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Certain Modifications of (p, q)-Szász-Mirakyan Operator

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Abstract. In this work, we present Chlodowsky variation of Szász-Mirakyan-Stancu operators via (p, q) calculus and Szász-Mirakyan-Baskakov-Stancu type operator. Here, we have calculated the moments and then formulated few properties which involves weighted approximation and direct estimates. For the particular case $\alpha = 0, \beta = 0, b_n = 1$, the previously known results for two parameter quantum-Szász-Mirakyan operators are obtained.

Key Words and Phrases: Szász-Mirakyan operators, approximation theory, linear positive operators, (p, q) calculus.

2010 Mathematics Subject Classifications: 41A25, 41A35, 41A36

1. Introduction

In last two decades q-calculus has played very important role in approximation theory. Various operators are introduced in q-form ([2, 3, 4, 7, 8, 9, 10, 11, 13, 15]). (p, q) calculus or two parameter quantum calculus is the extension of quantum calculus. Recently Mursaleen et al. [18, 19] introduced (p, q)-equivalent of Bernstein operators. Also in (p, q) calculus many operators has been defined and studied (see [1, 5, 6, 20, 21]). Basic notations and operations of two parameter quantum calculus are:
For $0 < q < p \leq 1$, the (p, q) integer $[n]_{p,q}$ is defined by

$$[n]_{p,q} := \frac{p^n - q^n}{p - q}.$$

(p, q) factorial is expressed as

$$[n]_{p,q}! = [n]_{p,q}[n-1]_{p,q}[n-2]_{p,q} \dots 1.$$

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(p,q) binomial coefficient is expressed as

$$\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} := \frac{[n]_{p,q}!}{[k]_{p,q}![n-k]_{p,q}!}.$$

(p,q) binomial expansion is defined as

$$(ax+by)^n_{p,q} := \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} a^{n-k} b^k x^{n-k} y^k.$$

$$(x+y)^n_{p,q} := (x+y)(px+qy)(p^2x+q^2y)\dots(p^{n-1}x+q^{n-1}y).$$

The definite integrals of the function f are given by

$$\int_0^a f(x) d_{p,q}x = (q-p)a \sum_{k=0}^{\infty} \frac{p^k}{q^{k+1}} f\left(\frac{p^k}{q^{k+1}}a\right), \qquad \left|\frac{p}{q}\right| < 1,$$

and

$$\int_0^a f(x) d_{p,q}x = (p-q)a \sum_{k=0}^{\infty} \frac{q^k}{p^{k+1}} f\left(\frac{q^k}{p^{k+1}}a\right), \qquad \left|\frac{p}{q}\right| > 1.$$

(p,q) power expansion is defined as

$$(x \oplus a)^n_{p,q} = \prod_{s=0}^{n-1}(p^s x + q^s a), \qquad (x \ominus a)^n_{p,q} = \prod_{s=0}^{n-1}(p^s x - q^s a).$$

Derivative of any function f is given by

$$D_{p,q}f(x) = \frac{f(px)-f(qx)}{(p-q)x}, \quad x \neq 0.$$

We have two types of the exponential function in two parameter calculus (see [1]):

$$e_{p,q}(x) = \sum_{n=0}^{\infty} \frac{p^{(n(n-1)/2} x^n}{[n]_{p,q}!}, \qquad E_{p,q}(x) = \sum_{n=0}^{\infty} \frac{q^{n(n-1)/2} x^n}{[n]_{p,q}!}.$$

Further (p,q) analysis can be found in [14, 16].

2. Construction of operators

Mohapatra and Walczak had studied the Szász-Mirakyan type operators in [17]. Recently in [1], in two parameter calculus, Szász-Mirakyan operators have been defined. Motivated by this, we define the Chlodowsky type variation of (p,q) Szász-Mirakyan Stancu operators and (p,q) Szász-Mirakyan Baskakov operators, as

2.1. (p,q) **Chlodowsky variant of Szász-Mirakyan-Stancu operator**

$$T^{p,q)}_{n,\alpha,\beta}(f;x) = \frac{1}{E_{p,q}} \frac{[n]_{p,q}x}{b_n} \sum_{n=0}^{\infty} \frac{q^{k(k-1)/2}(\frac{[n]_{p,q}x}{b_n})^k}{[k]_{p,q}!} f\left(\frac{[k](p,q)+\alpha}{q^{k-2}[n]_{p,q}+\beta} b_n\right), \qquad (1)$$

where $x \in [0, b_n],\ p,q \in (0,1],\ q < p,\ 0 \leq \alpha \leq \beta,\ \alpha,\beta \in N_0$, and b_n is an increasing sequence of positive terms s.t. $b_n \to \infty$ and $\frac{b_n}{n} \to 0$ as $n \to \infty$. For the special case where $p = 1,\ \alpha = \beta = 0,\ b_n = 1$, the operator (1) is reduced to q-Szász-Mirakyan operator.

2.2. (p,q) **Szász-Mirakyan-Baskakov-Stancu type operator**

$$U^{p,q}_{n,\alpha,\beta}(f;x) = [n-1]_{p,q} \sum_{n=0}^{\infty} s^{p,q}_{n,k}(x) q^{[k(k+1)-2]/2} p^{(k+1)(k+2)/2} \times$$
$$\times \int_0^{\infty} b^{p,q}_{n,k}(t) f\left(\frac{[n]_{p,q}p^k t + \alpha}{[n]_{p,q}+\beta}\right) d_{p,q}t, \qquad (2)$$

where
$s^{p,q}_{n,k}(x) = E^{-1}_{p,q}([n]_{p,q}x)\frac{q^{k(k-1)/2}[n]^k_{p,q}x^k}{[k]_{p,q}!},\ b^{p,q}_{n,k}(t) = \left(\begin{array}{c} n+k-1 \\ k \end{array}\right)_{p,q} \frac{t^k}{(1 \oplus pt)^{k+n}_{p,q}}.$
For the particular case where $p = q = 1, \alpha = \beta = 0$, the operator (2) has been well studied earlier.

3. Moments

Using the results of [1], we can easily obtain the first three moments of operators (1), as

Lemma 1. *Let* $x \in [0, b_n),\ p,q \in (0,1],\ q < p$. *Then we have*

1. $T^{p,q}_{n,\alpha,\beta}(1;x) = 1$,
2. $T^{p,q}_{n,\alpha,\beta}(t;x) = \frac{1}{(q^{k-2}[n]_{p,q}+\beta)}(\alpha b_n + q^{k-1}[n]_{p,q}x)$,
3. $T^{p,q}_{n,\alpha,\beta}(t^2;x) = \frac{1}{(q^{k-2}[n]_{p,q}+\beta)^2}(\alpha^2 b_n^2 + (2\alpha + q^{k-1})q^{k-1}[n]_{p,q}b_n x + pq^{2k-3}[n]^2_{p,q}x^2)$.

Proof. In view of Lemma 2, of Szász-Mirakyan operator defined in [1], we have

1. $T^{p,q}_{n,\alpha,\beta}(1;x) = 1$,

2. $T^{p,q}_{n,\alpha,\beta}(t;x) = \frac{b_n}{(q^{k-2}[n]_{p,q}+\beta)}\left(\alpha + q^{k-2}[n]_{p,q}S_{n,p,q}(t;\frac{x}{b_n})\right)$,

3. $T^{p,q}_{n,\alpha,\beta}(t^2;x) = \frac{b_n^2}{(q^{k-2}[n]_{p,q}+\beta)^2}(\alpha^2 + 2\alpha q^{k-2}[n]_{p,q}S_{n,p,q}(t;\frac{x}{b_n}) + q^{2(k-2)}[n]^2_{p,q} S_{n,p,q}(t;\frac{x^2}{b_n^2}))$.

Hence by substituting the moments from [1], we get the desired result. ◄

Similarly using the results for (p,q) Szász-Mirakyan Baskakov, we can easily obtain the first three moments of operators (2), as

Lemma 2. *For positive x and $q \in (0,1)$; $p \in (q,1]$ we have*

1. $U^{p,q}_{n,\alpha,\beta}(1;x) = 1$,

2. $U^{p,q}_{n,\alpha,\beta}(t;x) = \frac{\alpha}{([n]_{p,q}+\beta)} + \frac{[n]_{p,q}}{qp^2([n]_{p,q}+\beta)[n-2]_{p,q}} + \frac{[n]^2_{p,q}}{pq^2([n]_{p,q}+\beta)[n-2]_{p,q}}x$,

3. $U^{p,q}_{n,\alpha,\beta}(t^2;x) = \frac{\alpha^2}{([n]_{p,q}+\beta)^2} + \frac{2\alpha[n]_{p,q}}{qp^2([n]_{p,q}+\beta)^2[n-2]_{p,q}} + \frac{[2]_{p,q}[n]^2_{p,q}}{p^5q^3([n]_{p,q}+\beta)^2[n-2]_{p,q}[n-3]_{p,q}} + \frac{[n]^2_{p,q}}{([n]_{p,q}+\beta)^2pq^2[n-2]_{p,q})}\left[\frac{[q(p+[2]_{p,q})+p^2][n]_{p,q}}{p^3q^3[n-3]_{p,q}} + 2\alpha\right]x + \frac{[n]^4_{p,q}}{([n]_{p,q}+\beta)^2pq^6[n-2]_{p,q}[n-3]_{p,q}}x^2$.

4. Approximation properties of $T^{p,q}_{n,\alpha,\beta}$

4.1. Direct Results

Theorem 1. *Assuming $q \in (0,1)$, $p \in (q,1]$, the approximating operator maps space C_B into C_B and*

$$\|T^{p,q}_{n,\alpha,\beta}(f)\|_{C_B} \le \|f\|_{C_B},$$

where $C_B[0,\infty)$ is the continuous, bounded function space on $[0,\infty) \to \mathbb{R}$ and $\|f|_{C_B} = \sup_{x\in[0,\infty)}|f(x)|$.

Proof. From Lemma 1, we have

$$\begin{aligned}
|T^{p,q}_{n,\alpha,\beta}(f;x)| &\le \frac{1}{E_{p,q}(\frac{[n]_{p,q}x}{b_n})}\sum_{n=0}^{\infty}\frac{q^{\frac{k(k-1)}{2}}(\frac{[n]_{p,q}x}{b_n})^k}{[k]_{p,q}!}\left|f\left(\frac{[k]_{p,q}+\alpha}{q^{k-2}[n]_{p,q}+\beta)}b_n\right)\right| \\
&\le \sup_{x\in[0,\infty)}|f(x)|\frac{1}{E_{p,q}(\frac{[n]_{p,q}x}{b_n})}\sum_{n=0}^{\infty}\frac{q^{\frac{k(k-1)}{2}}(\frac{[n]_{p,q}x}{b_n})^k}{[k]_{p,q}!} \\
&= \sup_{x\in[0,\infty)}|f(x)|T^{p,q}_{n,\alpha,\beta}(1;x) \\
&= \|f\|_{C_B}.
\end{aligned}$$ ◄

Theorem 2. *If* $f \in C_B[0,\infty)$, $g \in C_B^2[0,\infty)$, *then*

$$|\hat{T}^{p,q}_{n,\alpha,\beta}(g;x) - g(x)| \le \|g^{''}\|(\delta_{n,p,q}(x) + \beta^2_{n,p,q}(x)).$$

where $\hat{T}^{p,q}_{n,\alpha,\beta}(g;x) = T^{p,q}_{n,\alpha,\beta}(g;x) + g(x) - g(T^{p,q}_{n,\alpha,\beta}(g;x))$.

Proof. Using Lemma 1 and definition of $T^{p,q}_{n,\alpha,\beta}(g;x)$, we have

$$\hat{T}^{p,q}_{n,\alpha,\beta}(t-x;x) = 0.$$

The Taylor's formula of expansion for any function g is

$$g(t) = g(x) + g(x)(t-x) + \int_x^t (t-v)g^{''}(v)dv.$$

Now on operating with $T^{p,q}_{n,\alpha,\beta}(g;x)$ in above equation, we have

$$\begin{aligned} T^{p,q}_{n,\alpha,\beta}(g;x) - g(x) &= T^{p,q}_{n,\alpha,\beta}\Big(\int_x^t (t-v)g^{''}(v)dv, x) \\ &= T^{p,q}_{n,\alpha,\beta}\Big(\int_x^t (t-v)g^{''}(v)dv, x) \\ &- \int_x^{T^{p,q}_{n,\alpha,\beta}(t;x)} \left(\frac{1}{q^{k-2}[n]_{p,q}+\beta}(\alpha b_n + q^{k-1}[n]_{p,q}x) - v\right) g^{''}(v)dv \end{aligned} \tag{3}$$

Here

$$\left|\int_x^t (t-v)g^{''}(v)dv\right| \le \left|\int_x^t |t-v|.|g^{''}(v)|dv\right| \le (t-x)^2\|g^{''}\|_{C_B},$$

and

$$\begin{aligned} &\left|\int_x^{T^{p,q}_{n,\alpha,\beta}(t;x)} \left(\frac{1}{q^{k-2}[n]_{p,q}+\beta}(\alpha b_n + q^{k-1}[n]_{p,q}x) - v\right) g^{''}(v)\ dv\right| \\ &\le \left(\frac{1}{q^{k-2}[n]_{p,q}+\beta}(\alpha b_n + q^{k-1}[n]_{p,q}x) - v\right)^2 \|g^{''}\|_{C_B} \\ &= \beta^2_{n,p,q}(x)\|g^{''}\|_{C_B}. \end{aligned}$$

Using the last two results, we get

$$\hat{T}^{p,q}_{n,\alpha,\beta}(g;x) - g(x) \le \|g^{''}\|_{C_B}\left(T^{p,q}_{n,\alpha,\beta}((t-x)^2;x) + \beta^2_{n,p,q}(x)\right).$$

Assuming

$$\delta_{n,p,q}(x) = \max_{x\in[0,\infty)} T^{p,q}_{n,\alpha,\beta}((t-x)^2;x),$$

we obtain the desired result. ◀

4.2. Korovkin Type Approximation

Define $C_{x^2}[0,\infty) = \{f : |f(x)| \leq M_f(1+x^2),\ M_f > 0,\ f \text{ is continuous}\}$, and $C^*_{x^2}[0,\infty) = \{g : g \in C_{x^2}[0,\infty),\ \lim_{|x|\to\infty} \frac{g(x)}{(1+x^2)} \text{ is finite}\}$. Hence for $f \in C^*_{x^2}[0,\infty)$ we have

$$\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{(1+x^2)}.$$

Consider $\hat{U}^{p,q}_{n,\alpha,\beta}(f;x) = \begin{cases} T^{p,q}_{n,\alpha,\beta}(f;x), & \text{if } x \in [0,b_n] \\ f(x), & \text{if } x \in (b_n,\infty) \end{cases}$.

Let's state Korovkin type weighted approximation theorem, as in [12], for $x \in [0,\infty)$.

Theorem 3. *Let $p = (p_n)$ and $q = (q_n)$ satisfy $0 < q_n < p_n \leq 1$ and for large values of n, $p_n \to 1$, $q_n \to 1$, with $p_n^n \to 1$, $q_n^n \to 1$, $\lim_{n\to\infty} [n]_{p_n,q_n} = \infty$.*
For $f \in C^*_{x^2}[0,\infty)$, we have

$$\lim_{n\to\infty} \|\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(f) - f\|_{x^2} = 0.$$

Proof. It is adequate enough to verify the following 3 equations (see [12]):

$$\lim_{n\to\infty} \|\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(t^i) - x^i\|_{x^2} = 0, \qquad i = 0,1,2. \tag{4}$$

Since $T^{p_n,q_n}_{n,\alpha,\beta}(1;x) = 1$, the foremost condition of (4) is well justified for $i = 0$.

$$\begin{aligned} \|\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(t) - x\|_{x^2} &\leq \frac{\alpha b_n}{q^{k-2}[n]_{p,q} + \beta} + \left| \frac{q^{k-1}[n]_{p,q}}{q^{k-2}[n]_{p,q} + \beta} - 1 \right| x, \\ \|\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(t^2) - x^2\|_{x^2} &\leq \frac{\alpha^2 b_n^2}{(q^{k-2}[n]_{p,q} + \beta)^2} \\ &+ \left| \frac{(2\alpha + q^{k-1})q^{k-1}[n]_{p,q} b_n}{(q^{k-2}[n]_{p,q} + \beta)^2} \right| \sup_{x\in[0,\infty)} \frac{x}{(1+x^2)} \\ &+ \left| \frac{pq^{2k-3}[n]^2_{p,q}}{(q^{k-2}[n]_{p,q} + \beta)^2} - 1 \right| \frac{x^2}{(1+x^2)} \end{aligned}$$

which results in

$$\lim_{n\to\infty} \|\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(t^i) - x^i\|_{x^2} = 0$$

for $i = 1,2$. This concludes the proof of Theorem 3.

4.3. Voronovskaja-type result

Theorem 4. *Let $p=(p_n)$ and $q=(q_n)$ satisfy $0<q_n<p_n\leq 1$ and for n large, $p_n\to 1,\ q_n\to 1$ and $p_n^n\to a$, $q_n^n\to b$. So, $\lim\limits_{n\to\infty}[n]_{p_n,q_n}=\infty$.*
For any $g\in C^*_{x^2}[0,\infty)$ such that $g^{'},\ g^{''}\in C^*_{x^2}[0,\infty)$, we can have

$$\lim_{n\to\infty}[n]_{p_n,q_n}|\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(g;x)-g(x)|=g^{'}(x)[\alpha b_n+Ax]+\frac{1}{2}g^{''}(x)[b_n+Bx]x,$$

where

$$A=\lim_{n\to\infty}[n]_{p_n,q_n}\left(\frac{q_n^{k-1}[n]_{p_n,q_n}}{q_n^{k-2}[n]_{p_n,q_n}+\beta}-1\right),$$
$$B=\lim_{n\to\infty}[n]_{p_n,q_n}\left(\frac{[n]^2_{p_n,q_n}p_nq_n^{2k-3}}{(q_n^{k-2}[n]_{p_n,q_n}+\beta)^2}-\frac{2q_n^{k-1}[n]_{p_n,q_n}}{(q_n^{k-2}[n]_{p_n,q_n}+\beta)}+1\right).$$

Proof. Taylor formula for any function g where $x\in[0,\infty)$ is

$$g(t)=g(x)+g^{'}(x)(t-x)+\frac{1}{2}g^{''}(x)(t-x)^2+h(t,x)(t-x)^2,$$

Here $h(t,x)$ is the remainder of Peano form, $h(\cdot,x)\in C^*_{x^2}[0,\infty)$ and $\lim\limits_{t\to x}h(t,x)=0$. Applying the operator $\hat{U}$ to both sides of the above equation, we get

$$[\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(g;x)-g(x)]=g^{'}(x)\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(t-x;x)+\frac{1}{2}g^{''}(x)\hat{U}^{p_n,q_n}_{n,\alpha,\beta}((t-x)^2;x)$$
$$+\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(h(t,x)(t-x)^2;x). \tag{5}$$

Using Cauchy-Schwarz inequality in last term, we have

$$\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(h(t,x)(t-x)^2;x)\leq\sqrt{\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(h^2(t,x);x)}\sqrt{\hat{U}^{p_n,q_n}_{n,\alpha,\beta}((t-x)^4;x)}.$$

Since we know $h^2(x,x)=0$ and $h^2(\cdot,x)\in C^*_{x^2}[0,\infty)$.
Hence from Theorem 3, $\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(h^2(t,x);x)=0$ uniformly for $x\in[0,K]$. So

$$\lim_{n\to\infty}\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(h(t,x)(t-x)^2;x)=0.$$

Therefore

$$\lim_{n\to\infty}[n]_{p_n,q_n}|\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(g;x)-g(x)|$$
$$=g^{'}(x)\lim_{n\to\infty}[n]_{p_n,q_n}\hat{U}^{p_n,q_n}_{n,\alpha,\beta}(t-x;x)+g^{''}(x)\lim_{n\to\infty}[n]_{p_n,q_n}\hat{U}^{p_n,q_n}_{n,\alpha,\beta}((t-x)^2;x)$$

$$= g^{'}(x)\lim_{n\to\infty}\left[\frac{[n]_{p_n,q_n}\alpha b_n}{q_n^{k-2}[n]_{p_n,q_n}+\beta)}+[n]_(p_n,q_n)\left(\frac{q_n^{k-1}[n]_{p_n,q_n}}{q_n^{k-2}[n]_{p_n,q_n}+\beta}-1\right)x\right]$$

$$+\frac{1}{2}g^{''}(x)\lim_{n\to\infty}\left[\frac{[n]_{p_n,q_n}\alpha^2 b_n^2}{(q_n^{k-2}[n]_{p_n,q_n}+\beta)^2}\right.$$

$$+\left(\frac{[n]_{p_n,q_n}^2(2\alpha+q_n^{k-1})q_n^{k-1}[n]_{p_n,q_n}b_n}{(q_n^{k-2}[n]_{p_n,q_n}+\beta)^2}-\frac{2\alpha[n]_{p_n,q_n}b_n}{q_n^{k-2}[n]_{p_n,q_n}+\beta}\right)x$$

$$\left.+[n]_{p_n,q_n}\left(\frac{p_n q_n^{2k-3}[n]_{p_n,q_n}^2}{(q_n^{k-2}[n]_{p_n,q_n}+\beta)^2}-\frac{2q_n^{k-1}[n]_{p_n,q_n}}{q_n^{k-2}[n]_{p_n,q_n}+\beta}+1\right)x^2\right]$$

$$= g^{'}(x)[\alpha b_n+Ax]+\frac{1}{2}g^{''}(x)[b_n+Bx]x.$$

which is the desired result.

5. Approximation properties of $U_{n,\alpha,\beta}^{p,q}$

5.1. Direct Results

Theorem 5. *With $q\in(0,1)$; $p\in(q,1]$, the operator $U_{n,\alpha,\beta}^{p,q}$ maps C_B into C_B and*

$$\|U_{n,\alpha,\beta}^{p,q}(f)\|_{C_B}\le\|f\|_{C_B},$$

where $C_B[0,\infty)$ is the continuous functions space on $[0,\infty)$ and $\|f\|_{C_B}=\sup_{x\in[0,\infty)}|f(x)|$.

Proof. From Lemma 2, we have

$$|U_{n,\alpha,\beta}^{p,q}(f;x)|$$

$$\le [n-1]_{p,q}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)q^{\frac{[k(k+1)-2]}{2}}p^{\frac{(k+1)(k+2)}{2}}\int_0^{\infty}b_{n,k}^{p,q}(t)\left|f\left(\frac{[n]_{p,q}p^k t+\alpha}{[n]_{p,q}+\beta}\right)\right|d_{p,q}t$$

$$\le \sup_{x\in[0,\infty)}|f(x)|[n-1]_{p,q}\sum_{k=0}^{\infty}s_{n,k}^{p,q}(x)q^{\frac{[k(k+1)-2]}{2}}p^{\frac{(k+1)(k+2)}{2}}\int_0^{\infty}b_{n,k}^{p,q}(t)d_{p,q}t$$

$$= \sup_{x\in[0,\infty)}|f(x)|U_n^{p,q}(1;x)$$

$$= \|f\|_{C_B}.\ \blacktriangleleft$$

Theorem 6. *Let* $q \in (0,1);\ p \in (q,1]$. *If* $f \in C_B[0,\infty)$, *then*

$$\begin{aligned}
&|U^{p,q}_{n,\alpha,\beta}(f;x)-f(x)|\\
&\le 5\omega\left(f,\frac{1}{\sqrt{[n-2]_{p,q}}}\right)\left(\frac{\alpha\sqrt{[n-2]_{p,q}}}{([n]_{p,q}+\beta)}+\frac{[n]_{p,q}}{qp^2([n]_{p,q}+\beta)\sqrt{[n-2]_{p,q}}}\right.\\
&\left.+\left|\frac{[n]^2_{p,q}}{pq^2([n]_{p,q}+\beta)\sqrt{([n-2]_{p,q}}}-\sqrt{[n-2]_{p,q}}\right|x\right)+\frac{9}{2}\omega_2\left(f,\frac{1}{\sqrt{[n-2]_{p,q}}}\right)\\
&\left[\left(\left(\frac{[n]^4_{p,q}}{([n]_{p,q}+\beta)^2pq^6[n-3]_{p,q}}-\frac{2[n]^2_{p,q}}{pq^2([n]_{p,q}+\beta)}+[n-2]_{p,q}\right)x^2\right.\right.\\
&+\left(\frac{2\alpha[n]^2_{p,q}}{([n]_{p,q}+\beta)^2pq^2}+\frac{[n]^2_{p,q}[q(p+[2]_{p,q})+p^2]}{([n]_{p,q}+\beta)^2p^4q^5[n-3]_{p,q}}\right.\\
&\left.-\frac{2\alpha[n-2]_{p,q}}{([n]_{p,q}+\beta)}-\frac{2[n]_{p,q}}{qp^2([n]_{p,q}+\beta)}\right)x\\
&\left.\left.+\frac{2\alpha[n]_{p,q}}{([n]_{p,q}+\beta)^2qp^2}+\frac{[2]_{p,q}[n]^2_{p,q}}{([n]_{p,q}+\beta)^2p^5q^3[n-3]_{p,q}}+\frac{[n-2]_{p,q}\alpha^2}{([n]_{p,q}+\beta)^2}\right)+2\right].
\end{aligned}$$

Proof. For positive x and natural number n, applying the Steklov-mean f_h, we have

$$|U^{p,q}_{n,\alpha,\beta}(f;x)-f(x)|\le U^{p,q}_{n,\alpha,\beta}(|f-f_h|;x)+|U^{p,q}_{n,\alpha,\beta}(f_h-f_h(x);x)|+|f_h(x)-f(x)|. \tag{6}$$

Using the property of Steklov mean and Theorem 1, we get

$$U^{p,q}_{n,\alpha,\beta}(|f-f_h|;x)\le\|U^{p,q}_{n,\alpha,\beta}(f-f_h)\|_{C_B}\le\|f-f_h\|_{C_B}\le\omega_2(f,h).$$

Now using Taylor expansion

$$|U^{p,q}_{n,\alpha,\beta}(f_h-f_h(x);x)|\le\|f'\|_{C_B}U^{p,q}_{n,\alpha,\beta}(t-x;x)+\frac{1}{2}\|f''\|_{C_B}U^{p,q}_{n,\alpha,\beta}((t-x)^2;x).$$

From the properties of Steklov mean and Lemma 2, we obtain

$$\begin{aligned}
&|U^{p,q}_{n,\alpha,\beta}(f_h-f_h(x);x)|\le\frac{5}{h}\omega(f,h)\left(\frac{\alpha}{([n]_{p,q}+\beta)}+\frac{[n]_{p,q}}{qp^2([n]_{p,q}+\beta)[n-2]_{p,q}}\right.\\
&\left.+\left|\frac{[n]^2_{p,q}}{pq^2([n]_{p,q}+\beta)[n-2]_{p,q}}-1\right|x\right)+\frac{9}{2h^2}\omega_2(f,h)U^{p,q}_{n,\alpha,\beta}((t-x)^2;x),
\end{aligned}$$

where

$$U^{p,q}_{n,\alpha,\beta}((t-x)^2;x)$$

$$= \left(\frac{[n]_{p,q}^4}{([n]_{p,q}+\beta)^2 pq^6 [n-2]_{p,q}[n-3]_{p,q}} - \frac{2[n]_{p,q}^2}{pq^2([n]_{p,q}+\beta)[n-2]_{p,q}} + 1 \right) x^2$$
$$+ \left(\frac{2\alpha[n]_{p,q}^2}{([n]_{p,q}+\beta)^2 pq^2[n-2]_{p,q}} + \frac{[n]_{p,q}^3[q(p+[2]_{p,q})+p^2]}{([n]_{p,q}+\beta)^2 p^4 q^5 [n-2]_{p,q}[n-3]_{p,q}} \right.$$
$$\left. - \frac{2\alpha}{([n]_{p,q}+\beta)} - \frac{2[n]_{p,q}}{qp^2([n]_{p,q}+\beta)[n-2]_{p,q}} \right) x + \frac{2\alpha[n]_{p,q}}{([n]_{p,q}+\beta)^2 qp^2[n-2]_{p,q}}$$
$$+ \frac{[2]_{p,q}[n]_{p,q}^2}{([n]_{p,q}+\beta)^2 p^5 q^3 [n-2]_{p,q}[n-3]_{p,q}} + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}.$$

Replacing $h = \sqrt{\frac{1}{[n-2]_{p,q}}}$ and substituting the above estimated values in (6), we get the above mentioned result.

5.2. Korovkin Type Weighted Approximation

Define $C_{x^2}[0,\infty) = \{f : |f(x)| \le M_f(1+x^2),\ M_f > 0,\ f \text{ is continuous}\}$, and $C_{x^2}^*[0,\infty) = \{g : g \in C_{x^2}[0,\infty),\ \lim_{|x|\to\infty} \frac{g(x)}{(1+x^2)} \text{ is finite}\}$. Hence for $f \in C_{x^2}^*[0,\infty)$ we have

$$\|f\|_{x^2} = \sup_{x\in[0,\infty)} \frac{|f(x)|}{(1+x^2)}.$$

Let's, we state the Korovkin type weighted approximation theorem, as in [12], for $x \in [0,\infty)$.

Theorem 7. *Let $p = (p_n)$ and $q = (q_n)$ satisfy $0 < q_n < p_n \le 1$ and for n large, $p_n \to 1,\ q_n \to 1$ and $p_n^n \to a,\ q_n^n \to b,\ \lim_{n\to\infty} [n]_{p_n,q_n} = \infty$. For $f \in C_{x^2}^*[0,\infty)$, we have*

$$\lim_{n\to\infty} \|U_{n,\alpha,\beta}^{p_n,q_n}(f) - f\|_{x^2} = 0.$$

Proof. It is adequate to examine the following 3 conditions (see [12]):

$$\lim_{n\to\infty} \|U_{n,\alpha,\beta}^{p_n,q_n}(t^i) - x^i\|_{x^2} = 0, i = 0,1,2. \tag{7}$$

Since $U_{n,\alpha,\beta}^{p_n,q_n}(1,x) = 1$, the first condition of (7) is satisfied for $i = 0$. For $n > 3$, we can say

$$\|U_{n,\alpha,\beta}^{p_n,q_n}(t) - x\|_{x^2}$$
$$\le \frac{\alpha}{([n]_{p,q}+\beta)} + \frac{[n]_{p,q}}{qp^2([n]_{p,q}+\beta)[n-2]_{p,q}} + \left| \frac{[n]_{p,q}^2}{pq^2([n]_{p,q}+\beta)[n-2]_{p,q}} - 1 \right| x,$$

$$\|U_{n,\alpha,\beta}^{p_n,q_n}(t^2)-x^2\|_{x^2}$$

$$\leq \frac{1}{([n]_{p,q}+\beta)^2}\left(\alpha^2+\frac{2\alpha[n]_{p,q}}{qp^2[n-2]_{p,q}}+\frac{[2]_{p,q}[n]_{p,q}^2}{p^5q^3[n-2]_{p,q}[n-3]_{p,q}}\right)^2$$

$$+\frac{[n]_{p,q}^2}{([n]_{p,q}+\beta)^2pq^2[n-2]_{p,q}}\left|\frac{[q[n]_{p,q}(p+[2]_{p,q})+p^2]}{p^3q^3[n-3]_{p,q}}+2\alpha\right|\sup_{x\in[0,\infty)}\frac{x}{1+x^2}$$

$$+\left|\frac{[n]_{p,q}^4}{([n]_{p,q}+\beta)^2pq^6[n-2]_{p,q}[n-3]_{p,q}}-1\right|\frac{x^2}{1+x^2},$$

which results in

$$\lim_{n\to\infty}\|U_{n,\alpha,\beta}^{p_n,q_n}(t^i)-x^i\|_{x^2}=0,$$

for $i=1,2$. This concludes the proof of Theorem 7. ◄

5.3. Voronovskaja-type result

Theorem 8. *Let $p=(p_n)$ and $q=(q_n)$, satisfy $0<q_n<p_n\leq 1$ and for n large, $p_n\to 1$, $q_n\to 1$ and $p_n^n\to a$, $q_n^n\to b$. So, $\lim_{n\to\infty}[n]_{p_n,q_n}\to\infty$. Then for any $g\in C_{x^2}^*[0,\infty)$ such that $g^{'},g^{''}\in C_{x^2}^*[0,\infty)$, we have*

$$\lim_{n\to\infty}[n]_{p_n,q_n}|U_{n,\alpha,\beta}^{p_n,q_n}(g;x)-g(x)|=g^{'}(x)[1+\alpha+Ax]+\frac{1}{2}g^{''}(x)[1+Bx]x,$$

uniformly on any $[0,K]$, $K>0$, where

$$A=\lim_{n\to\infty}[n]_{p_n,q_n}\left(\frac{[n]_{p_n,q_n}}{[n-2]_{p_n,q_n}}-1\right),$$

$$B=\lim_{n\to\infty}[n]_{p_n,q_n}\left(\frac{[n]_{p_n,q_n}^3}{([n]_{p_n,q_n}+\beta)[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}-1\right).$$

Proof. For $x\in[0,\infty)$, the Taylor's formula for function g is given by

$$g(t)=g(x)+g^{'}(x)(t-x)+\frac{1}{2}g^{''}(x)(t-x)^2+H(t,x)(t-x)^2, \tag{8}$$

where $H(t,x)$ is the remainder of Peano form, $H(\cdot,x)\in C_{x^2}^*[0,\infty)$ and $\lim_{t\to x}H(t,x)=0$. Applying the operator U to both sides of the above equation, we get

$$[U_{n,\alpha,\beta}^{p_n,q_n}(g;x)-g(x)]=g^{'}(x)U_{n,\alpha,\beta}^{p_n,q_n}(t-x;x)$$

$$+\frac{1}{2}g^{''}(x)U_{n,\alpha,\beta}^{p_n,q_n}((t-x)^2;x)+U_{n,\alpha,\beta}^{p_n,q_n}(H(t,x)(t-x)^2;x). \tag{9}$$

Using Cauchy-Schwarz inequality, we have

$$U_{n,\alpha,\beta}^{p_n,q_n}(H(t,x)(t-x)^2;x) \leq \sqrt{U_{n,\alpha,\beta}^{p_n,q_n}(H^2(t,x);x)}\sqrt{U_{n,\alpha,\beta}^{p_n,q_n}((t-x)^4;x)} \qquad (10)$$

Since we know $H^2(x,x)=0$ and $H^2(\cdot,x) \in C^*_{x^2}[0,\infty)$. Hence from above theorem, $U_{n,\alpha,\beta}^{p_n,q_n}(H^2(t,x);x)=0$ uniformly for $x \in [0,K]$. So,

$$\lim_{n\to\infty} U_{n,\alpha,\beta}^{p_n,q_n}(H(t,x)(t-x)^2;x)=0.$$

Therefore

$$\begin{aligned}
&\lim_{n\to\infty}[n]_{p_n,q_n}|U_{n,\alpha,\beta}^{p_n,q_n}(g;x)-g(x)| = g^{'}(x)\lim_{n\to\infty}[n]_{p_n,q_n}U_{n,\alpha,\beta}^{p_n,q_n}(t-x;x)\\
&+g^{''}(x)\lim_{n\to\infty}[n]_{p_n,q_n}U_{n,\alpha,\beta}^{p_n,q_n}((t-x)^2;x)\\
&= g^{'}(x)\lim_{n\to\infty}\left[\alpha+\frac{[n]_{p_n,q_n}}{q_np_n^2[n-2]_{p_n,q_n}}+[n]_{p_n,q_n}\left(\frac{[n]_{p_n,q_n}}{p_nq_n^2[n-2]_{p_n,q_n}}-1\right)x\right]\\
&+\frac{1}{2}g^{''}(x)\lim_{n\to\infty}\left[\frac{1}{([n]_{p_n,q_n}+\beta)^2}\left(\alpha^2+\frac{2\alpha[n]_{p_n,q_n}}{q_np_n^2[n-2]_{p_n,q_n}}\right.\right.\\
&\left.+\frac{((p_n+q_n)[n]^2_{p_n,q_n}}{p_n^5q_n^3[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}\right)^2+\frac{[n]^2_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)^2p_nq_n^2[n-2]_{p_n,q_n}}\\
&\left(\frac{[q_n[n]_{p_n,q_n}(2p_n+q_n)+p_n^2]}{p_n^3q_n^3[n-3]_{p_n,q_n}}+2\alpha\right)x-\frac{2[n]_{p_n,q_n}}{q_np_n^2[n-2]_{p_n,q_n}}x-2\alpha x\\
&+\left(\frac{[n]^4_{p_n,q_n}}{([n]_{p_n,q_n}+\beta)p_nq_n^6[n-2]_{p_n,q_n}[n-3]_{p_n,q_n}}+[n]_{p_n,q_n}\right.\\
&\left.\left.-\frac{2[n]^2_{p_n,q_n}}{p_nq_n^2[n-2]_{p_n,q_n}}\right)x^2\right] = g^{'}(x)[1+\alpha+Ax]+\frac{1}{2}g^{''}(x)[1+Bx]x.
\end{aligned}$$

which is the desired result. ◀

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