, \vec{m})$ *are equal to the given numbers* $\lambda_1^*, \lambda_2^*, ..., \lambda_n^*, \lambda_{n+1}^*, ..., \lambda_{2n}^*$.

In this case $\vec{\lambda}^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_n^*, \lambda_{n+1}^*, ..., \lambda_{2n}^*) \in \mathbb{C}^{2n}$ will be called the spectral data of MPISP. By the solution of MPISP we will mean a set of parameters $\vec{k}, \vec{m} \in \mathbb{C}^n$, such that $dim\left(ker(L(\lambda_j^*, \vec{k}, \vec{m}))\right) > 0$ for $\forall j = \overline{1, 2n}$.

Further we'll show that MPISP for the operator pencil $L(\lambda, \vec{k}, \vec{m})$ can be represented as MPISP for a finite-dimensional operator. Let us rename the matrices and parameters for convenience.

$$m_1 = p_1, m_2 = p_2 ..., m_n = p_n, \quad k_1 = p_{n+1}, ..., k_n = p_l, \quad l = 2n,$$

$$B_0(\lambda) = I, \quad B_k(\lambda) = \begin{pmatrix} 0 & ... & g_0(M_1, M_k, \lambda) & ... & 0 \\ 0 & ... & g_0(M_2, M_k, \lambda) & ... & 0 \\ ... & ... & ... & ... & ... \\ 0 & ... & g_0(M_n, M_k, \lambda) & ... & 0 \end{pmatrix},$$

$$B_{n+k}(\lambda) = \lambda \cdot B_k(\lambda) \quad k = \overline{1, n}.$$

Then the system (16) can be written in the form

$$B(\vec{p},\lambda)\vec{u}(\lambda) = (B_0(\lambda) + p_1B_1(\lambda) + ... + p_lB_l(\lambda))\,\vec{u}(\lambda) = 0, \qquad (19)$$

where $\vec{p} = (p_1, p_2, ..., p_l) \in \mathbb{C}^l$.

Note that $B(\vec{p},\lambda) : \mathbb{C} \to \mathbb{E}^n$ for each fixed vector $\vec{p}$ is a meromorphic function in $\lambda \in \mathbb{C}$ with poles at the points of the spectrum of the unperturbed pencil $L_0(\lambda)$. Thereby, we will further require that the spectral data $\vec{\lambda}^* = (\lambda_1^*, \lambda_2^*, ..., \lambda_l^*) \in \mathbb{C}^l$ of MPISP do not intersect with the spectrum of an unperturbed pencil of operators $L_0(\lambda)$.

Let us reformulate the initial MPISP for an infinite-dimensional pencil $L(\lambda,\vec{k},\vec{m}) : H_1 \to H_1$ equivalent to the MPISP for a finite-dimensional matrix pencil $B(\vec{p},\lambda) : \mathbb{E}^n \to \mathbb{E}^n$.

MPISPf *It is required to find possible values of the vector $\vec{p}$ from the space $\mathbb{C}^l$, for which the given numbers $\lambda_1, \lambda_2, ..., \lambda_l$ are the eigenvalues of the operator $B(\vec{p},\lambda)$, i.e. $det(B(\vec{p},\lambda)) = 0$.*

Thus, under the assumption that λ_k^* is not an eigenvalue of the pencil $L_0(\lambda)$, the inverse spectral problem for the operator pencil is reduced to a system of $l = 2n$ polynomial equations of $l = 2n$ variables $p_1, .., p_l$

$$det(B(\vec{p},\lambda_j^*)) = 0, \quad j = \overline{1,l}.$$

It can be argued that the solution of the inverse spectral problem for $L(\lambda,\vec{k},\vec{m})$ is the union of algebraic manifolds in $\mathbb{C}^l$ and a finite set of points in $\mathbb{C}^l$.

The existence and isolation problems of the solutions of MPISP for oscillations of elastic plates and shells with dotted fastenings will be discussed in detail in a separate paper.

In conclusion, note that the mathematical model proposed in this paper for describing an elastic plate with point fixings can also be used to study the elastic oscillations of beams, shells, and bulk bodies with additional point fixings.

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