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Fuzzy Normed Linear Spaces and Fuzzy Frames

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Abstract. In this paper, we show that every classical inner product on a linear space induces the fuzzy inner product and fuzzy norm in the sense of Bag and Samanta. We prove the Cauchy-Schwarz inequality on fuzzy Hilbert spaces. We also define fuzzy frame on fuzzy Hilbert spaces. As is known, $C^\infty(\Omega)$ is not normable in classical Hilbert space, but in this paper we show that $C^\infty(\Omega)$ is normable in fuzzy Hilbert space and so defining fuzzy frame on $C^\infty(\Omega)$ is possible.

Key Words and Phrases: fuzzy norm, fuzzy inner product space, fuzzy Hilbert space, fuzzy frame.

2010 Mathematics Subject Classifications: 46S40, 54A40

1. Introduction

The idea of fuzzy norms on a linear space was first introduced by Katsaras [17] in 1984. Later on, many authors, such as Felbin [16], Cheng, Mordeson [8], Bag and Samanta [2] etc. gave different definitions of fuzzy normed linear spaces. R. Biswas [7] and A. M. El-Abyad and H. M. El-Hamouly [15] tried to give a meaningful definition of fuzzy inner product space and associated fuzzy norm function by restricting to the real linear space. P. Mazumder and S. K. Samanta introduced the definition of fuzzy inner product space in the sense of Bag and Samanta fuzzy norm [2]. Moreover, B. T. Bilalov et al. investigated the intuitionistic fuzzy normed space of coefficients [4], and B. T. Bilalov and F. A. Guliyeva studied the basicity and weak basicity of system in the intuitionistic fuzzy metric space in [5, 6]. Recently, B. Daraby and et al. [10] studied some properties of fuzzy Hilbert spaces and showed that all results in classical Hilbert spaces are immediate consequences of the corresponding results for Felbin-fuzzy

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Hilbert spaces. Also, they provided an example where the spectrum of the category of Felbin- fuzzy Hilbert spaces is broader than that of the category of classical Hilbert spaces [11].

One of the important concepts in the study of vector spaces is basis, which allows every vector to be uniquely represented as a linear combination of basis elements. The main feature of a basis $\{x_k\}$ in a Hilbert space H is that every $x \in H$ can be represented as a linear combination of elements x_k in the form:

$$x = \sum_{k=1}^{\infty} c_k(x) x_k. \tag{1}$$

The coefficients $c_k(x)$ are unique. However, the linear independence property for a basis which implies the uniqueness of coefficients is restrictive in applications; sometimes it is impossible to find vectors which both fulfill the basis requirements and satisfy external conditions demanded by applied problems. For such purposes, a more flexible types of spanning sets are needed. Frames provide these alternatives. Frames are used in signal and image processing, non-harmonic Fourier series, data compression, and sampling theory. Today, frame theory has ever increasing applications to problems in both pure and applied mathematics, physics, engineering, computer science and etc.

Many physical systems are inherently nonlinear functions and must be described by non-linear models. But some systems have an uncertain structure and it is not possible to provide an accurate mathematical model. Therefore, the conventional control models cannot be applied to these systems. To solve these problems, we need to use a new concept, namely, fuzzy frames theory and fuzzy waveletes. Fuzzy frame and fuzzy wavelet came from frame theory, wavelet theory and fuzzy concepts. For more information on approximation functions, control and identification of nonlinear systems, see [3, 20]. It not only retains the frame and wavelet properties, but also has some advantages such as simple structure for approximation and good interpretability approximation of non-linear functions.

In this paper, we define fuzzy inner product satisfying (FIP8) and (FIP9) on linear spaces, so we get the fuzzy norm in the sense of Bag and Samanta. In Section 4, we introduce fuzzy frame and show that in fuzzy Hilbert space, $C^\infty(\Omega)$ is normable and so we can define fuzzy frame on $C^\infty(\Omega)$.

2. Preliminaries

In this section, some definitions and preliminary results are given which will be used in this paper.

Definition 1. *[2]. Let U be a linear space over the field F. A fuzzy subset N of $U \times \mathbb{R}$ is called a fuzzy norm on U if for all $x, u \in U$ and $c \in F$, the following conditions are satisfied:*

(N1) $\forall t \in \mathbb{R}$ *with* $t \leq 0$, $N(x,t) = 0$;

(N2) $(\forall t \in \mathbb{R}, t > 0, N(x,t) = 1)$ *iff* $x = \underline{0}$;

(N3) $\forall t \in \mathbb{R}, t > 0, N(cx,t) = N(x, \frac{t}{|c|})$ *if* $c \neq 0$;

(N4) $\forall s,t \in \mathbb{R}, x,u \in U, N(x+u, s+t) \geq \min\{N(x,s), N(u,t)\}$;

(N5) $N(x,.)$ *is a non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t) = 1$.

The pair (U,N) will be referred to as a fuzzy normed linear space.

Theorem 1. *[2] Let (U,N) be a fuzzy normed linear space. Assume further that,*

(N6) $\forall t > 0, N(x,t) > 0 \Rightarrow x = \underline{0}$.

Define $\|x\|_\alpha = \bigwedge\{t > 0 : N(x,t) \geq \alpha\}$, $\alpha \in (0,1)$. Then $\{\|.\|_\alpha : \alpha \in (0,1)\}$ is an ascending family of norms on U and they are called α- norms on U corresponding to the fuzzy norm N on U.

Definition 2. *[1] Let (U,N) be a fuzzy normed linear space. Let $\{x_n\}$ be a sequence in U. Then $\{x_n\}$ is said to be convergent if there exists $x \in U$ such that $\lim_{n\to\infty} N(x_n - x, t) = 1$, for all $t > 0$. In that case, x is called the limit of the sequence $\{x_n\}$ and denoted by $\lim x_n$.*

Proposition 1. *[8] Let (U,N) be a fuzzy normed linear space satisfying (N_6) and $\{x_n\}$ be a sequence in U. Then $\{x_n\}$ converges to x iff $x_n \to x$ w.r.t. $\|.\|_\alpha$, for all $\alpha \in (0,1)$.*

Definition 3. *[1] Let (U,N) be a fuzzy normed linear space and $\alpha \in (0,1)$. A sequence $\{x_n\}$ in U is said to be α-convergent in U if there exists $x \in U$ such that $\lim_{n\to\infty} N(x_n - x, t) > \alpha$, for all $t > 0$ and x is called the limit of $\{x_n\}$.*

Proposition 2. *[19] Let (U,N) be a fuzzy normed linear space satisfying (N_6). If $\{x_n\}$ is an α-convergent sequence in (U,N), then $\|x_n - x\|_\alpha \to 0$ as $n \to \infty$.*

Definition 4. *[18] Let U be a linear space over the field $\mathbb{C}$ of complex numbers. Let $\mu : U \times U \times \mathbb{C} \longrightarrow I = [0,1]$ be a mapping such that the following conditions and statements hold:*

(FIP1) *for* $s,t \in \mathbb{C}, \mu(x+y, z, |t|+|s|) \geq \min\{\mu(x,z,|t|), \mu(y,z,|s|)\}$;

(FIP2) *for* $s, t \in \mathbb{C}, \mu(x, y, |st|) \geq \min\left\{\mu\left(x, x, |s|^2\right), \mu\left(y, y, |t|^2\right)\right\}$*;*

(FIP3) *for* $t \in \mathbb{C}, \mu(x, y, t) = \mu\left(y, x, \overline{t}\right)$*;*

(FIP4) $\mu(\alpha x, y, t) = \mu(x, y, \frac{t}{|\alpha|}), \alpha\,(\neq 0) \in \mathbb{C}, t \in \mathbb{C}$*;*

(FIP5) $\mu(x, x, t) = 0, \forall t \in \mathbb{C}\backslash\mathbb{R}^+$*;*

(FIP6) $(\mu(x, x, t) = 1, \forall t > 0)$ *iff* $x = 0$*;*

(FIP7) $\mu(x, x, .) : \mathbb{R} \longrightarrow I$ *is a monotonic non-decreasing function on* $\mathbb{R}$ *and* $\lim_{t\to\infty} \mu(\alpha x, x, t) = 1$*.*

We call μ *fuzzy inner product function on* U *and* (U, μ) *fuzzy inner product space (FIP space).*

Theorem 2. *[18] Let* U *be a linear space over* $\mathbb{C}$*. Let* μ *be a FIP on* U*. Then*

$$N(x,t) = \begin{cases} \mu(x, x, t^2) & \text{if } t \in \mathbb{R}, t > 0, \\ 0 & \text{if } t \leq 0. \end{cases}$$

is a fuzzy norm on U*. Now if* μ *satisfies the following conditions:*

(FIP8) $\left(\mu\left(x, x, t^2\right) > 0, \forall t > 0\right) \Rightarrow x = 0$ *and*

(FIP9) *for all* $x, y \in U$ *and* $p, q \in \mathbb{R}$*,*

$$\mu\left(x + y, x + y, 2q^2\right) \bigwedge \mu\left(x - y, x - y, 2p^2\right) \geq \mu\left(x, x, p^2\right) \bigwedge \mu\left(y, y, q^2\right),$$

then $\|x\|_\alpha = \bigwedge\{t > 0 : N(x,t) \geq \alpha\}, \alpha \in (0,1)$ *is an ordinary norm satisfying parallelogram law. By using polarization identity, we can get ordinary inner product, called the* $\langle ., . \rangle_\alpha$*-inner product, as follows:*

$$\langle x, y\rangle_\alpha = \frac{1}{4}\left(\|x + y\|_\alpha^2 - \|x - y\|_\alpha^2\right) + \frac{1}{4}i\left(\|x + iy\|_\alpha^2 - \|x - iy\|_\alpha^2\right), \forall\alpha \in (0,1).$$

Definition 5. *[18] Let* (U, μ) *be a FIP space satisfying* $(FIP8)$*. The linear space* U *is said to be level complete if for any* $\alpha \in (0,1)$*, every Cauchy sequence converges w.r.t.* $\|.\|_\alpha$ *(the* α*-norm generated by the fuzzy norm* N *which is induced by fuzzy inner product* μ *).*

Definition 6. *[1] Let* $T : (U, N_1) \longrightarrow (V, N_2)$ *be a linear operator where* (U, N_1) *and* (V, N_2) *are fuzzy normed linear spaces. The mapping* T *is said to be strongly fuzzy bounded on* U *if and only if there exists a positive real number* M *such that*

$$N_2(T(x), s) \geq N_1(x, \tfrac{s}{M}), \quad \forall x \in U, \forall s \in \mathbb{R}.$$

Definition 7. *[1] Let $T : (U, N_1) \longrightarrow (V, N_2)$ be a linear operator where (U, N_1) and (V, N_2) are fuzzy normed linear spaces. The mapping T is said to be uniformly bounded if there exists $M > 0$ such that*

$$\|Tx\|_\alpha^2 \leq M\|x\|_\alpha^1 \quad \forall \alpha \in (0, 1),$$

where $\|.\|_\alpha^1$ and $\|.\|_\alpha^2$ are α-norms on N_1 and N_2, respectively.

Remark 1. *Let us denote the set of all strongly fuzzy bounded linear operators from a fuzzy normed linear space (U, N_1) to (V, N_2) by $B(U, V)$.*

Theorem 3. *[1] Let $T : (U, N_1) \longrightarrow (V, N_2)$ be a linear operator where (U, N_1) and (V, N_2) are fuzzy normed linear spaces satisfying (N_6). Then T is strongly fuzzy bounded if and only if it is uniformly bounded with respect to α-norms of N_1 and N_2.*

Definition 8. *[1] Let (U, N_1) and (V, N_2) be two fuzzy normed linear spaces satisfying (N_6). For $T \in B(U, V)$, let*

$$\|T\|_\beta^{'} = \bigvee_{x \in U, x \neq \underline{0}} \frac{\|Tx\|_\beta^2}{\|x\|_\beta^1}, \quad \beta \in (0, 1),$$

and

$$\|T\|_\alpha = \bigvee_{\beta \leq \alpha} \|T\|_\beta^{'}, \quad \alpha \in (0, 1).$$

Then $\{\|.\|_\alpha : \alpha \in (0, 1)\}$ is an ascending family of norms in $B(U, V)$.

Definition 9. *[18] Let (U, μ) be a FIP space. The linear space U is said to be a fuzzy Hilbert space if it is level complete.*

Definition 10. *[18] Let $\alpha \in (0, 1)$ and (U, μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$. Now, if $x, y \in U$ are such that $\langle x, y\rangle_\alpha = 0$, then we say that x, y are α-fuzzy orthogonal to each other and denote it by $x \perp_\alpha y$. Let M be a subset of U and $x \in U$. Now if $\langle x, y\rangle_\alpha = 0$ for all $y \in M$, then we say that x is α-fuzzy orthogonal to M and denote it by $x \perp_\alpha M$. The set of all α-fuzzy orthogonal elements to M is called α-fuzzy orthogonal set.*

Definition 11. *[18] Let (U, μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$. Now if $x, y \in U$ are such that $\langle x, y\rangle_\alpha = 0$, for all $\alpha \in (0, 1)$, then we say that x, y are fuzzy orthogonal to each other and denote it by $x \perp y$. Thus $x \perp y$ if and only if $x \perp_\alpha y$, for all elements $\alpha \in (0, 1)$. The set of all elements fuzzy orthogonal to each other is called fuzzy orthogonal set.*

Definition 12. *[19] Let (U, μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$ and $\alpha \in (0,1)$. An α-fuzzy orthogonal set M in U is said to be α-fuzzy orthonormal if the elements have α-norm $1, \alpha \in (0,1)$, that is for all $x, y \in M$,*

$$\langle x, y\rangle_\alpha = \begin{cases} 1 & , \quad x = y \\ 0 & , \quad x \neq y, \end{cases}$$

where $\langle ., . \rangle_\alpha$ is an inner product induced by μ.

Definition 13. *[19] Let (U, μ) be a FIP space satisfying $(FIP8)$ and $(FIP9)$. A fuzzy orthonormal set M in U is said to be fuzzy orthonormal if the elements have α-norm 1 for all $\alpha \in (0,1)$, that is for all $x, y \in M$*

$$\langle x, y\rangle_\alpha = \begin{cases} 1 & , \quad x = y \\ 0 & , \quad x \neq y, \end{cases}$$

where $\langle ., . \rangle_\alpha$ is an inner product induced by μ.

Proposition 3. *[19] An α-fuzzy orthonormal set and a fuzzy orthonormal set in a FIP space are linearly independent.*

Proposition 4. *[19] Let $\{e_k\}_{k=1}^\infty$ be a fuzzy orthonormal sequence in a fuzzy Hilbert space (U, μ) satisfying $(FIP8)$ and $(FIP9)$. Then the series $\sum_{k=1}^\infty \alpha_k e_k$ converges if and only if $\sum_{k=1}^\infty |\alpha_k|^2$ converges.*

Proposition 5. *[19] Let (U, μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$, $\alpha \in (0,1)$ and $\{e_k\}_{k=1}^\infty$ be an α-fuzzy orthonormal sequence in U. If the series $\sum_{k=1}^\infty \beta_k e_k$ is α-convergent w.r.t. N induced by μ, then the coefficients $\beta_k = \langle x, e_k\rangle_\alpha$, where x denotes the sum $\sum_{k=1}^\infty \beta_k e_k$ and hence $x = \sum_{k=1}^\infty \langle x, e_k\rangle_\alpha e_k$.*

Theorem 4. *[19] Let (U, μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$, $\alpha \in (0,1)$ and $\{e_k\}_{k=1}^\infty$ be an α-fuzzy orthonormal sequence in U. If the series $\sum_{k=1}^\infty \gamma_k e_k$ converges w.r.t. N induced by μ, then*

$$\gamma_k = \langle x, e_k\rangle_\alpha = \langle x, e_k\rangle_\beta, \quad \forall \alpha, \beta \in (0,1),$$

where $\langle ., . \rangle$ denotes the α-inner product induced by μ, x denotes the sum of $\sum_{k=1}^\infty \gamma_k e_k$. Hence

$$x = \textstyle\sum_{k=1}^\infty \langle x, e_k\rangle_\alpha e_k = \sum_{k=1}^\infty \langle x, e_k\rangle_\beta e_k, \quad \forall \alpha, \beta \in (0,1).$$

Theorem 5. *[19] Let (U, μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$ and $\{e_k\}_{k=1}^\infty$ be a fuzzy orthonormal sequence in U. Then the following statements are equivalent:*

(i) $\{e_k\}_{k=1}^{\infty}$ *is complete fuzzy orthonormal;*

(ii) *if* $x \perp e_i$ *for* $i = 1, 2, \ldots,$ *then* $x = 0$*;*

(iii) *For every* $x \in U, x = \sum_{k=1}^{\infty} \langle x, e_i \rangle_\alpha e_i$ *for all* $\alpha \in (0,1)$ *and hence*

$$\langle x, e_k \rangle_\alpha = \langle x, e_k \rangle_\beta, \quad \forall \alpha, \beta \in (0,1)\,;$$

i.e. x *is independent of* α*.*

(iv) *For every* $x \in U$*,* $\|x\|_\alpha^2 = \sum_{k=1}^{\infty} |\langle x, e_i \rangle_\alpha|$ *for all* $\alpha \in (0,1)$ *and hence*

$$\|x\|_\alpha^2 = \|x\|_\beta^2, \quad \forall \alpha, \beta \in (0,1)\,.$$

We denote by U^* the set of all strongly fuzzy bounded linear functionals over (U, N_1) (where N_1 is a norm induced by fuzzy inner product).

Theorem 6. *[19] (Riesz Representation Theorem) Let* (U, μ) *be a fuzzy Hilbert space satisfying* $(FIP8)$ *and* $(FIP9)$ *and* $f \in U^*$*. Then for each* $\alpha \in (0,1)$*, there is* $y_\alpha \in U$ *such that* $f(x) = \langle x, y_\alpha \rangle_\alpha$*, where* y_α *depends on* f *and* $\|f\|_\alpha^* \geq \|y_\alpha\|_\alpha$ *when* $\alpha \geq \dfrac{1}{2}$ *and* $\|f\|_{1-\alpha}^* \leq \|y_\alpha\|_{1-\alpha}$ *when* $\alpha \leq \dfrac{1}{2}$*.*

3. Some properties of fuzzy inner product spaces

In this section, we present some properties of the space $(B(U,V), \|.\|_\alpha)$ and some properties of fuzzy linear spaces similar to those of ordinary normed spaces.

Proposition 6. *Let* (U, μ) *be a FIP space satisfying* $(FIP8)$ *and* $(FIP9)$*. A fuzzy inner product space* (U, μ) *with its corresponding norm* N *satisfies the Schwartz inequality*

$$|\langle x, y \rangle_\alpha| \leq \|x\|_\alpha \|y\|_\alpha \quad \forall \alpha \in (0,1].$$

Proof. First, we show that for all $\alpha \in (0,1)$, $\langle x, x \rangle_\alpha = \|x\|_\alpha^2$.
According to the definition of α-fuzzy inner product, by supposing $x = y$ we have:

$$\begin{aligned}
\langle x, x \rangle_\alpha &= \frac{1}{4}(\|x+x\|_\alpha^2 - \|x-x\|_\alpha^2) + \frac{i}{4}(\|x+ix\|_\alpha^2 - \|x-ix\|_\alpha^2) \\
&= \frac{1}{4}(4\|x\|_\alpha^2 - 0) + \frac{i}{4}x(\|1+i\|_\alpha^2 - \|1-i\|_\alpha^2) \\
&= \|x\|_\alpha^2.
\end{aligned}$$

Therefore $\langle x, x\rangle_\alpha = \|x\|_\alpha^2$. Let $x, y \in U$ be arbitrary. In the special case where $y = 0$, the assertion is trivially true. Assume that $y \neq 0$. Considering $\lambda \in \mathbb{C}$ and $\lambda = \frac{\langle x,y\rangle_\alpha}{\|y\|_\alpha^2}$, for all $\alpha \in (0, 1)$, we have:

$$\begin{aligned} 0 &\leq \|x - \lambda y\|_\alpha^2 \\ &= \langle x, x\rangle_\alpha - \langle \lambda y, x\rangle_\alpha - \langle x, \lambda y\rangle_\alpha + \langle \lambda y, \lambda y\rangle_\alpha \\ &= \langle x, x\rangle_\alpha - \lambda\langle y, x\rangle_\alpha - \overline{\lambda}\langle x, y\rangle_\alpha + \lambda\overline{\lambda}\langle y, y\rangle_\alpha \\ &= \|x\|_\alpha^2 - \lambda\overline{\langle x, y\rangle_\alpha} - \overline{\lambda}\langle x, y\rangle_\alpha + \lambda\overline{\lambda}\|y\|_\alpha^2 \\ &= \|x\|_\alpha^2 - \frac{|\langle x, y\rangle_\alpha|^2}{\|y\|_\alpha^2} - \frac{|\langle x, y\rangle_\alpha|^2}{\|y\|_\alpha^2} + \frac{|\langle x, y\rangle_\alpha|^2}{\|y\|_\alpha^2} \\ &= \|x\|_\alpha^2 - \frac{|\langle x, y\rangle_\alpha|^2}{\|y\|_\alpha^2}. \end{aligned}$$

Therefore

$$0 \leq \|x\|_\alpha^2 - \frac{|\langle x, y\rangle_\alpha|^2}{\|y\|_\alpha^2},$$

It follows that $|\langle x, y\rangle_\alpha| \leq \|x\|_\alpha\|y\|_\alpha$. ◀

Example 1. *Let $(U, \langle ., .\rangle)$ be a real inner product space. Define a function $\mu : U \times U \times \mathbb{C} \to [0, 1]$ by*

$$\mu(x, y, t) = \begin{cases} \dfrac{|t|}{|t| + \|x\|\|y\|} & \text{if } t > \|x\|\|y\|, \\ 0 & \text{if } t \leq \|x\|\|y\|, \\ 0 & \text{if } t \in \mathbb{C} \setminus \mathbb{R}^+. \end{cases}$$

We verify the following conditions.

(FIP1) If at least one of $|t|$ and $|s|$ is zero, then the result is obvious. Suppose that $|t|$ and $|s|$ are non-zero. Let us assume without loss of generality that $\mu(x, z, |t|) \leq \mu(y, z, |s|)$. So, we have

$$\frac{|t|}{|t| + \|x\|\|z\|} \leq \frac{|s|}{|s| + \|y\|\|z\|}.$$

It follows that

$$1 + \frac{\|x\|\|z\|}{|t|} \geq 1 + \frac{\|y\|\|z\|}{|s|}.$$

Now, we have

$$\frac{|s|}{|t|}\|x\|\|z\| \geq \|y\|\|z\|.$$

By adding $\|x\|\|z\|$ to both sides of the above inequality, we get

$$\frac{|s|}{|t|}\|x\|\|z\| \geq \|x+y\|\|z\| - \|x\|\|z\|.$$

Therefore

$$(\frac{|s|}{|t|}+1)\|x\|\|z\| \geq \|x+y\|\|z\|.$$

It follows that

$$\frac{\|x\|\|z\|}{|t|} \geq \frac{\|x+y\|\|z\|}{|t|+|s|}.$$

By adding 1 to both sides of the last inequality, we get

$$1+\frac{\|x\|\|z\|}{|t|} \geq 1+\frac{\|x+y\|\|z\|}{|t|+|s|}.$$

So, we have

$$\frac{|t|}{|t|+\|x\|\|z\|} \leq \frac{|t|+|s|}{|t|+|s|+\|x+y\|\|z\|}.$$

Hence $\mu\,(x+y,z,|t|+|s|) \geq \min\{\mu(x,z,|t|),\mu(y,z,|s|)\}$.

(FIP2) If at least one of $|t|$ and $|s|$ is zero, then the condition obviously holds. Let none of $|t|$ and $|s|$ be non-zero. Without loss of generality we assume $\mu(x,x,|s|^2) \leq \mu(y,y,|t|^2)$. Theno we have

$$\frac{|s|^2}{|s|^2+\|x\|^2} \leq \frac{|t|^2}{|t|^2+\|y\|^2}.$$

It follows that

$$\frac{\|x\|^4}{|s|^4} \geq \frac{\|y\|^2\|x\|^2}{|t|^2|s|^2}.$$

Therefore, we have

$$\frac{\|x\|^2}{|s|^2} \geq \frac{\|y\|\|x\|}{|t||s|}.$$

By adding 1 *to both sides of the above inequality, we get*

$$1 + \frac{\|x\|^2}{|s|^2} \geq 1 + \frac{\|y\|\|x\|}{|t||s|},$$

so

$$\frac{|s|^2 + \|x\|^2}{|s|^2} \geq \frac{|t||s| + \|y\|\|x\|}{|t||s|}.$$

Hence $\mu\,(x, y, |ts|) \geq \min\Big\{\mu(x, x, |s|^2), \mu(y, y, |t|^2)\Big\}$.

(FIP3) For $t \in \mathbb{C}$, $\mu(x, y, t) = \dfrac{|t|}{|t| + \|x\|\|y\|} = \dfrac{|\bar{t}|}{|\bar{t}| + \|x\|\|y\|} = \mu(y, x, \bar{t})$.

(FIP4) Suppose that $0 \neq \alpha \in \mathbb{C}$ *and* $t \in \mathbb{C}$. *So we have the following cases:*
Case (i) If $t \neq 0$, *then we consider two following cases:*
(a) If $\min\{\ \|x\|, \|y\|\} = 0$, *then we have*

$$\mu(\alpha x, y, t) = \frac{|t|}{|t| + \|\alpha x\|\|y\|} = \frac{|t|}{|t|} = 1 = \frac{\frac{|t|}{|\alpha|}}{\frac{|t|}{|\alpha|} + \|x\|\|y\|} = \mu(x, y, \frac{t}{|\alpha|}).$$

Thus the condition holds.
(b) If $\min\{\ \|x\|, \|y\|\} \neq 0$, *then we have*

$$\mu(\alpha x, y, t) = \frac{|t|}{|t| + \|\alpha x\|\|y\|} = \frac{|t|}{|t| + |\alpha|\|x\|\|y\|} = \frac{\frac{|t|}{|\alpha|}}{\frac{|t|}{|\alpha|} + \|x\|\|y\|} = \mu(x, y, \frac{t}{|\alpha|}).\ \textit{So,}$$

$\mu(\alpha x, y, t) = \mu(x, y, \dfrac{t}{|\alpha|})$.

Case (ii) If $t = 0$, *then* $\mu(\alpha x, y, t) = \mu(x, y, \dfrac{t}{|\alpha|}) = 0$.

(FIP5) It follows from definition.

(FIP6) Suppose that $x = 0$.

$$x = 0 \Leftrightarrow \|x\|^2 = 0 \Leftrightarrow \forall t > 0, \mu(x, x, t) = \frac{|t|}{|t| + \|x\|^2} = \frac{|t|}{|t|} = 1.$$

(FIP7) For all $t \in \mathbb{R}$ *and* $t > 0$,
$\mu(x,x,t) = \dfrac{|t|}{|t| + \|x\|^2} = \dfrac{1}{1 + \dfrac{\|x\|^2}{|t|}} \to 1$ *as* $t \to \infty$ *and it is obviously monotonicallly non-decreasing.*

Hence, we conclude that every classic inner product induces the fuzzy inner product. In the sequel, we will show that $(FIP8)$ *also holds. So, we have fuzzy norm in the sense of Bag and Samanta.*

(FIP8) $\mu(x,x,t^2) > 0, \forall t > 0 \Rightarrow t > \|x\|^2 \quad \forall t > 0 \Rightarrow x = 0;$

$$\begin{aligned} \|x\|_\alpha &= \bigwedge \left\{t : \mu(x,x,t^2) \geq \alpha\right\} \\ &= \bigwedge \left\{t : \frac{|t|^2}{|t|^2 + \|x\|^2} \geq \alpha\right\} \\ &= \sqrt{\frac{\alpha}{1-\alpha}}\|x\|. \end{aligned}$$

It is clear that $(FIP9)$ *holds. Using polarization identity, the* α*-inner product follows from classic inner product.*

$$\begin{aligned} \|x-y\|_\alpha^2 + \|x+y\|_\alpha^2 &= \frac{\alpha}{1-\alpha}\|x-y\|^2 + \frac{\alpha}{1-\alpha}\|x+y\|^2 \\ &= \frac{\alpha}{1-\alpha}(\|x-y\|^2 + \|x+y\|^2) \\ &= \frac{\alpha}{1-\alpha}(2\|x\|^2 + 2\|y\|^2) \\ &= 2(\|x\|_\alpha^2 + \|y\|_\alpha^2). \end{aligned}$$

It follows that

$$\begin{aligned} \langle x,y\rangle_\alpha &= \frac{1}{4}(\|x+y\|_\alpha^2 - \|x-y\|_\alpha^2) + \frac{i}{4}(\|x+iy\|_\alpha^2 - \|x-iy\|_\alpha^2) \\ &= \frac{\alpha}{4(1-\alpha)}(\|x+y\|^2 - \|x-y\|^2) + \frac{\alpha i}{4(1-\alpha)}(\|x+iy\|^2 - \|x-iy\|^2) \\ &= \frac{\alpha}{1-\alpha}\langle x,y\rangle. \end{aligned}$$

Definition 14. *Let* (U,μ) *and* (V,μ) *be two fuzzy Hilbert spaces satisfying* $(FIP8)$ *and* $(FIP9)$*, where* μ *is the same fuzzy inner product. Let* T *be a fuzzy bounded linear operator from* U *to* V*. If there exists an operator* T^* *from* V *to* U *such that for all* $\alpha \in (0,1)$

$$\langle Tx, y\rangle_\alpha = \langle x, T^* y\rangle_\alpha, \quad \forall x \in U, y \in V,$$

then the operator T^* is called fuzzy adjoint of T.

Theorem 7. *If $T : (U, N_1) \longrightarrow (V, N_2)$ is a strongly fuzzy bounded operator, where (U, N_1) and (V, N_2) are fuzzy normed linear spaces, N_1 and N_2 are the norms induced by fuzzy inner products on U and V, respectively, then there exists $T^* : (V, N_2) \longrightarrow (U, N_1)$ such that for all $x \in U, y \in V$ and for all $\alpha \in (0, 1)$*

$$\langle x, T(y)\rangle_\alpha = \langle T^*(x), y\rangle_\alpha . \tag{2}$$

Proof. To demonstrate the existence of T^*, we have to show that for every $x \in U$, there is a vector $z \in U$, depending linearly on x, such that

$$\langle z, y\rangle_\alpha = \langle T^*(x), y\rangle_\alpha \quad \forall\alpha \in (0, 1).$$

By Theorem 3, T is uniformly bounded and there exists $M > 0$ such that

$$\|T(x)\|_\alpha^2 \le M\|x\|_\alpha^1 \quad \forall\alpha \in (0, 1).$$

Suppose that $\alpha \in (0, 1)$, and for fixed x consider the mapping φ_x, defined by

$$\varphi_x(y) = \langle x, T(y)\rangle_\alpha .$$

The mapping φ_x is a fuzzy bounded linear functional on U crossponding to α, i.e. $\varphi_x \in U_\alpha^*$ and $\|\varphi_x\|_\alpha \le M\|x\|_\alpha$. By the Riesz Representation Theorem, there is a unique $z \in U$ such that $\varphi_x(y) = \langle z, y\rangle_\alpha$.Thus, the equality (2) holds. So, we set $T^*(x) = z$. The linearity of T^* follows from its uniqueness by Riesz Representation Theorem and from the linearity of the inner product. Since we have

$$\begin{aligned} \|T^*(x)\|_\alpha = \|z\| &= \bigvee_{\|y\|_\alpha=1} |\langle y, z\rangle_\alpha| \\ &= \bigvee_{\|y\|_\alpha=1} |\langle T(y), x\rangle_\alpha| \\ &\le \bigvee_{\|y\|_\alpha=1} \|T(y)\|_\alpha \|x\|_\alpha \\ &\le \bigvee_{\|y\|_\alpha=1} \|T\|_\alpha \|y\|_\alpha \|x\|_\alpha = \|T\|_\alpha \|x\|_\alpha, \end{aligned}$$

it follows that T^* is bounded and $\|T^*\|_\alpha \le \|T\|_\alpha$ for any $\alpha \in (0, 1)$. Finally, we show that T^* is unique. Suppose that $S \in B(U, V)$ and $\langle T(x), y\rangle_\alpha = \langle S(x), y\rangle_\alpha$ for all $x \in U$, $y \in V$ and $\alpha \in (0, 1)$. For each fixed y and for every x, we have $\langle x, S(y) - T^*(y)\rangle_\alpha = 0$. It follows that $S(y) - T^*(y) = 0$. Hence $S = T^*$. ◄

Proposition 7. *If $T \in B(U,V)$, then for all $\alpha \in (0,1)$, $\|T^*\|_\alpha = \|T\|_\alpha$.*

Proof. In Theorem 7, we already showed that

$$\|T^*\|_\alpha \leq \|T\|_\alpha \quad \forall \alpha \in (0,1). \tag{3}$$

For $x \in U$, we have

$$\begin{aligned} \|T(x)\|_\alpha^2 &= \langle T(x), T(x)\rangle_\alpha \\ &= \langle T^*T(x), x\rangle_\alpha \\ &\leq \|T^*T(x)\|_\alpha \|x\|_\alpha \\ &\leq \|T^*\|_\alpha \|T(x)\|_\alpha \|x\|_\alpha. \end{aligned}$$

Hence $\|T(x)\|_\alpha \leq \|T^*\|_\alpha \ \|x\|_\alpha$. It follows that

$$\|T\|_\alpha \leq \|T^*\|_\alpha \quad \forall \alpha \in (0,1). \tag{4}$$

From the inequalites (3) and (4) we have

$$\|T\|_\alpha = \|T^*\|_\alpha \quad \forall \alpha \in (0,1).$$

◄

Theorem 8. *Let (U,μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$ and $\alpha \in (0,1)$. Let T be a fuzzy operator on (U,μ). Then T^* is also a fuzzy linear operator on (U,μ) and following properties hold:*

i) $(T^*)^* = T$;

ii) $(T_1 + T_2)^* = T_1^* + T_2^*$;

iii) $(\lambda T)^* = \overline{\lambda} T^*, \quad \forall \lambda \in \mathbb{C}$;

iv) $(ST)^* = T^* S^*$.

Proof. Suppose that $y_1, y_2 \in U$ and $\lambda, \beta \in \mathbb{C}$. For each $x \in U$, we have

$$\begin{aligned} \langle x, T^*(\lambda y_1 + \beta y_2)\rangle_\alpha &= \langle Tx, \lambda y_1 + \beta y_2\rangle_\alpha \\ &= \overline{\lambda}\langle Tx, y_1\rangle_\alpha + \overline{\beta}\langle Tx, y_2\rangle_\alpha \\ &= \langle x, \lambda T^* y_1 + \beta T^* y_2\rangle_\alpha. \end{aligned}$$

It follows that $T^*(\lambda y_1 + \beta y_2) = \lambda T^* y_1 + \beta T^* y_2$, that is, T^* is linear.

For each $x, y \in U$,

$$\langle y,(T^*)^*x\rangle_\alpha=\langle T^*y,x\rangle_\alpha=\overline{\langle x,T^*y\rangle_\alpha}=\overline{\langle Tx,y\rangle_\alpha}=\langle y,Tx\rangle_\alpha.$$

Hence $(T^*)^*=T$, so we have (i).
To prove (ii), we note that

$$\begin{aligned}\langle x,(T_1+T_2)^*y\rangle_\alpha &= \langle (T_1+T_2)x,y\rangle_\alpha\\ &= \langle T_1x,y\rangle_\alpha+\langle T_2x,y\rangle_\alpha\\ &= \langle x,T_1^*y\rangle_\alpha+\langle x,T_2^*y\rangle_\alpha\\ &= \langle x,(T_1^*+T_2^*)y\rangle_\alpha.\end{aligned}$$

(iii) For each $\alpha\in(0,1]$ and $\lambda\in\mathbb{C}$, we have

$$\langle \lambda Tx,y\rangle_\alpha=\lambda\langle Tx,y\rangle_\alpha=\lambda\langle x,T^*y\rangle_\alpha=\langle x,\overline{\lambda}T^*y\rangle_\alpha,\text{ so we get }(iii).$$

For each $x,y\in U$,

$$\langle STx,y\rangle_\alpha=\langle Tx,S^*y\rangle_\alpha=\langle x,T^*S^*y\rangle_\alpha.$$

Therefore $(ST)^*=T^*S^*$. ◄

Corollary 1. *Let (U,μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$ and $\alpha\in(0,1)$. Let T be a fuzzy operator on (U,μ). Then*

$$\|T^*T\|_\alpha=\|TT^*\|_\alpha=\|T\|_\alpha^2.$$

Proof. By Theorem 8, proof is straightforward. For all $x\in U$,

$$\|T^*Tx\|_\alpha\le\|T^*\|_\alpha\|Tx\|_\alpha\le\|T\|_\alpha^2\|x\|_\alpha$$

and therefore $\|T^*T\|_\alpha\le\|T\|_\alpha^2$.
Also, we can write

$$\begin{aligned}\|T^*T\|_\alpha &= \bigvee_{\beta\le\alpha}\|T^*Tx\|_\beta^{'}\\ &= \bigvee_{\beta\le\alpha}\left(\bigvee_{x\in U,x\neq0}\frac{\|T^*Tx\|_\beta^2}{\|x\|_\beta^1}\right)\\ &\geqslant \bigvee_{\beta\le\alpha}\left(\bigwedge_{x\in U,x\neq0}\frac{\|T^*Tx\|_\beta^2}{\|x\|_\beta^1}\right)\\ &= \bigvee_{\beta\le\alpha}\left(\bigwedge_{x\in U,x\neq0}\frac{\|T\|_\beta^2\|Tx\|_\beta^2}{\|x\|_\beta^1}\right)\end{aligned}$$

$$= \bigvee_{\beta \leq \alpha} \frac{\|T^2 x\|_\beta^2}{\|x\|_\beta^1}$$
$$= \|T\|_\alpha^2.$$

Hence we get the assertion. ◀

4. Fuzzy frames theory

In this section, after a brief trip back to the history of frame, we define fuzzy frame and prove some new results.

Frames were introduced in 1952 by Duffin and Schaeffer in their fundamental paper [14]; they used frames as a tool in the study of nonharmonic Fourier series, i.e., sequences of the type $\{e^{i\lambda_n}x\}_{n\in Z}$, where $\{\lambda_n\}_{n\in Z}$ is a family of real or complex numbers. Apparently, the importance of the concept was not realized by the mathematical community; at least it took almost 30 years before the next publication appeared. Frames were presented in the abstract setting, and again used in the context of nonharmonic Fourier series. Then, in 1985, as the wavelet era began, Daubechies, Grossmann and Meyer [12, 13] observed that frames can be used to find series expansions of functions in $L^2(\mathbb{R})$ which are very similar to expansions using orthogonal bases.

Recall that for a Hilbert space H and a countable index set I, a family of vectors $\{x_i\}_{i\in I} \subseteq H$ is called a discrete frame for H, if there exist constants $0 < A \leq B < +\infty$ such that

$$A\|x\|^2 \leq \sum_{i\in I} |\langle x, x_i\rangle|^2 \leq B\|x\|^2, \quad x \in H,$$

where the constants A and B are called frame bounds. The frame $\{x_i\}_{i\in I}$ is called tight if $A = B$ and Parseval if $A = B = 1$. For a very good and useful reference, we refer to the comprehensive book by Christensen [9]. The concept of frame has been improved and generalized to Banach spaces, Frechet spaces and a lot of papers have been published in both pure and applied mathematics concerning frames. In this manuscript, we will try to present the fuzzy frame version of frame theorems and related concepts.

In a fuzzy Hilbert space (U, μ) satisfying $(FIP8)$ and $(FIP9)$ when $\alpha \in (0, 1)$ and $\{e_k\}_{k=1}^\infty$ is an α-fuzzy orthonormal sequence in U, we say that $\{e_k\}_{k=1}^\infty$ is a basis for U if the following two conditions are satisfied:

(i) $U = span\,\{e_k\}_{k=1}^\infty$

(ii) $\{e_k\}_{k=1}^\infty$ is linearly independent.

So, every $x \in U$ has a unique representation in terms of the elements in the basis, i.e., there exist unique coefficients $\{\beta_k\}_{k=1}^{\infty}$ such that $x = \sum_{k=1}^{\infty} \beta_k e_k$. By Proposition 5 and Theorem 4, if (U, μ) is a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$ and $x \in U$, then $\{e_k\}_{k=1}^{\infty}$ is a fuzzy orthonormal sequence in U. Then, since $U = span\, \{e_k\}_{k=1}^{\infty}$, we can write $x = \sum_{k=1}^{\infty} \beta_k e_k$ and $\beta_k = \langle x, e_k \rangle_\alpha$.

Definition 15. *Let (U, μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$. A countable family of elements $\{x_k\}_{k=1}^{\infty}$ in U is a fuzzy frame for U if there exist constants $A, B > 0$ such that for all $x \in U$ and $\alpha \in (0, 1)$:*

$$A\|x\|_\alpha^2 \leq \sum_{k=1}^{\infty} |\langle x, x_k \rangle_\alpha|^2 \leq B\|x\|_\alpha^2. \tag{5}$$

The numbers A and B are called fuzzy frame bounds. Fuzzy frame bounds are not unique. The optimal lower frame bound is supremum over all lower frame bounds, and the optimal upper frame bound is the infimum over all upper frame bounds. Note that the optimal fuzzy frame bounds are actually fuzzy frame bounds. If $\|x_k\|_\alpha = 1$, the fuzzy frame is normalized. A fuzzy frame $\{x_k\}_{k=1}^{\infty}$ is tight if $A = B$ and in case $A = B = 1$, we call it Parseval fuzzy frame. In case the upper inequality in (5) is satisfied, $\{x_k\}_{k=1}^{\infty}$ is called fuzzy Bessel sequence. It follows from the definition that if $\{x_k\}_{k=1}^{\infty}$ is a fuzzy frame for (U, μ), then $\overline{span}\, \{x_k\}_{k=1}^{\infty} = U$.

Theorem 9. *Let (U, μ) be a fuzzy Hilbert space satisfying $(FIP8)$ and $(FIP9)$, $\alpha \in (0, 1)$ and $\{e_k\}_{k=1}^{\infty}$ be an α-fuzzy orthonormal sequence in U. Then for every $x \in U$,*

$$\sum_{k=1}^{\infty} |\langle x, x_k \rangle_\alpha|^2 \leq B\|x\|_\alpha^2.$$

Proof. Since α-fuzzy orthonormal sequence is orthonormal sequence in $(U, \langle ., . \rangle_\alpha)$, so by Bessel's inequality in crisp inner product we have

$$\sum_{k=1}^{\infty} |\langle x, x_k \rangle_\alpha|^2 \leq B\|x\|_\alpha^2 \quad \forall x \in U. \ ◄$$

Example 2. *Let $(U, \langle\ .\ \rangle)$ be a classic Hilbert space and $\{x_n\}_{n=1}^{\infty}$ be a frame for U with frame bounds A and B. Then $\{x_n\}_{n=1}^{\infty}$ is a fuzzy frame in fuzzy Hilbert space $(U, \langle\ .\ \rangle)$ satisfying (FIP8) and (FIP9) and $\alpha \in (0, 1)$.*

Since $\{x_n\}_{n=1}^{\infty}$ is a frame for U, there exist constants A, $B > 0$ such that

$$A\|x\|^2 \le \sum_{n=1}^{\infty} |\langle x, x_n\rangle|^2 \le B\|x\|^2, \quad \forall x \in U.$$

From Example 1, for all $x \in U$ and for any $\alpha \in (0,1)$ we have

$$\frac{\alpha}{1-\alpha}A\|x\|^2 \le \sum_{n=1}^{\infty} \frac{\alpha}{1-\alpha}\langle x, x_n\rangle|^2 \le \frac{\alpha}{1-\alpha}B\|x\|^2.$$

It follows that

$$A\|x\|_\alpha^2 \le \textstyle\sum_{n=1}^{\infty} |\langle x, x_n\rangle_\alpha|^2 \le B\|x\|_\alpha^2, \quad \forall x \in U, \forall \alpha \in (0,1).$$

We define $l^2(\mathbb{N}) = \left\{\{\beta_k\}_{k=1}^{\infty} \mid \sum_{k=1}^{\infty} |\beta_k|^2 < \infty\right\}$ and consider fuzzy inner product defined as follows:

$$\mu(x,y,t) = \begin{cases} 1 & , \quad t > \|x\|_{l^2(\mathbb{N})}\|y\|_{l^2(\mathbb{N})} \\ 0 & , \quad t \le \|x\|_{l^2(\mathbb{N})}\|y\|_{l^2(\mathbb{N})}. \end{cases}$$

Now we have:

$$\begin{aligned} \|x\|_\alpha &= \bigwedge \left\{t > 0 \mid \mu(x,x,t^2) \ge \alpha\right\} \\ &= \bigwedge \left\{t > 0 \mid t^2 \ge \|x\|_{l^2(\mathbb{N})}^2\right\} \\ &= \|x\|_{l^2(\mathbb{N})}. \end{aligned}$$

Therefore for all $\alpha \in (0,1)$, $\|x\|_\alpha = \|x\|_{l^2(\mathbb{N})}$. As well as on $(l^2(\mathbb{N}), \mu)$ satisfying parallelogram law.

Proposition 8. *Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in the fuzzy Hilbert space (U, μ) satisfying $(FIP8)$ and $(FIP9)$, and suppose that $\sum_{k=1}^{\infty} \beta_k x_k$ is α-convergent. Then*

$$T : l^2(\mathbb{N}) \longrightarrow U, \quad T\{\beta_k\} = \sum_{k=1}^{\infty} \beta_k x_k.$$

defines a fuzzy bounded linear operator. The adjoint operator is given by

$$T^* : U \longrightarrow l^2(\mathbb{N}), \quad T^*x = \{\langle x, x_k\rangle_\alpha\}_{k=1}^{\infty}.$$

Furthermore,

$$\sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2 \le \|T\|_\alpha^2 \|x\|_\alpha^2, \quad \forall x \in U.$$

Proof. Consider the sequence of fuzzy bounded linear operators T_n, $n \in \mathbb{N}$ defined by

$$T_n : l^2(\mathbb{N}) \longrightarrow U, \quad T_n \{\beta_k\}_{k=1}^{\infty} = \sum_{k=1}^{n} \beta_k x_k.$$

Clearly, $T_n \to T$, so by uniform boundedness principle theorem the map T defines a bounded linear operator. In order to find the expression of T^*, let $x \in U$ and $\{\beta_k\} \in l^2(\mathbb{N})$. Then

$$\langle x, T\{\beta_k\}\rangle_\alpha = \langle x, \sum_{k=1}^{\infty} \beta_k x_k\rangle_\alpha = \sum_{k=1}^{\infty} \langle x, x_k\rangle_\alpha \beta_k.$$

Proposition 2 and Proposition 1 imply that the series $\sum_{k=1}^{\infty} \langle x, x_k\rangle_\alpha \beta_k$ for all $\{\beta_k\} \in l^2(\mathbb{N})$ is convergent and according to Theorem 4 the series $\sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|$ is convergent. Therefore $\{\langle x, x_k\rangle_\alpha\}_{k=1}^{\infty} \in l^2(\mathbb{N})$. It follows that the series $\sum_{k=1}^{\infty} \langle x, x_k\rangle_\alpha \beta_k$ for all $\{\beta_k\} \in l^2(\mathbb{N})$ is α-convergent. Thus, we can write

$$\langle x, T\{\beta_k\}\rangle_U = \langle \{\langle x, x_k\rangle_\alpha\}, \{\beta_k\}\rangle_{l^2(\mathbb{N})}$$

and conclude that

$$T^* x = \{\langle x, x_k\rangle_\alpha\}_{k=1}^{\infty}.$$

Alternatively, when $T : l^2(\mathbb{N}) \longrightarrow U$ is fuzzy bounded, by Theorem 7 T^* is a fuzzy bounded operator from U to $l^2(\mathbb{N})$. By Proposition 7, $\|T\|_\alpha = \|T^*\|_\alpha$. Therefore

$$\sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2 = \|T^* x\|_\alpha^2 \le \|T\|_\alpha^2 \|x\|_\alpha^2, \quad \forall x \in U. \blacktriangleleft$$

Consider now a vector space U equipped with a fuzzy frame $\{x_k\}_{k=1}^{\infty}$, and define a linear mapping

$$T : l^2(\mathbb{N}) \longrightarrow U, \quad T\{\beta_k\}_{k=1}^{\infty} = \sum_{k=1}^{\infty} \beta_k x_k.$$

T is usually called the pre-fuzzy frame operator or the **fuzzy synthesis operator**. The adjoint operator is given by

$$T^* : U \longrightarrow l^2(\mathbb{N}), \quad T^* x = \{\langle x, x_k\rangle_\alpha\}_{k=1}^{\infty},$$

and is called the **fuzzy analysis operator**. Composing T with its adjoint T^*, we obtain the fuzzy frame operator,

$$S : U \longrightarrow U, \quad Sx = TT^* x = \sum_{k=1}^{\infty} \langle x, x_k\rangle_\alpha x_k.$$

Note that in terms of the fuzzy frame operator, we have

$$\langle Sx, x\rangle_\alpha = \sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2, \quad \forall x \in U.$$

Similar to Theorem 3.2.3 of [9], the following Theorem shows that for given fuzzy Bessel sequence its pre-frame operator is bounded and vice-versa.

Theorem 10. *Let $\{x_k\}_{k=1}^{\infty}$ be a sequence in the fuzzy Hilbert space (U, μ) satisfying $(FIP8)$ and $(FIP9)$ and $B > 0$ be given. Then $\{x_k\}_{k=1}^{\infty}$ is a fuzzy Bessel sequence with fuzzy Bessel bound B if and only if the operator*

$$T : \{\beta_k\}_{k=1}^{\infty} \longrightarrow \sum_{k=1}^{\infty} \beta_k x_k$$

defines a fuzzy bounded linear operator from $l^2(\mathbb{N})$ into U and $\|T\|_\alpha \leq \sqrt{B}$ for all $\alpha \in (0, 1)$.

Proof. Assume that $\{x_k\}$ is a fuzzy Bessel sequence with Bessel bound B and $\{\beta_k\}_{k=1}^{\infty} \in l^2(\mathbb{N})$. We want to show that $T\{\beta_k\}_{k=1}^{\infty}$ is well-defined, i.e. $\sum_{k=1}^{\infty} \beta_k x_k$ is α-convergent. Consider $n, m \in \mathbb{N}, n > m$. Then

$$\|\sum_{k=1}^{n} \beta_k x_k - \sum_{k=1}^{m} \beta_k x_k\|_\alpha = \|\sum_{k=m+1}^{n} \beta_k x_k\|_\alpha.$$

Using $\|x\|_\alpha = \bigvee_{\|y\|_\alpha=1} |\langle x, y\rangle_\alpha|$ and Cauchy-Schwarz' inequality, we obtain

$$\begin{aligned}
\|\sum_{k=1}^{n} \beta_k x_k - \sum_{k=1}^{m} \beta_k x_k\|_\alpha &= \bigvee_{\|y\|_\alpha=1} |\langle \sum_{k=m+1}^{n} \beta_k x_k, y\rangle_\alpha| \\
&\leq \bigvee_{\|y\|_\alpha=1} \sum_{k=m+1}^{n} |\beta_k \langle x_k, y\rangle_\alpha| \\
&\leq \left(\sum_{k=m+1}^{n} |\beta_k|^2\right)^{\frac{1}{2}} \bigvee_{\|y\|_\alpha=1} \left(\sum_{k=m+1}^{n} |\langle x_k, y\rangle_\alpha|^2\right)^{\frac{1}{2}} \\
&\leq \sqrt{B}\left(\sum_{k=m+1}^{n} |\beta_k|^2\right)^{\frac{1}{2}}.
\end{aligned}$$

Since $\{\beta_k\}_{k=1}^{\infty} \in l^2(\mathbb{N})$, $\left\{\sum_{k=1}^{n} |\beta_k|^2\right\}_{n=1}^{\infty}$ is a Cauchy sequence in $\mathbb{C}$. Thus $\{\sum_{k=1}^{n} \beta_k x_k\}_{n=1}^{\infty}$ is a Cauchy sequence in U and therefore convergent. Hence

$T\{\beta_k\}_{k=1}^{\infty}$ is well-defined. Clearly T is a fuzzy bounded linear operator and $\|T\|_\alpha \leq \sqrt{B}$.

To prove the opposite assertion, suppose that T defines a fuzzy bounded operator with $\|T\|_\alpha \leq \sqrt{B}$. Then Proposition 8 shows that $\{x_k\}_{k=1}^{\infty}$ is a fuzzy Bessel sequence with a fuzzy Bessel bound B. ◄

One of the most important and useful operators in the study of frames in Hilbert spaces is the so called **frame operator** and the invertibility of this operator has a lot of applications. The following Theorem shows some of its properties in fuzzy version.

Theorem 11. *Let $\{x_k\}_{k=1}^{\infty}$ be a fuzzy frame with fuzzy frame bounds A, B in fuzzy Hilbert space (U, μ) satisfying $(FIP8)$ and $(FIP9)$. Then fuzzy frame operator S is fuzzy bounded, invertible and self-adjoint.*

Proof. Since S is a composition of two fuzzy bounded operators, it is bounded.

$$\|S\|_\alpha = \|TT^*\|_\alpha \leq \|T\|_\alpha \|T^*\|_\alpha = \|T\|_\alpha^2 \leq B.$$

Since $S^* = (TT^*)^* = TT^* = S$, the operator S is self-adjoint. For invertibility of S, firstly, we show that S is injective. By definition, one has to show that: if $Sx = 0$, then $x = 0$.
If $Sx = 0$, then $0 = \langle Sx, x\rangle_\alpha = \sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2$.

$$A\|x\|_\alpha^2 \leq \sum_{k=1}^{\infty} |\langle x, x_k\rangle_\alpha|^2 = 0$$

$$A\|x\|_\alpha^2 = 0 \Rightarrow \|x\|_\alpha^2 = 0 \Rightarrow x = 0.$$

Hence S is injective and it follows that S^* is surjective and $S = S^*$. Thus S is surjective. The fuzzy frame condition implies that $\overline{span}\{x_k\}_{k=1}^{\infty} = U$. So the fuzzy synthesis operator T is surjective. For a given $x \in U$, we can find $y \in l^2(\mathbb{N})$ such that $Ty = x$. We can choose $y \in N_T^{\perp} = R_{T^*}$, so it follows that $R_S = R_{TT^*} = U$. This shows that S is invertible. ◄

We know that $C^\infty(\Omega)$ is not normable in classical Hilbert spaces, so in this case, defining frame on $C^\infty(\Omega)$ is not possible. In the next proposition, we show that $C^\infty(\Omega)$ is normable on fuzzy Hilbert space satisfying (FIP8) and (FIP9) properties. So we can define fuzzy frame on $C^\infty(\Omega)$.

Proposition 9. *The linear space $C^\infty(\Omega)$ (the vector space of all complex valued continous functions on Ω) is normable on fuzzy Hilbert space satisfying (FIP8) and (FIP9).*

Proof. For $\alpha \in (0,1]$, there exists $n \in \mathbb{N}$ such that $\frac{1}{n+1} < \alpha \leq \frac{1}{n}$. It is well known that Ω is a countable union of the sets $k_n = [\frac{1}{n}, 1 - \frac{1}{n}] \neq \emptyset$. For any $f \in C^{\infty}(\Omega)$ and $\frac{1}{n+1} < \alpha \leq \frac{1}{n}$, we define

$$f^{\alpha} = \begin{cases} \|f\|_{k_n} & \text{if } t \in k_n, \\ 0 & \text{if } otherwise. \end{cases}$$

g^{α} is defined similarly. Let

$$\mu(f,g,t) = 1 - inf\{\alpha \in (0,1] : f^{\alpha} g^{\alpha} \leq t\} \quad \forall f,g \in C^{\infty}(\Omega).$$

We define the norm by

$$\|f\|_{k_n}^2 = \langle f, f\rangle_{k_n} = \int_{k_n} |f(t)|^2 dt$$

and define the inner product by

$$\langle f, g\rangle_{k_n} = \int_{k_n} f(t)g(t)dt.$$

We show that μ is Bag-Samanta fuzzy inner product on $C^{\infty}(\Omega)$.
$FIP1$): Define

$$A = \{\alpha \in (0,1] : (f^{\alpha})^2 \leq |t|^2\},$$

$$B = \{\beta \in (0,1] : (g^{\beta})^2 \leq |s|^2\},$$

$$C = \{\gamma \in (0,1] : (f^{\gamma} g^{\gamma})^2 \leq (|t| + |s|)^2\}.$$

We show

$$1 - infC \geq \min\{1 - infA, 1 - infB\}.$$

Suppose

$$1 - infA \leq 1 - infB \text{ (a.e. } infB \leq infA).$$

Then there exists $\beta \in B$ such that for all $\alpha \in A$, $\beta < infA$, therefore $\beta < \alpha$. thus

$$f^{\alpha}h^{\alpha} \leq |t|, g^{\beta}h^{\beta} \leq |s|, g^{\alpha} \leq g^{\beta}.$$

Also

$$(f+g)^{\alpha}h^{\alpha} = f^{\alpha}h^{\alpha} + g^{\alpha}h^{\alpha} \leq f^{\alpha}h^{\alpha} + g^{\beta}h^{\beta} \leq |s| + |t|.$$

Hence for all $\alpha \in A$, we have $\alpha \in C$ a.e. $A \subset C$.
It follows that $infC \leq infA$. So $1 - infA \leq 1 - infC$. Hence for $s, t \in \mathbb{C}$, $\mu(f+g, h, |t|+|s|) \geq \min\{\mu(f, h, |t|), \mu(g, h, |s|)\}$.
$FIP2$): Define

$$A = \{\alpha \in (0,1] : (f^{\alpha})^2 \leq |t|^2\},$$

$$B = \{\beta \in (0,1] : (g^{\beta})^2 \leq |s|^2\},$$

$$C = \{\gamma \in (0,1] : (f^{\gamma} g^{\gamma})^2 \leq |ts|\}.$$

We show that

$$1 - infC \geq \min\{1 - infA, 1 - infB\}.$$

Suppose that

$$1 - infA \leq 1 - infB \text{ (a.e. } infB \leq infA).$$

Then there exists $\beta \in B$ such that for all $\alpha \in A$, $\beta < infA$, therefore $\beta < \alpha$. thus

$$f^{\alpha} \leq |t|, g^{\beta} \leq |s|, g^{\alpha} \leq g^{\beta}, f^{\alpha} g^{\alpha} \leq f^{\alpha} g^{\beta} \leq |st|.$$

Hence for all $\alpha \in A$ we have $\alpha \in C$ a.e. $A \subset C$. It follows that $infC \leq infA$. So $1 - infA \leq 1 - infC$. Hence for $s, t \in \mathbb{C}$,

$$\mu(f, g, |ts|) \geq \min\{\mu(f, f, |t|^2), \mu(g, g, |s|^2)\}.$$

FIP3): It is obvious.
FIP4): We have

$$\begin{aligned} \mu(\alpha f, g, t) &= 1 - inf\{\alpha \in (0,1] : |\alpha| f^{\alpha} g^{\alpha} \leq t\} \\ &= 1 - inf\{\alpha \in (0,1] : f^{\alpha} g^{\alpha} \leq \frac{t}{|\alpha|}\} \\ &= \mu(f, g, \frac{t}{|\alpha|}). \end{aligned}$$

Hence $\mu(\alpha f, g, t) = \mu(f, g, \frac{t}{|\alpha|})$, $\alpha (\neq 0) \in \mathbb{C}$, $t \in \mathbb{C}$.
$FIP5$): It is obvious.
$FIP6$): We show that $(\mu(f, f, t) = 1, \forall t > 0)$ iff $f = 0$.
If $\mu(f, f, t) = 1, \forall t > 0$, then

$$1 - inf\{\alpha \in (0,1] : f^{\alpha} \leq t\} = 1,$$

therefore

$$inf\{\alpha \in (0,1] : f^\alpha \leq t\} = 0.$$

As a result, for all $\alpha \in (0,1]$, we have $f^\alpha = 0$, thus $f = 0$. Opposite assertion is following directly.
$FIP7$): Is is obvious.
According to our definition of norm,

$$\begin{aligned}
\|f\|_\alpha &= inf\{t : \mu(f,f,t^2) \geq \alpha\} \\
&= inf\{t : 1 - inf\{\beta \in (0,1] : f^\beta \leq t\} \geq \alpha\} \\
&= inf\{t : inf\{\beta \in (0,1] : f^\beta \leq t\} \geq 1-\alpha\} \\
&= f^{1-\alpha}.
\end{aligned}$$

Then $\|f\|_\alpha = f^{1-\alpha} = \|f\|_{k_n}$. Using polarization identity, we can get ordinary inner product, called the α-inner product, as follows:

$$\begin{aligned}
\langle f, g\rangle_\alpha &= \frac{1}{4}\left(\|f+g\|_\alpha^2 - \|f-g\|_\alpha^2\right) + \frac{i}{4}\left(\|f+ig\|_\alpha^2 - \|f-ig\|_\alpha^2\right) \\
&= \frac{1}{4}\left[\left((f+g)^{1-\alpha}\right)^2 - \left((f-g)^{1-\alpha}\right)^2\right] \\
&+ \frac{i}{4}\left[\left((f+ig)^{1-\alpha}\right)^2 - \left((f-ig)^{1-\alpha}\right)^2\right] \\
&= \langle f, g\rangle_{k_n}. \blacktriangleleft
\end{aligned}$$

In the following example, we give a frame on $C^\infty(\Omega)$.

Example 3. *Suppose that $\{g_k(.)\}_{k=1}^\infty$ is a sequence on $C^\infty(\Omega)$, where $g_k(x) = sin\frac{x}{k}$ and $0 < x \leq 1$. We show that $\{g_k(.)\}_{k=1}^\infty$ is a fuzzy frame on $C^\infty(\Omega)$.*
If $\alpha \in (0,1]$, then there exists n such that $\frac{1}{n+1} < 1-\alpha \leq \frac{1}{n}$. Consider $k_n = [\frac{1}{n}, 1-\frac{1}{n}]$. Then $\Omega = \bigcup_{n=1}^\infty k_n$. For $f \in C^\infty(\Omega)$

$$\begin{aligned}
\sum_{k=1}^\infty |\langle f, g_k\rangle_\alpha|^2 &= \sum_{k=1}^\infty \left(\int_{k_n} f g_k d\mu\right)^2 \\
&\leq \sum_{k=1}^\infty \int_{k_n} f^2 d\mu \int_{k_n} g_k^2 d\mu \\
&= \|f\|_{k_n}^2 \mu(k_n) \sum_{k=1}^\infty \frac{1}{k^2}
\end{aligned}$$

$$
\begin{aligned}
&= ((1-\frac{2}{n})\frac{\pi^2}{6})\|f\|_{k_n}^2 \\
&= B\|f\|_{k_n}^2.
\end{aligned}
$$

With $B = ((1-\frac{2}{n})\frac{\pi^2}{6})$*, the sequence* $\{g_k(.)\}_{k=1}^{\infty}$ *is a fuzzy Bessel sequence on* $C^{\infty}(\Omega)$*.*
For the lower bound, we consider

$$A := \sum_{k=1}^{\infty} |\langle f, g_k\rangle_\alpha|^2 = \bigwedge \left\{\sum_{k=1}^{\infty} |\langle f, g_k\rangle_\alpha|^2 : f \in C^{\infty}(\Omega), \|f\|_\alpha = 1\right\}.$$

It is clear that $A > 0$*. Now given* $f \in C^{\infty}(\Omega)$ *and* $f \neq 0$*, we have*

$$
\begin{aligned}
\sum_{k=1}^{\infty} |\langle f, g_k\rangle_\alpha|^2 &= \sum_{k=1}^{\infty} \left(\int_{k_n} f g_k d\mu\right)^2 \\
&= \sum_{k=1}^{\infty} \left(\int_{k_n} \frac{f}{\|f\|_\alpha} g_k d\mu\right)^2 \|f\|_\alpha^2 \\
&\geq A\|f\|_\alpha^2.
\end{aligned}
$$

Hence $\{g_k(.)\}_{k=1}^{\infty}$ *is a fuzzy frame on* $C^{\infty}(\Omega)$*.*

5. Conclusion

$C^{\infty}(\Omega)$ was not normable on classical Hilbert spaces, but in this paper we show that $C^{\infty}(\Omega)$ is normable on fuzzy Hilbert spaces and we provide an example of fuzzy frame on $C^{\infty}(\Omega)$.

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