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A Numerical Computation of Zeros and Determinant Forms for Some New Families of q-special Polynomials

M. Riyasat*, S. Khan, N.I. Mahmudov

Abstract. The main aim of this article is to investigate some new families of q-special polynomials and to study their properties using different approaches. The 2-iterated q-Appell polynomial family is introduced and some properties of these polynomials are considered under q-umbral calculus methods. Some 2-iterated q-Appell and hybrid q-special polynomials are studied as members of this family. The numbers related to these polynomials are obtained. The graphical representation of the 2-iterated and hybrid q-Appell polynomials is presented. The zeros of these polynomials are investigated for some values of index n using numerical computation. The approximate solutions of the real zeros of these polynomials are also considered. The determinant forms for the 2-iterated q-Appell family and for the 2-iterated and hybrid q-special polynomials are established using linear algebraic approach.

Key Words and Phrases: q-Appell polynomials, 2-iterated q-Appell polynomials, hybrid q-special polynomials, determinant.

2010 Mathematics Subject Classifications: 33D45, 33D99, 33E20

1. Introduction and preliminaries

The area of q-calculus has in the last twenty years served as a bridge between mathematics and physics. Recently, there has been a significant increase of activities in the area of q-calculus due to its applications in various fields such as mechanics, mathematics and physics.

The definitions and notations of q-calculus reviewed here are taken from [3]. The q-analogue of the shifted factorial $(a)_n$ is defined by

*Corresponding author.

$$(a;q)_0 = 1, (a;q)_n = \prod_{m=0}^{n-1} (1 - q^m a),\ n \in \mathbb{N}.$$

The q-analogues of a complex number a and of the factorial function are defined by

$$[a]_q = \frac{1 - q^a}{1 - q}, \quad q \in \mathbb{C} - \{1\}; \quad a \in \mathbb{C},$$

$$[n]_q! = \prod_{m=1}^{n} [m]_q = [1]_q [2]_q \cdots [n]_q = \frac{(q;q)_n}{(1-q)^n},$$

$$q \neq 1;\ n \in \mathbb{N}, \quad [0]_q! = 1,\ q \in \mathbb{C};\ 0 < q < 1.$$

The Gauss q-binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}_q$ is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q! [n-k]_q!} = \frac{(q;q)_n}{(q;q)_k (q;q)_{n-k}}, \quad k = 0, 1, \ldots, n.$$

The q-exponential function is defined as:

$$e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{[n]_q!},\ 0 < |q| < 1.$$

The q-derivative $D_q f$ of a function f at a point $0 \neq z \in \mathbb{C}$ is defined as:

$$D_q f(z) = \frac{f(qz) - f(z)}{qz - z}, \quad 0 < |q| < 1.$$

The q-analogue of Taylor series expansion of an arbitrary function $f(z)$ for $0 < q < 1$ is defined as:

$$f(z) = \sum_{n=0}^{\infty} \frac{(1-q)^n}{(q;q)_n} D_q^n f(a) (z-a)_q^n,$$

where $D_q^n f(a)$ is the n^{th} q-derivative of the function f at point a.

In 1985, Roman proposed an approach similar to the umbral approach under the area of nonclassical umbral calculus which is called q-umbral calculus [11, 9]. Let $\mathbb{P}$ be the algebra of polynomials in the single variable x over the field of complex numbers. Let $\mathbb{P}^\star$ be the vector space of all linear functionals on $\mathbb{P}$. Let $\langle L \mid p(x) \rangle$ be the action of a linear functional L on a polynomial $p(x)$. Let $\mathfrak{F}$ denote the algebra of formal power series

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{k!}.$$

This kind of algebra is called an umbral algebra. Each $f(t) \in \mathfrak{F}$ defines a linear functional on $\mathbb{P}$ given by $\langle f(t)|x^k\rangle = a_k$ for all $k \geq 0$. In particular, $\langle t^k|x^n\rangle = n!\delta_{n,k}$ for all $n, k \geq 0$, where $\delta_{n,k}$ is the Kronecker delta. Let c_n be a fixed sequence of nonzero constants. Then the algebra $\mathfrak{F}$ of all formal power series in t is isomorphic onto the vector space $\mathbb{P}^\star$ by setting

$$\langle t^k|x^n\rangle = c_n\delta_{n,k}.$$

If

$$f(t) = \sum_{k=0}^{\infty} a_k \frac{t^k}{c_k},$$

then

$$\langle f(t)|x^k\rangle = a_k.$$

The q-umbral calculus is defined by setting

$$c_n = \frac{(1-q)(1-q^2)\dots(1-q^n)}{(1-q)^n}.$$

Then

$$\frac{c_n}{c_{n-1}} = \frac{1-q^n}{1-q}.$$

The functional $e_q(yt)$ satisfies

$$\langle e_q(yt)|x^n\rangle = \left\langle \sum_{k=0}^{\infty} \frac{(yt)^k}{[k]_q!}|x^n \right\rangle = y^n.$$

Then

$$\langle e_q(yt)|p(x)\rangle = p(y),$$

for all polynomials $p(x) \in \mathbb{P}$. Also, the operator $e_q(yt)$ satisfies

$$e_q(yt)x^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k\, x^{n-k}.$$

Let $f(t),\ g(t) \in \mathfrak{F}$. Then we have

$$\langle f(t)g(t)|p(x)\rangle = \langle f(t) \mid g(t)p(y)\rangle.$$

In 1967, Al-Salaam [1] introduced the family of q-Appell polynomials $\{A_{n,q}(x)\}_{n\geq 0}$ and studied some of its properties. The n-degree polynomials

$A_{n,q}(x)$ are called q-Appell if and only if these are defined by means of the following generating function [1]:

$$A_q(t)e_q(xt) = \sum_{n=0}^{\infty} A_{n,q}(x)\frac{t^n}{[n]_q!}, \qquad 0 < q < 1, \tag{1}$$

where

$$A_q(t) := \sum_{n=0}^{\infty} A_{n,q}\frac{t^n}{[n]_q!}, \quad A_{0,q} = 1; \; A_q(t) \neq 0. \tag{2}$$

It is to be noted that $A_q(t)$ is an analytic function at $t = 0$ and $A_{n,q} := A_{n,q}(0)$. Also, there exists a sequence of numbers $\{A_{n,q}\}_{n\geq 0}$, such that the polynomials $A_{n,q}(x)$ satisfy the following relation [1]:

$$A_{n,q}(x) = A_{n,q} + \begin{bmatrix} n \\ 1 \end{bmatrix}_q A_{n-1,q}\; x + \begin{bmatrix} n \\ 2 \end{bmatrix}_q A_{n-2,q}\; x^2 + \cdots + A_{0,q}\; x^n, \; n = 0, 1, 2, \ldots \;.$$

The q-Appell polynomials $A_{n,q}(x)$ are characterized by Roman [10] under q-umbral calculus. Let $A_{n,q}(x)$ be q-Appell polynomials for $g(t)$. Then

$$e_q(yt) = \sum_{n=0}^{\infty} \frac{\langle e_q(yt)|A_{n,q}(x)\rangle}{[n]_q!} g_q(t)t^n = \sum_{n=0}^{\infty} \frac{A_{n,q}(y)}{[n]_q!} g_q(t)t^n.$$

The polynomials $A_{n,q}(x)$ are the q-Appell polynomials for $g_q(t)$ if and only if

i.

$$\frac{1}{g_q(t)} e_q(xt) = \sum_{n=0}^{\infty} A_{n,q}(x)\frac{t^n}{[n]_q!},$$

ii.

$$A_{n,q}(x) = (g_q(t))^{-1}x^n, \tag{3}$$

iii.

$$tA_{n,q}(x) = [n]_q A_{n-1,q}(x),$$

iv.

$$e_q(yt)A_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A_{n-k,q}(x)y^k.$$

Al-Salaam showed that the class of all q-Appell polynomials is a maximal commutative subgroup of the group of all polynomial sets, *i.e.* the class of all q-Appell sequences is closed under the operation of q-umbral composition of polynomial sequences. If $A_{n,q}(x)=\sum\limits_{k=0}^{n} a_{n,k;q}\, x^k$ and $B_{n,q}(x)=\sum\limits_{k=0}^{n} b_{n,k;q}\, x^k$ are sequences of q-polynomials, then the q-umbral composition of $A_{n,q}(x)$ with $B_{n,q}(x)$ is defined to be the sequence

$$(A_{n,q}\circ B_q)(x)=\sum_{k=0}^{n} a_{n,k;q}\, B_{k,q}(x)=\sum_{0\le k\le l\le n} a_{n,k;q}b_{k,l;q}x^l.$$

Under this operation, the set of all q-Appell sequences is an abelian group and it can be seen by considering the fact that every q-Appell sequence is of the form

$$A_{n,q}(x)=\left(\sum_{k=0}^{\infty}\frac{a_{k,q}}{[k]_q!}D_q^k\right)x^n$$

and that the umbral composition of q-Appell sequences corresponds to multiplication of these formal q-power series in the operator D_q.

Based on appropriate selection for the function $A_q(t)$ or $g_q(t)^{-1}$, different members belonging to the family of q-Appell polynomials can be obtained. These members are listed in Table 1.

Table 1. Some members belonging to the q-Appell family

S. No.	Name of the q-special polynomials and related numbers	$A_q(t)$ or $g_q(t)^{-1}$	Generating function	Series definition
I.	q-Bernoulli polynomials and numbers [4, 2]	$\left(\frac{t}{e_q(t)-1}\right)$	$\left(\frac{t}{e_q(t)-1}\right)e_q(xt)= =\sum\limits_{n=0}^{\infty}B_{n,q}(x)\frac{t^n}{[n]_q!}$ $\left(\frac{t}{e_q(t)-1}\right)=\sum\limits_{n=0}^{\infty}B_{n,q}\frac{t^n}{[n]_q!}$ $B_{n,q}:=B_{n,q}(0)$	$B_{n,q}(x)=\sum\limits_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q B_{k,q}x^{n-k}$
II.	q-Euler polynomials and numbers [4, 8]	$\left(\frac{2}{e_q(t)+1}\right)$	$\left(\frac{2}{e_q(t)+1}\right)e_q(xt)= =\sum\limits_{n=0}^{\infty}E_{n,q}(x)\frac{t^n}{[n]_q!}$ $\left(\frac{2}{e_q(t)+1}\right)=\sum\limits_{n=0}^{\infty}E_{n,q}\frac{t^n}{[n]_q!}$ $E_{n,q}:=E_{n,q}(0)$	$E_{n,q}(x)=\sum\limits_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q E_{k,q}x^{n-k}$
III.	q-Hermite polynomials and numbers [6, 8]	$\sum\limits_{n=0}^{\infty}(-1)^n\, q^{n(n-1)}\times\frac{t^{2n}}{[2n]_q!!}$	$\sum\limits_{n=0}^{\infty}(-1)^n\, q^{n(n-1)}\frac{t^{2n}}{[2n]_q!!}\times$ $\times e_q(xt)=\sum\limits_{n=0}^{\infty}H_{n,q}(x)\frac{t^n}{[n]_q!}$ $\sum\limits_{n=0}^{\infty}(-1)^n\, q^{n(n-1)}\frac{t^{2n}}{[2n]_q!!}= =\sum\limits_{n=0}^{\infty}H_{n,q}\frac{t^n}{[n]_q!}$ $H_{n,q}:=H_{n,q}(0)$	$H_{n,q}(x)=\sum\limits_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_q H_{k,q}x^{n-k}$

The article is organized as follows. In Section 2, the 2-iterated q-Appell polynomials are introduced by means of generating function and series definition. Some properties of these polynomials are considered under q-umbral calculus. The members belonging to the 2-iterated q-Appell family and some hybrid q-special polynomials and related numbers are also explored. In Section 3, the zeros of the 2-iterated q-Appell and hybrid q-Appell polynomials are investigated using numerical computation and their graphs are drawn for suitable values of indices. In Section 4, the determinant forms of the 2-iterated and hybrid q-Appell polynomials are established.

2. 2-iterated q-Appell polynomials

In order to introduce the 2IqAP, two different sets of q-Appell polynomials $A^{I}_{n,q}(x)$ and $A^{II}_{n,q}(x)$ are considered. Thus, from definitions (1) and (2), it follows that

$$A^{I}_{q}(t)e_q(xt) = \sum_{n=0}^{\infty} A^{I}_{n,q}(x)\frac{t^n}{[n]_q!},\ 0 < q < 1, \tag{4}$$

where

$$A^{I}_{q}(t) := \sum_{n=0}^{\infty} A^{I}_{n,q}\frac{t^n}{[n]_q!};\ A^{I}_{n,q} := A^{I}_{n,q}(0);\ A^{I}_{0,q} = 1;\ A^{I}_{q}(t) \neq 0 \tag{5}$$

and

$$A^{II}_{q}(t)e_q(xt) = \sum_{n=0}^{\infty} A^{II}_{n,q}(x)\frac{t^n}{[n]_q!},\ 0 < q < 1, \tag{6}$$

where

$$A^{II}_{q}(t) := \sum_{n=0}^{\infty} A^{II}_{n,q}\frac{t^n}{[n]_q!};\ A^{II}_{n,q} := A^{II}_{n,q}(0);\ A^{II}_{0,q} = 1;\ A^{II}_{q}(t) \neq 0.$$

2.1. Generating function and other properties

The generating function for the 2IqAP is derived by using a different approach based on replacement techniques. For this, the following result is proved:

Theorem 1. *The polynomials* $A^{[2]}_{n,q}(x)$ *are the 2-iterated* q*-Appell polynomials for* $A^{I}_{q}(t)$ *and* $A^{II}_{q}(t)$ *(or for* $g^{I}(t)$ *and* $g^{II}(t)$*) if and only if*

$$G_q(x,t) := A^{I}_{q}(t)A^{II}_{q}(t)e_q(xt) = \sum_{n=0}^{\infty} A^{[2]}_{n,q}(x)\frac{t^n}{[n]_q!}, \ 0<q<1, \tag{7}$$

or, equivalently,

$$G_q(x,t) := \frac{1}{g^{I}_{q}(t)g^{II}_{q}(t)}e_q(xt) = \sum_{n=0}^{\infty} A^{[2]}_{n,q}(x)\frac{t^n}{[n]_q!}, \ 0<q<1. \tag{8}$$

Proof. Expanding the q-exponential function $e_q(xt)$ in the l.h.s. of equation (4) and then replacing the powers of x, *i.e.* $x^0,\ x^1,\ x^2,\ldots,x^n$ by the corresponding polynomials $A^{II}_{0,q}(x),\ A^{II}_{1,q}(x),\ \ldots, A^{II}_{n,q}(x)$ in the l.h.s. and replacing x by the polynomial $A^{II}_{1,q}(x)$ in the r.h.s. of the resultant equation, it follows that

$$A^{I}_{q}(t)\Big[1 + A^{II}_{1,q}(x)\frac{t}{[1]_q!} + A^{II}_{2,q}(x)\frac{t^2}{[2]_q!} + \ldots + A^{II}_{n,q}(x)\frac{t^n}{[n]_q!} + \ldots\Big] =$$

$$= \sum_{n=0}^{\infty} A^{I}_{n,q}\{A^{II}_{1,q}(x)\}\frac{t^n}{[n]_q!}.$$

Summing up the series in l.h.s. and then using equation (6) and denoting the resultant 2IqAP in the r.h.s. by $A^{[2]}_{n,q}(x)$, that is

$$A^{[2]}_{n,q}(x) = A^{I}_{n,q}\{A^{II}_{1,q}(x)\}, \tag{9}$$

assertion (7) is proved. In view of equations (1) and (3), assertion (8) follows. ◄

Next, the series definition for the 2-iterated q-Appell polynomials is obtained by proving the following result:

Theorem 2. *The polynomials* $A^{[2]}_{n,q}(x)$ *are the 2-iterated* q*-Appell polynomials for* $A^{I}_{n,q}(x)$ *and* $A^{II}_{n,q}(x)$ *if and only if*

$$A^{[2]}_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A^{I}_{k,q}\ A^{II}_{n-k,q}(x). \tag{10}$$

Proof. Using equations (5) and (6) in the l.h.s. of generating function (7) and then using Cauchy-product rule in the l.h.s. of the resultant equation, we obtain

$$\sum_{n=0}^{\infty}\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A^{I}_{k,q}\ A^{II}_{n-k,q}(x)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} A^{[2]}_{n,q}(x)\frac{t^n}{[n]_q!}. \tag{11}$$

Equating the coefficients of same powers of t in both sides of equation (11), assertion (10) follows. ◀

Further, as consequences of some results mentioned in Section 1, the following theorems are deduced:

Theorem 3. *The polynomials $A^{[2]}_{n,q}(x)$ are the 2-iterated q-Appell polynomials if and only if*

$$\langle e_q(yt) \mid A^{[2]}_{n,q}(x)\rangle = \left\langle \sum_{k=0}^{\infty} \frac{(yt)^k}{[k]_q!} \mid A^{[2]}_{n,q}(x) \right\rangle = A^{[2]}_{n,q}(y).$$

Theorem 4. *The polynomials $A^{[2]}_{n,q}(x)$ are the 2-iterated q-Appell polynomials for $g^I_q(t)$ and $g^{II}_q(t)$, if and only if*

$$e_q(yt) = \sum_{k=0}^{\infty} \frac{\langle e_q(yt) \mid A^{[2]}_{k,q}(x)\rangle}{[k]_q!} g^I_q(t) g^{II}_q(t) t^k = \sum_{k=0}^{\infty} \frac{A^{[2]}_{k,q}(y)}{[k]_q!} g^I_q(t) g^{II}_q(t) t^k.$$

Theorem 5. *The polynomials $A^{[2]}_{n,q}(x)$ are the 2-iterated q-Appell polynomials for $g^I_q(t)$ and $g^{II}_q(t)$, if and only if*

$$A^{[2]}_{n,q}(x) = (g^I_q(t) g^{II}_q(t))^{-1}\ x^n.$$

Theorem 6. *The polynomials $A^{[2]}_{n,q}(x)$ are the 2-iterated q-Appell polynomials, if and only if*

$$t\ A^{[2]}_{n,q}(x) = [n]_q\ A^{[2]}_{n-1,q}(x).$$

Theorem 7. *The polynomials $A^{[2]}_{n,q}(x)$ are the 2-iterated q-Appell polynomials, if and only if*

$$e_q(yt) A^{[2]}_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q y^k\ A^{[2]}_{n-k,q}(x).$$

Theorem 8. *The polynomials $A^{[2]}_{n,q}(x)$ are the 2-iterated q-Appell polynomials for $A^I_{n,q}(x)$ and $A^{II}_{n,q}(x)$ if and only if*

$$A^{[2]}_{n,q}(x) = \left(\sum_{k=0}^{\infty} \frac{A^I_{k,q}}{[k]_q!} D^k_q \right) A^{II}_{n,q}(x).$$

2.2. Particular members

By making suitable selections for the functions $A_q^I(t)$ and $A_q^{II}(t)$ in equations (7) and (10), the generating function and series definition for the corresponding member belonging to the 2-iterated q-Appell family can be obtained. These resultant 2-iterated q-Appell polynomials along with their notations, names, generating functions and series definitions are given in Table 2.

Table 2. Some members belonging to the 2-Iterated q-Appell family

S. No.	$A_q^I(t) = A_q^{II}(t)$	**Resultant 2IqAP**	**Generating function**	**Series definition**
I.	$\left(\frac{t}{e_q(t)-1}\right)$	$B_{n,q}^{[2]}(x):=$ 2-iterated q-Bernoulli polynomials (2IqBP)	$\left(\frac{t}{e_q(t)-1}\right)^2 e_q(xt) = \sum\limits_{n=0}^{\infty} B_{n,q}^{[2]}(x)\frac{t^n}{[n]_q!}$	$B_{n,q}^{[2]}(x) = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q} B_{n-k,q}(x)$
II.	$\left(\frac{2}{e_q(t)+1}\right)$	$E_{n,q}^{[2]}(x):=$ 2-iterated q-Euler polynomials (2IqEP)	$\left(\frac{2}{e_q(t)+1}\right)^2 e_q(xt) = \sum\limits_{n=0}^{\infty} E_{n,q}^{[2]}(x)\frac{t^n}{[n]_q!}$	$E_{n,q}^{[2]}(x) = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q} E_{n-k,q}(x)$
III.	$\sum\limits_{n=0}^{\infty} (-1)^n q^{n(n-1)} \frac{t^{2n}}{[2n]_q!!}$	$H_{n,q}^{[2]}(x):=$ 2-iterated q-Hermite polynomials (2IqHP)	$\left(\sum\limits_{n=0}^{\infty} (-1)^n q^{n(n-1)} \frac{t^{2n}}{[2n]_q!!}\right)^2 \times e_q(xt) = \sum\limits_{n=0}^{\infty} H_{n,q}^{[2]}(x)\frac{t^n}{[n]_q!}$	$H_{n,q}^{[2]}(x) = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q H_{k,q} H_{n-k,q}(x)$

The combinations of any two different members of the q-Appell family in the 2-iterated q-Appell family yields a new hybrid q-special polynomial. Thus, by making suitable selections for the functions $A_q^I(t)$ and $A_q^{II}(t)$ in equations (7) and (10), the generating function and series definition of the resultant hybrid q-special polynomials can be obtained. The possible combinations of the q-Bernoulli, q-Euler and q-Hermite polynomials (Table 1 (I-III)) are considered. The resultant hybrid q-special polynomials are given in Table 3.

Remark 1. *Note that*

$$_BE_{n,q}(x) \equiv {}_EB_{n,q}(x); \quad {}_BH_{n,q}(x) \equiv {}_HB_{n,q}(x); \quad {}_EH_{n,q}(x) \equiv {}_HE_{n,q}(x),$$

where $_EB_{n,q}(x)$, $_HB_{n,q}(x)$ and $_HE_{n,q}(x)$ are the q-Euler-Bernoulli polynomials (qEBP), q-Hermite-Bernoulli polynomials (qHBP) and q-Hermite-Euler polynomials (qHEP). This confirms the fact that the set of all q-Appell sequences is *closed under the operation of q-umbral composition of polynomial sequences and forms an abelian group.*

Table 3. Some hybrid q-special polynomials

S. No.	$A_q^I(t)$; $A_q^{II}(t)$	Hybrid polynomial	Generating function	Series definition
I.	$\left(\frac{t}{e_q(t)-1}\right)$; $\left(\frac{2}{e_q(t)+1}\right)$	${}_BE_{n,q}(x) :=$ q-Bernoulli-Euler polynomials (qBEP)	$\frac{2t}{(e_q(t)-1)(e_q(t)+1)}e_q(xt) = \sum_{n=0}^{\infty} {}_BE_{n,q}(x)\frac{t^n}{[n]_q!}$	${}_BE_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}B_{n-k,q}(x)$
II.	$\left(\frac{t}{e_q(t)-1}\right)$; $\sum_{n=0}^{\infty}(-1)^n q^{n(n-1)}\frac{t^{2n}}{[2n]_q!!}$	${}_BH_{n,q}(x):=$ q-Bernoulli-Hermite polynomials (qBHP)	$\frac{t}{(e_q(t)-1)}\sum_{n=0}^{\infty}(-1)^n \times q^{n(n-1)}\frac{t^{2n}}{[2n]_q!!}e_q(xt) = \sum_{n=0}^{\infty} {}_BH_{n,q}(x)\frac{t^n}{[n]_q!}$	${}_BH_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q H_{k,q}B_{n-k,q}(x)$
III.	$\left(\frac{2}{e_q(t)+1}\right)$; $\sum_{n=0}^{\infty}(-1)^n q^{n(n-1)}\frac{t^{2n}}{[2n]_q!!}$	${}_EG_{n,q}(x):=$ q-Euler-Hermite polynomials (qEHP)	$\frac{2}{e_q(t)+1}\sum_{n=0}^{\infty}(-1)^n \times q^{n(n-1)}\frac{t^{2n}}{[2n]_q!!}e_q(xt) = \sum_{n=0}^{\infty} {}_EH_{n,q}(x)\frac{t^n}{[n]_q!}$	${}_EH_{n,q}(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q H_{k,q}E_{n-k,q}(x)$

2.3. Related numbers

The numbers related to the members of the 2IqAP $A^{[2]}_{n,q}(x)$ given in Tables 2 and 3 are obtained. By taking $x = 0$ in series definitions of 2IqBP $B^{[2]}_{n,q}(x)$, 2IqEP $E^{[2]}_{n,q}(x)$ and 2IqHP $H^{[2]}_{n,q}(x)$ given in Table 2 (I-III) and using notations from Table 1, the 2-iterated q-Bernoulli, q-Euler and q-Hermite numbers are obtained. These numbers are given in Table 4.

Table 4. 2-Iterated q-numbers

S.No.	Notation and name of the 2-iterated q-number	Series definition
I.	$B^{[2]}_{n,q} := B^{[2]}_{n,q}(0)$ 2-iterated q-Bernoulli numbers (2IqBN)	$B^{[2]}_{n,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q B_{k,q}B_{n-k,q}$
II.	$E^{[2]}_{n,q} := E^{[2]}_{n,q}(0)$ 2-iterated q-Euler numbers (2IqEN)	$E^{[2]}_{n,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q}E_{n-k,q}$
III.	$H^{[2]}_{n,q} := H^{[2]}_{n,q}(0)$ 2-iterated q-Hermite numbers (2IqHN)	$H^{[2]}_{n,q} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q H_{k,q}H_{n-k,q}$

Similarly, the numbers corresponding to the hybrid q-special polynomials given in Table 3 can be obtained. Taking $x = 0$ in series definitions of the qBEP ${}_BE_{n,q}(x)$, qBHP ${}_BH_{n,q}(x)$ and qEHP ${}_EH_{n,q}(x)$ (Table 3 (I-III)) and using notations from Table 1, the q-Bernoulli-Euler, q-Bernoulli-Hermite and q-Euler-Hermite numbers are obtained. These numbers are given in Table 5.

Table 5. Hybrid q-numbers

S.No.	Notation and name of the mixed type q-numbers	Series definition
I.	${}_BE_{n,q} := {}_BE_{n,q}(0)$ q-Bernoulli-Euler numbers (qBEN)	${}_BE_{n,q} = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q E_{k,q} B_{n-k,q}$
II.	${}_BH_{n,q} := {}_BH_{n,q}(0)$ q-Bernoulli-Hermite numbers (qBHN)	${}_BH_{n,q} = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q H_{k,q} B_{n-k,q}$
III.	${}_EH_{n,q} := {}_EH_{n,q}(0)$ q-Euler-Hermite numbers (qEHN)	${}_EH_{n,q} = \sum\limits_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q H_{k,q} E_{n-k,q}$

Remark 2. *Also note that*

$$
\begin{gathered}
{}_BE_{n,q} \equiv {}_EB_{n,q} := {}_EB_{n,q}(0); \quad {}_BH_{n,q} \equiv {}_HB_{n,q} := \\
:= {}_HB_{n,q}(0); \quad {}_EH_{n,q} \equiv {}_HE_{n,q} := {}_HE_{n,q}(0),
\end{gathered} \tag{12}
$$

where ${}_EB_{n,q}$, ${}_HB_{n,q}$ and ${}_HE_{n,q}$ are the q-Euler-Bernoulli numbers (qEBN), q-Hermite-Bernoulli numbers (qHBN) and q-Hermite-Euler numbers (qHEN), respectively.

In the next section, the shapes of the 2IqBP $B^{[2]}_{n,q}(x)$, 2IqEP $E^{[2]}_{n,q}(x)$, 2IqHP $H^{[2]}_{n,q}(x)$, qBEP ${}_BE_{n,q}(x)$, qBHP ${}_BH_{n,q}(x)$ and qEHP ${}_EH_{n,q}(x)$ are displayed. The zeros of these polynomials are also investigated using numerical computation.

3. Zeros and approximate solutions

There has been increasing interest in solving mathematical problems with the aid of computers. The numerical investigation of the zeros of some q-polynomials are considered in [12, 13, 14, 7]. First, we give the shapes of the q-special polynomials.

3.1. Shapes of the 2-iterated and hybrid q-Appell polynomials

The plots of the 2IqBP $B^{[2]}_{n,q}(x)$, 2IqEP $E^{[2]}_{n,q}(x)$, 2IqHP $H^{[2]}_{n,q}(x)$, qBEP ${}_BE_{n,q}(x)$, qBHP ${}_BH_{n,q}(x)$ and qEHP ${}_EH_{n,q}(x)$ are drawn for $n = 1, 2, 3, 4$ and

$q = \frac{1}{2}$ $(0 < q < 1)$. This shows the four plots combined into one for each of these polynomials. For this, the values of the first five $B_{n,q}$, $E_{n,q}$ and $H_{n,q}$ are required. The values of first five $B_{n,q}$, $E_{n,q}$ and $H_{n,q}$ [4, 6] are mentioned in Table 6.

Table 6. Values of first five $B_{n,q}$, $E_{n,q}$ and $H_{n,q}$

n	0	1	2	3	4
$B_{n,q}$	1	$-(1+q)^{-1}$	$q^2([3]_q!)^{-1}$	$(1-q)q^3([2]_q)^{-1}([4]_q)^{-1}$	$q^4(1-q^2-2q^3-q^4+q^6)([2]_q^2[3]_q[5]_q)^{-1}$
$E_{n,q}$	1	$-\frac{1}{2}$	$\frac{1}{4}(-1+q)$	$\frac{1}{8}(-1+2q+2q^2-q^3)$	$\frac{1}{16}(q-1)[3]_q!(q^2-4q+1)$
$H_{n,q}$	1	0	-1	0	$[3]_q\, q^2$

The expressions of the first five $B_{n,q}(x)$, $E_{n,q}(x)$ and $H_{n,q}(x)$ are obtained by making use of the values of the first five $B_{n,q}$, $E_{n,q}$ and $H_{n,q}$ in the series definitions given in Table 1 (I-III). The expressions of first five $B_{n,q}(x)$, $E_{n,q}(x)$ and $H_{n,q}(x)$ are mentioned in Table 7.

Table 7. Expressions of first five $B_{n,q}(x)$, $E_{n,q}(x)$ and $H_{n,q}(x)$

n	0	1	2	3	4
$B_{n,q}(x)$	1	$x - \frac{1}{1+q}$	$x^2 - \frac{[2]_q}{1+q}x + \frac{q^2}{[3]_q[2]_q}$	$x^3 - \frac{[3]_q x^2}{1+q} + \frac{q^2 x}{[2]_q} + \frac{(1-q)q^3}{[2]_q[4]_q}$	$x^4 - \frac{[4]_q}{1+q}x^3 + \frac{[4]_q q^2}{[2]_q^2}x^2 + q^4(1-q^2-2q^3-q^4+q^6)([2]_q^2[3]_q[5]_q)^{-1}$
$E_{n,q}(x)$	1	$x - \frac{1}{2}$	$x^2 - \frac{[2]_q}{2}x + \frac{1}{4}(-1+q)$	$x^3 - \frac{[3]_q}{2}x^2 + \frac{[3]_q}{4}(-1+q)x + \frac{1}{8}(-1+2q+2q^2-q^3)$	$x^4 - \frac{[4]_q}{2}x^3 + \frac{[4]_q[3]_q(q-1)}{4[2]_q}x^2 + \frac{[4]_q(-1+2q+2q^2-q^3)}{8}x + \frac{(q-1)[3]_q!(q^2-4q+1)}{16}$
$H_{n,q}(x)$	1	x	$x^2 - 1$	$x^3 - [3]_q\, x$	$x^4 - \left(1+q^2\right)[3]_q\, x^2 + [3]_q\, q$

By making appropriate substitutions from Tables 6 and 7 in the series definitions given in Tables 2 and 3, the expressions of the first five 2IqBP $B^{[2]}_{n,q}(x)$, 2IqEP $E^{[2]}_{n,q}(x)$, 2IqHP $H^{[2]}_{n,q}(x)$, qBEP ${}_BE_{n,q}(x)$, qBHP ${}_BH_{n,q}(x)$ and qEHP ${}_EH_{n,q}(x)$ for $q = \frac{1}{2}$ are obtained. These expressions are given in Table 8.

With the help of Matlab and by using the expressions for the first five $B^{[2]}_{n,1/2}(x)$, $E^{[2]}_{n,1/2}(x)$, $H^{[2]}_{n,1/2}(x)$, ${}_BE_{n,1/2}(x)$, ${}_BH_{n,1/2}(x)$ and ${}_EH_{n,1/2}(x)$ from Table 8 for $n = 1, 2, 3, 4$, the following graphs are drawn:

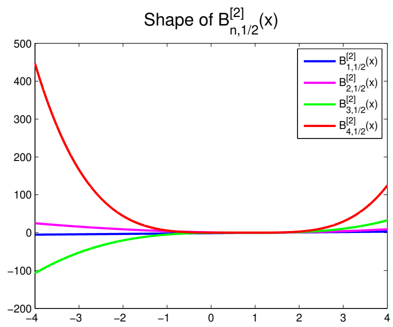

Figure 3.1

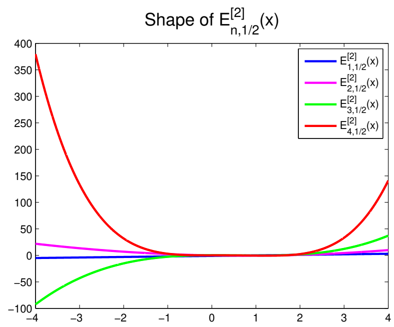

Figure 3.2

Table 8. Expressions of first five $B^{[2]}_{n,1/2}(x)$, $E^{[2]}_{n,1/2}(x)$, $H^{[2]}_{n,1/2}(x)$, ${}_BE_{n,1/2}(x)$, ${}_BH_{n,1/2}(x)$, ${}_EH_{n,1/2}(x)$

n	0	1	2	3	4
$B^{[2]}_{n,1/2}(x)$	1	$x-\frac{4}{3}$	$x^2-2x+\frac{6}{7}$	$x^3-\frac{7}{3}x^2+\frac{3}{2}x-\frac{8}{45}$	$x^4-\frac{5}{2}x^3+\frac{45}{24}x^2-\frac{8}{24}x-\frac{221}{7812}$
$E^{[2]}_{n,1/2}(x)$	1	$x-1$	$x^2-\frac{3}{2}x-\frac{1}{16}$	$x^3-\frac{7}{4}x^2+\frac{7}{32}x+\frac{5}{16}$	$x^4-\frac{15}{8}x^3+\frac{35}{128}x^2+\frac{125}{1024}x+\frac{71}{1024}$
$H^{[2]}_{n,1/2}(x)$	1	x	x^2-2	$x^3-\frac{7}{2}x$	$x^4-\frac{35}{8}x^2+\frac{56}{16}$
${}_BE_{n,1/2}(x)$	1	$x-\frac{7}{6}$	$x^2-\frac{3}{2}x+\frac{51}{168}$	$x^3-\frac{49}{24}x^2+\frac{79}{96}x+\frac{379}{2880}$	$x^4-\frac{35}{16}x^3-\frac{445}{384}x^2-\frac{461}{1536}x-\frac{402305}{9999360}$
${}_BH_{n,1/2}(x)$	1	$x-\frac{3}{2}$	$x^2-x-\frac{19}{21}$	$x^3-\frac{7}{6}x^2-\frac{19}{12}x+\frac{107}{90}$	$x^4-\frac{5}{4}x^3-\frac{100}{48}x^2+\frac{107}{48}x+\frac{14554}{62496}$
${}_EH_{n,1/2}(x)$	1	$x-\frac{1}{2}$	$x^2-\frac{3}{4}x-\frac{9}{8}$	$x^3-\frac{7}{8}x^2-\frac{63}{32}x+\frac{59}{64}$	$x^4-\frac{15}{16}x^3-\frac{315}{128}x^2+\frac{885}{512}x+\frac{791}{1024}$

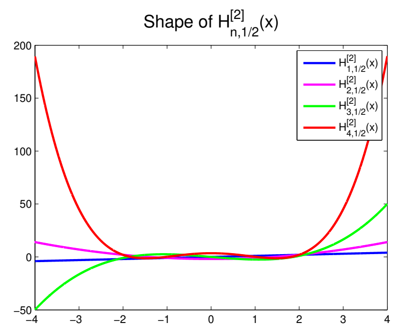

Figure 3.3

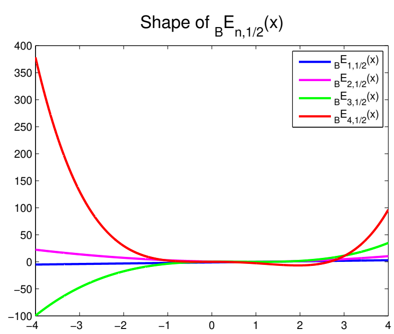

Figure 3.4

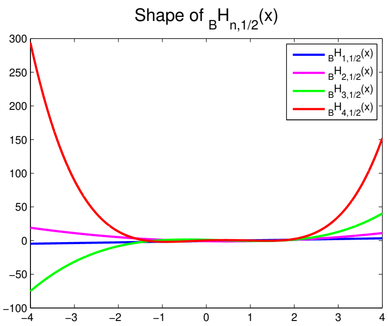

Figure 3.5

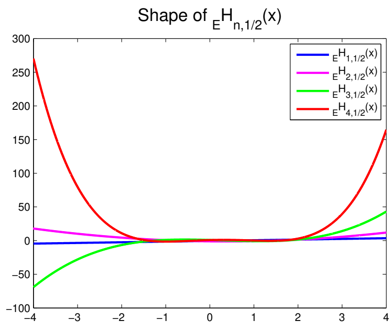

Figure 3.6

3.2. Zeros and approximate solutions of the 2-iterated and hybrid q-Appell polynomials

We find the real and complex zeros of the polynomials given in Table 8. The manual computation of these zeros is too complicated, therefore, we use Matlab to investigate these zeros. The investigation in this direction will lead to a new approach employing numerical methods in the field of these q-special polynomials.

The real zeros of $B^{[2]}_{n,1/2}(x)$, $E^{[2]}_{n,1/2}(x)$, $H^{[2]}_{n,1/2}(x)$, $_BE_{n,1/2}(x)$, $_BH_{n,1/2}(x)$ and $_EH_{n,1/2}(x)$ are computed by using Matlab. These zeros are given in Table 9.

Again, with the help of Matlab, we find the complex zeros of these polynomials. These complex zeros are given in Table 10.

Table 9. Real zeros of $B^{[2]}_{n,1/2}(x)$, $E^{[2]}_{n,1/2}(x)$, $H^{[2]}_{n,1/2}(x)$, $_BE_{n,1/2}(x)$, $_BH_{n,1/2}(x)$, $_EH_{n,1/2}(x)$

Degree n	$B^{[2]}_{n,1/2}(x)$	$E^{[2]}_{n,1/2}(x)$	$H^{[2]}_{n,1/2}(x)$	$_BE_{n,1/2}(x)$	$_BH_{n,1/2}(x)$	$_EH_{n,1/2}(x)$
1	1.3333	1.0000	0.0000	1.1667	0.6667	0.5000
2	0.6220, 1.3780	−0.0406, 1.5406	1.4142, −1.4142	0.2411, 1.2589	1.5746, −0.5746	1.5000, −0.7500
3	0.1522, 0.9446, 1.2365	−3.2768, −0.2642, 2.3743	0.0000, 1.8708 −1.8708	−0.1213, 0.7910, 1.3719	−1.1392, 1.6874 0.6184	−1.2626, 1.7108, 0.4268
4	−0.0617, 0.3823	0.5479, 1.6488	−1.8224, −1.0266 1.8224, 1.0266	−0.2476, 2.6664	−1.3694, 1.6381, 1.0777, −0.0963	−1.4071, 1.6153, 1.0523, −0.3230

Remark 3. *From Tables 9 and 10, the following general relation is observed. The number of real zeros lying on the real plane $Im\ (x) = 0$, i.e.,*

$$\textit{Real zeros of } A^{[2]}_{n,q}(x) = n - \textit{Complex zeros of } A^{[2]}_{n,q}(x),$$

where n is the degree of the polynomial.

Table 10. Complex zeros of $B^{[2]}_{n,1/2}(x)$, $E^{[2]}_{n,1/2}(x)$, $H^{[2]}_{n,1/2}(x)$, ${}_BE_{n,1/2}(x)$, ${}_BH_{n,1/2}(x)$, ${}_EH_{n,1/2}(x)$

Degree n	$B^{[2]}_{n,1/2}(x)$	$E^{[2]}_{n,1/2}(x)$	$H^{[2]}_{n,1/2}(x)$	${}_BE_{n,1/2}(x)$	${}_BH_{n,1/2}(x)$	${}_EH_{n,1/2}(x)$
1	–	–	–	–	–	–
2	–	–	–	–	–	–
3	–	–	–	–	–	–
4	$1.0897 + 0.1112i$, $1.0897 - 0.1112i$	$-0.1609 + 0.225i$, $-0.1609 - 0.225i$	–	$-0.1157 + 0.2181i$, $-0.1157 - 0.2181i$	–	

In order to make the above discussion more clear, we draw the combined graphs of shapes and zeros of the polynomials $B^{[2]}_{n,1/2}(x)$, $E^{[2]}_{n,1/2}(x)$, $H^{[2]}_{n,1/2}(x)$, ${}_BE_{n,1/2}(x)$, ${}_BH_{n,1/2}(x)$ and ${}_EH_{n,1/2}(x)$ for $n = 4$.

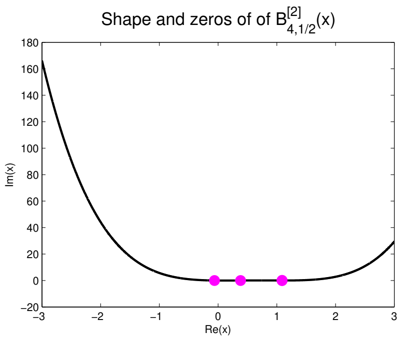

Figure 3.7

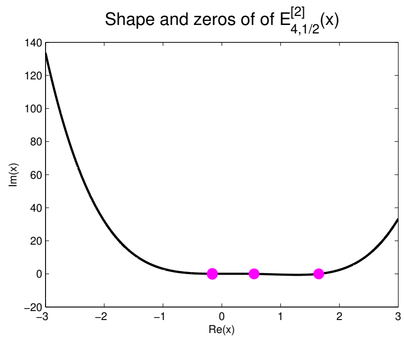

Figure 3.8

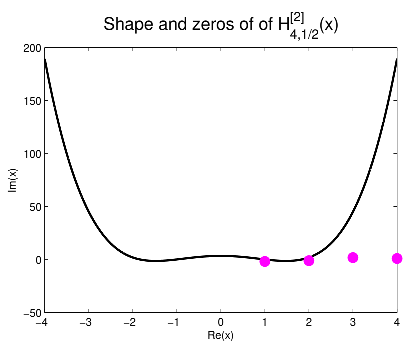

Figure 3.9

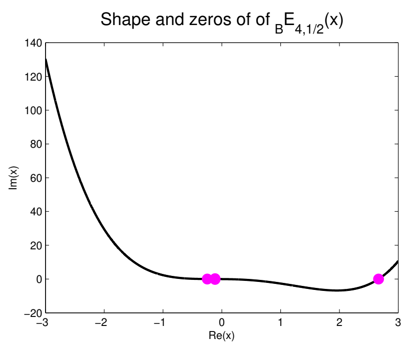

Figure 3.10

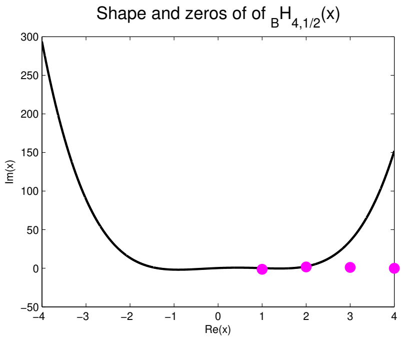

Figure 3.11

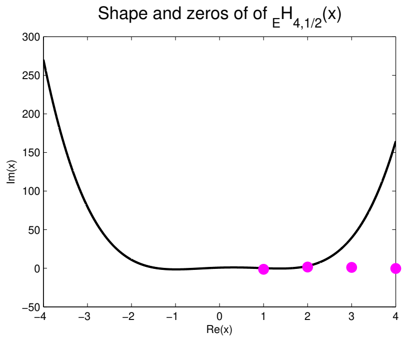

Figure 3.12

It should be noted that in Figures 3.7, 3.8 and 3.10 out of total two complex zeros only one, with positive imaginary part is visible, due to the absence of negative imaginary axis in these graphs.

The numerical results for approximate solutions of real zeros of $B^{[2]}_{n,1/2}(x)$, $E^{[2]}_{n,1/2}(x)$, $H^{[2]}_{n,1/2}(x)$, ${}_BE_{n,1/2}(x)$, ${}_BH_{n,1/2}(x)$ and ${}_EH_{n,1/2}(x)$ for $(n = 1, 2, 3, 4)$ are displayed in Table 9.

Also, we note that the real zeros of these polynomials as shown in Table 9 give the numerical results for the approximate solutions of $B^{[2]}_{n,1/2}(x) = 0$, $E^{[2]}_{n,1/2}(x) = 0$, $H^{[2]}_{n,1/2}(x) = 0$, ${}_BE_{n,1/2}(x) = 0$, ${}_BH_{n,1/2}(x) = 0$ and ${}_EH_{n,1/2}(x) = 0$ for $n = 1, 2, 3, 4$.

In the next section, the determinant forms for the 2-iterated q-Appell polynomials and some hybrid q-special polynomials are established.

4. Determinant forms

The q-Appell polynomials are studied using determinant and umbral approaches, see for example [5, 6]. Obtaining determinant forms for the q-Appell polynomials and their members is an important aspect of such study. The determinant forms can be helpful for computation purposes and can also be used in finding the solutions of general linear interpolation problems. This fact provides motivation to establish the determinant forms of the q-special polynomials introduced in previous section. In order to define the 2IqAP $A^{[2]}_{n,q}(x)$ by means of determinant, the following result is proved:

4.1. Determinant form for the 2-iterated q-Appell polynomials

Theorem 9. *The following determinant form for the 2-iterated q-Appell polynomials* $A^{[2]}_{n,q}(x)$ *of degree* n *holds true:*

$$A^{[2]}_{0,q}(x)=\tfrac{1}{\beta_{0,q}},$$
$$A^{[2]}_{n,q}(x)=\tfrac{(-1)^n}{(\beta_{0,q})^{n+1}}\begin{vmatrix} 1 & A^{II}_{1,q}(x) & A^{II}_{2,q}(x) & \cdots & A^{II}_{n-1,q}(x) & A^{II}_{n,q}(x)\\ \beta_{0,q} & \beta_{1,q} & \beta_{2,q} & \cdots & \beta_{n-1,q} & \beta_{n,q}\\ 0 & \beta_{0,q} & \left[{2 \atop 1}\right]_q\beta_{1,q} & \cdots & \left[{n-1 \atop 1}\right]_q\beta_{n-2,q} & \left[{n \atop 1}\right]_q\beta_{n-1,q}\\ 0 & 0 & \beta_{0,q} & \cdots & \left[{n-1 \atop 2}\right]_q\beta_{n-3,q} & \left[{n \atop 2}\right]_q\beta_{n-2,q}\\ . & . & . & \cdots & . & .\\ . & . & . & \cdots & . & .\\ 0 & 0 & 0 & \cdots & \beta_{0,q} & \left[{n \atop n-1}\right]_q\beta_{1,q}\end{vmatrix}, \tag{13}$$

where $n=1,2,\dots$ *and* $A^{II}_{n,q}(x)$ $(n=0,1,2,\dots)$ *are the q-Appell polynomials of degree n;* $\beta_{0,q}\neq 0$ *and*

$$\beta_{0,q}=\tfrac{1}{A^I_{0,q}},$$
$$\beta_{n,q}=-\tfrac{1}{A^I_{0,q}}\Big(\sum_{k=1}^{n}\left[{n \atop k}\right]_q A^I_{k,q}\,\beta_{n-k,q}\Big),\quad n=1,2,\dots\ .$$

Proof. Let $A^{[2]}_{n,q}(x)$ be a sequence of the 2IqAP defined by equation (7) and $A^I_{n,q}$, $\beta_{n,q}$, be two numerical sequences (the coefficients of q-Taylor's series expansions of functions) such that

$$A^I_q(t)=A^I_{0,q}+\frac{t}{[1]_q!}A^I_{1,q}+\frac{t^2}{[2]_q!}A^I_{2,q}+\cdots+\frac{t^n}{[n]_q!}A^I_{n,q}+\cdots,\ n=0,1,\dots;\ A^I_{0,q}\neq 0, \tag{14}$$

$$\hat{A}^I_q(t)=\beta_{0,q}+\frac{t}{[1]_q!}\beta_{1,q}+\frac{t^2}{[2]_q!}\beta_{2,q}+\cdots+\frac{t^n}{[n]_q!}\beta_{n,q}+\cdots,\ n=0,1,\dots;\ \beta_{0,q}\neq 0, \tag{15}$$

satisfying

$$A^I_q(t)\hat{A}^I_q(t)=1.$$

Then, according to the Cauchy-product rule, it follows that

$$A^I_q(t)\hat{A}^I_q(t)=\sum_{n=0}^{\infty}\sum_{k=0}^{n}\begin{bmatrix} n\\ k\end{bmatrix}_q A^I_{k,q}\,\beta_{n-k,q}\frac{t^n}{[n]_q!},$$

which gives

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q A^I_{k,q}\ \beta_{n-k,q} = \begin{cases} 1, & \text{if } n = 0, \\ 0, & \text{if } n > 0. \end{cases}$$

Consequently, the following holds:

$$\begin{cases} \beta_{0,q} = \frac{1}{A^I_{0,q}}, \\ \beta_{n,q} = -\frac{1}{A^I_{0,q}}\Big(\sum_{k=1}^{n} \left[{n \atop k}\right]_q A^I_{k,q}\ \beta_{n-k,q}\Big), \quad n = 1, 2, \ldots \quad . \end{cases} \tag{16}$$

Multiplication of equation (7) by $\hat{A}^I_q(t)$ gives

$$A^I_q(t)\hat{A}^I_q(t)A^{II}_q(t)e_q(xt) = \hat{A}^I_q(t)\sum_{n=0}^{\infty} A^{[2]}_{n,q}(x)\frac{t^n}{n!},$$

which in view of equations (14), (15) and (6) gives

$$\sum_{n=0}^{\infty} A^{II}_{n,q}(x)\frac{t^n}{[n]_q!} = \sum_{n=0}^{\infty} A^{[2]}_{n,q}(x)\frac{t^n}{[n]_q!}\sum_{k=0}^{\infty}\beta_{k,q}\frac{t^k}{[k]_q!}.$$

Again, multiplication of the series on the r.h.s. of equation (16) according to Cauchy-product rule leads to the following system of infinite equations in the unknowns $A^{[2]}_{n,q}(x)$ $(n = 0, 1, \ldots)$:

$$\begin{cases} A^{[2]}_{0,q}(x)\beta_{0,q} = 1, \\ A^{[2]}_{0,q}(x)\beta_{1,q} + A^{[2]}_{1,q}(x)\beta_{0,q} = A^{II}_{1,q}(x), \\ A^{[2]}_{0,q}(x)\beta_{2,q} + \left[{2 \atop 1}\right]_q A^{[2]}_{1,q}(x)\beta_{1,q} + A^{[2]}_{2,q}(x)\beta_{0,q} = A^{II}_{2,q}(x), \\ \vdots \qquad\qquad\qquad\qquad \vdots \\ A^{[2]}_{0,q}(x)\beta_{n-1,q} + \left[{n-1 \atop 1}\right]_q A^{[2]}_{1,q}(x)\beta_{n-2,q} + \cdots + A^{[2]}_{n,q}(x)\beta_{0,q} = A^{II}_{n-1,q}(x), \\ A^{[2]}_{0,q}(x)\beta_{n,q} + \left[{n \atop 1}\right]_q A^{[2]}_{1,q}(x)\beta_{n-1,q} + \cdots + A^{[2]}_{n,q}(x)\beta_{0,q} = A^{II}_{n,q}(x), \\ \vdots \qquad\qquad\qquad\qquad \vdots \quad . \end{cases}$$

First equation of system (17), proves the first part of assertion (13). Also, the special form of system ((17) (lower triangular) allows to work out the unknowns

$A^{[2]}_{n,q}(x)$. Operating with the first $n+1$ equations simply by applying the Cramer's rule, it follows that

$$A^{[2]}_{n,q}(x) =$$

$$= \frac{1}{\beta_{0,q}^{n+1}} \begin{vmatrix} \beta_{0,q} & 0 & 0 & \cdots & 0 & 1 \\ \beta_{1,q} & \beta_{0,q} & 0 & \cdots & 0 & A^{II}_{1,q}(x) \\ \beta_{2,q} & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \beta_{1,q} & \beta_{0,q} & \cdots & 0 & A^{II}_{2,q}(x) \\ . & . & . & \cdots & . & . \\ \beta_{n-1,q} & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \beta_{n-2,q} & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \beta_{n-3,q} & \cdots & \beta_{0,q} & A^{II}_{n-1,q}(x) \\ \beta_{n,q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \beta_{n-1,q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \beta_{n-2,q} & \cdots & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \beta_{1,q} & A^{II}_{n,q}(x) \end{vmatrix}$$

Now, bringing the $(n+1)$-th column to the first place by n transpositions of adjacent columns and in view of the fact that the determinant of a square matrix is the same as that of its transpose, second part of assertion (13) is proved. ◄

4.2. Determinant forms for the 2-iterated and hybrid q-Appell polynomials

We know that the Bernoulli polynomials $B_{n,q}(x)$ and Euler polynomials $E_{n,q}(x)$ are the two particular members of the q-Appell family $A_{n,q}(x)$. First, we establish determinant forms for the polynomials $B_{n,q}(x)$ and $E_{n,q}(x)$ by choosing suitable values of the coefficients $\beta_{0,q}$ and $\beta_{i,q}$ $(i = 1, 2, \cdots, n)$ in the determinant form of $A_{n,q}(x)$.

Taking $\beta_{0,q} = 1$, $\beta_{i,q} = \frac{1}{[i+1]_q}$ $(i = 1, 2, \cdots, n)$ in the determinant form of the q-Appell polynomials $A_{n,q}(x)$ [5, p.12(19)], the following determinant form of the q-Bernoulli polynomials $B_{n,q}(x)$ is obtained:

Definition 1. *The q-Bernoulli polynomials $B_{n,q}(x)$ of degree n are defined by*

$$B_{0,q}(x)=1,$$
$$B_{n,q}(x)= = (-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n & 1 \\ 1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & \cdots & \frac{1}{[n]_q} & \frac{1}{[n+1]_q} & 0 \\ 0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q \frac{1}{[2]_q} & \cdots & \begin{bmatrix}n-1\\1\end{bmatrix}_q \frac{1}{[n-1]_q} & \begin{bmatrix}n\\1\end{bmatrix}_q \frac{1}{[n]_q} & 0 \\ 0 & 0 & 1 & \cdots & \begin{bmatrix}n-1\\2\end{bmatrix}_q \frac{1}{[n-2]_q} & \begin{bmatrix}n\\2\end{bmatrix}_q \frac{1}{[n-1]_q} & \\ . & . & . & \cdots & . & . & \\ . & . & . & \cdots & . & . & \\ 0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}n\\n-1\end{bmatrix}_q \frac{1}{[2]_q} & \end{vmatrix}, \quad n=1,2,\cdots. \tag{17}$$

The particular case of Definition 1, for $n=4$, has been considered in [15, p.250].

Next, taking $\beta_{0,q}=1$, $\beta_{i,q}=\frac{1}{2}$ $(i=1,2,\cdots,n)$ in the determinant definition of the q-Appell polynomials $A_{n,q}(x)$ [5, p.12(19)], the following determinant form of the q-Euler polynomials $E_{n,q}(x)$ is obtained:

Definition 2. *The q-Euler polynomials $E_{n,q}(x)$ of degree n are defined by*

$$E_{0,q}(x)=1,$$
$$E_{n,q}(x)=(-1)^n \begin{vmatrix} 1 & x & x^2 & \cdots & x^{n-1} & x^n & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}_q & \cdots & \frac{1}{2}\begin{bmatrix}n-1\\1\end{bmatrix}_q & \frac{1}{2}\begin{bmatrix}n\\1\end{bmatrix}_q & 0 \\ 0 & 0 & 1 & \cdots & \frac{1}{2}\begin{bmatrix}n-1\\2\end{bmatrix}_q & \frac{1}{2}\begin{bmatrix}n\\2\end{bmatrix}_q & \\ . & . & . & \cdots & . & . & \\ . & . & . & \cdots & . & . & \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\begin{bmatrix}n\\n-1\end{bmatrix}_q & \end{vmatrix}, \quad n=1,2,\cdots. \tag{18}$$

Remark 4. *Taking suitable values of the coefficients $\beta_{0,q}$ and $\beta_{i,q}$ $(i=1,2,\cdots,n)$ in the determinant form of the 2IqAP family, the determinant forms for the 2-iterated q-members and hybrid q-special polynomials can be obtained.*

First, the determinant forms for the members of the 2IqAP given in Table 2 are obtained. Taking $\beta_{0,q} = 1$, $\beta_{i,q} = \frac{1}{[i+1]_q}$ $(i = 1, 2, \cdots, n)$ and $A^{II}_{n,q}(x) = B_{n,q}(x)$ in equation (13), the following determinant form of the 2IqBP $B^{[2]}_{n,q}(x)$ is obtained:

Definition 3. *The 2-iterated q-Bernoulli polynomials $B^{[2]}_{n,q}(x)$ of degree n are defined by*

$$\begin{aligned} &B^{[2]}_{0,q}(x) = 1,\\ &B^{[2]}_{n,q}(x) =\\ &= (-1)^n \begin{vmatrix} 1 & B_{1,q}(x) & B_{2,q}(x) & \cdots & B_{n-1,q}(x) & B_{n,q}(x) & 1 \\ 1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & \cdots & \frac{1}{[n]_q} & \frac{1}{[n+1]_q} & 0 \\ 0 & 1 & \begin{bmatrix} 2 \\ 1 \end{bmatrix}_q \frac{1}{[2]_q} & \cdots & \begin{bmatrix} n-1 \\ 1 \end{bmatrix}_q \frac{1}{[n-1]_q} & \begin{bmatrix} n \\ 1 \end{bmatrix}_q \frac{1}{[n]_q} & 0 \\ 0 & 0 & 1 & \cdots & \begin{bmatrix} n-1 \\ 2 \end{bmatrix}_q \frac{1}{[n-2]_q} & \begin{bmatrix} n \\ 2 \end{bmatrix}_q \frac{1}{[n-1]_q} & \\ . & . & . & \cdots & . & . & \\ . & . & . & \cdots & . & . & \\ 0 & 0 & 0 & \cdots & 1 & \begin{bmatrix} n \\ n-1 \end{bmatrix}_q \frac{1}{[2]_q} & \end{vmatrix}, \quad n = 1, 2, \cdots, \end{aligned} \tag{19}$$

where $B_{n,q}(x)$ $(n = 0, 1, 2, \ldots)$ are the q-Bernoulli polynomials of degree n.

Taking $\beta_{0,q} = 1$, $\beta_{i,q} = \frac{1}{2}$ $(i = 1, 2, \cdots, n)$ and $A^{II}_{n,q}(x) = E_{n,q}(x)$ in equation (13), the following determinant form of the 2IqEP $E^{[2]}_{n,q}(x)$ is obtained:

Definition 4. *The 2-iterated q-Euler polynomials $E^{[2]}_{n,q}(x)$ of degree n are defined by*

$$E^{[2]}_{0,q}(x) = 1,$$
$$E^{[2]}_{n,q}(x) =$$
$$= (-1)^n \begin{vmatrix} 1 & E_{1,q}(x) & E_{2,q}(x) & \cdots & E_{n-1,q}(x) & E_{n,q}(x) & 1 \\ 1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 1 & \frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}_q & \cdots & \frac{1}{2}\begin{bmatrix}n-1\\1\end{bmatrix}_q & \frac{1}{2}\begin{bmatrix}n\\1\end{bmatrix}_q & 0 \\ 0 & 0 & 1 & \cdots & \frac{1}{2}\begin{bmatrix}n-1\\2\end{bmatrix}_q & \frac{1}{2}\begin{bmatrix}n\\2\end{bmatrix}_q & \\ . & . & . & \cdots & . & . & \\ . & . & . & \cdots & . & . & \\ 0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\begin{bmatrix}n\\n-1\end{bmatrix}_q & \end{vmatrix}, \quad n = 1, 2, \cdots, \tag{20}$$

where $E_{n,q}(x)$ $(n = 0, 1, 2, \ldots)$ are the q-Euler polynomials of degree n.

Next, the determinant forms for the hybrid q-special polynomials given in Table 3 are obtained.

Replacing the powers of x, *i.e.* $x^0,\ x^1,\ x^2, \ldots, x^n$ by the corresponding polynomials $B_{0,q}(x),\ B_{1,q}(x), \ldots, B_{n,q}(x)$ in the r.h.s. and replacing x by the polynomial $B_{1,q}(x)$ and using the relation ${}_EB_{n,q}(x) \equiv {}_BE_{n,q}(x) = E_{n,q}\{B_{1,q}(x)\}$ in the l.h.s. of the equation (18), the following determinant form of the qBEP ${}_BE_{n,q}(x)$ (or qEBP ${}_EB_{n,q}(x)$) is obtained:

Definition 5. *The q-Bernoulli-Euler polynomials ${}_BE_{n,q}(x)$ (or q-Euler-Bernoulli polynomials ${}_EB_{n,q}(x)$) of degree n are defined by*

$$
\begin{aligned}
&{}_EB_{0,q}(x) \equiv {}_BE_{0,q}(x) = 1,\\
&{}_EB_{n,q}(x) \equiv {}_BE_{n,q}(x) =\\
&= (-1)^n \begin{vmatrix}
1 & B_{1,q}(x) & B_{2,q}(x) & \cdots & B_{n-1,q}(x) & B_{n,q}(x) & 1 \\
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & 0 \\
0 & 1 & \frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}_q & \cdots & \frac{1}{2}\begin{bmatrix}n-1\\1\end{bmatrix}_q & \frac{1}{2}\begin{bmatrix}n\\1\end{bmatrix}_q & 0 \\
0 & 0 & 1 & \cdots & \frac{1}{2}\begin{bmatrix}n-1\\2\end{bmatrix}_q & \frac{1}{2}\begin{bmatrix}n\\2\end{bmatrix}_q & \\
. & . & . & \cdots & . & . & \\
. & . & . & \cdots & . & . & \\
0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\begin{bmatrix}n\\n-1\end{bmatrix}_q &
\end{vmatrix}, \quad n = 1, 2, \cdots ,
\end{aligned} \tag{21}
$$

where $B_{n,q}(x)$ $(n = 0, 1, 2, \ldots)$ are the q-Bernoulli polynomials of degree n.

Next, replacing the powers of x, *i.e.* $x^0,\ x^1,\ x^2, \ldots, x^n$ by the corresponding polynomials $H_{0,q}(x),\ H_{1,q}(x),\ \ldots, H_{n,q}(x)$ in the r.h.s. and replacing x by the polynomial $H_{1,q}(x)$ and using the relation ${}_BH_{n,q}(x) \equiv {}_HB_{n,q}(x) = B_{n,q}\{H_{1,q}(x)\}$ in the l.h.s. of the equation (17), the following determinant form of the qHBP ${}_HB_{n,q}(x)$ (or qBHP ${}_BH_{n,q}(x)$) is obtained:

Definition 6. *The q-Hermite-Bernoulli polynomials ${}_HB_{n,q}(x)$ (or q-Bernoulli-Hermite polynomials ${}_BH_{n,q}(x)$) of degree n are defined by*

$$
\begin{aligned}
&{}_BH_{0,q}(x) \equiv {}_HB_{0,q}(x) = 1,\\
&{}_BH_{n,q}(x) \equiv {}_HB_{n,q}(x) =\\
&= (-1)^n \begin{vmatrix}
1 & H_1(x) & H_2(x) & \cdots & H_{n-1}(x) & H_n(x) & 1 \\
1 & \frac{1}{[2]_q} & \frac{1}{[3]_q} & \cdots & \frac{1}{[n]_q} & \frac{1}{[n+1]_q} & 0 \\
0 & 1 & \begin{bmatrix}2\\1\end{bmatrix}_q \frac{1}{[2]_q} & \cdots & \begin{bmatrix}n-1\\1\end{bmatrix}_q \frac{1}{[n-1]_q} & \begin{bmatrix}n\\1\end{bmatrix}_q \frac{1}{[n]_q} & 0 \\
0 & 0 & 1 & \cdots & \begin{bmatrix}n-1\\2\end{bmatrix}_q \frac{1}{[n-2]_q} & \begin{bmatrix}n\\2\end{bmatrix}_q \frac{1}{[n-1]_q} & \\
. & . & . & \cdots & . & . & \\
. & . & . & \cdots & . & . & \\
0 & 0 & 0 & \cdots & 1 & \begin{bmatrix}n\\n-1\end{bmatrix}_q \frac{1}{[2]_q} &
\end{vmatrix}, \quad n = 1, 2, \cdots .
\end{aligned} \tag{22}
$$

where $H_{n,q}(x)$ $(n = 0, 1, 2, \ldots)$ are the q-Hermite polynomials of degree n.

Further, replacing the powers of x, *i.e.* $x^0,\ x^1,\ x^2,\ldots,x^n$ by the corresponding polynomials $H_{0,q}(x),\ H_{1,q}(x),\ldots,H_{n,q}(x)$ in the r.h.s. and replacing x by the polynomial $H_{1,q}(x)$ and using the relation ${}_EH_{n,q}(x)\equiv{}_HE_{n,q}(x)=E_{n,q}\{H_{1,q}(x)\}$ in the l.h.s. of the equation (18), the following determinant form of the qHEP ${}_HE_{n,q}(x)$ (or qEHP ${}_EH_{n,q}(x)$) is obtained:

Definition 7. *The q-Hermite-Euler polynomials* ${}_HE_{n,q}(x)$ *(or q-Euler-Hermite polynomials* ${}_EH_{n,q}(x)$*) of degree n are defined by*

$$
\begin{aligned}
&{}_EH_{0,q}(x)\equiv{}_HE_{0,q}(x)=1,\\
&{}_EH_{n,q}(x)\equiv{}_HE_{n,q}(x)=\\
&=(-1)^n\begin{vmatrix}
1 & H_{1,q}(x) & H_{2,q}(x) & \cdots & H_{n-1,q}(x) & H_{n,q}(x) & 1\\
1 & \frac{1}{2} & \frac{1}{2} & \cdots & \frac{1}{2} & \frac{1}{2} & 0\\
0 & 1 & \frac{1}{2}\begin{bmatrix}2\\1\end{bmatrix}_q & \cdots & \frac{1}{2}\begin{bmatrix}n-1\\1\end{bmatrix}_q & \frac{1}{2}\begin{bmatrix}n\\1\end{bmatrix}_q & 0\\
0 & 0 & 1 & \cdots & \frac{1}{2}\begin{bmatrix}n-1\\2\end{bmatrix}_q & \frac{1}{2}\begin{bmatrix}n\\2\end{bmatrix}_q & \\
. & . & . & \cdots & . & . & \\
. & . & . & \cdots & . & . & \\
0 & 0 & 0 & \cdots & 1 & \frac{1}{2}\begin{bmatrix}n\\n-1\end{bmatrix}_q &
\end{vmatrix},\quad n=1,2,\cdots,
\end{aligned}
\tag{23}
$$

where $H_{n,q}(x)$ $(n=0,1,2,\ldots)$ *are the q-Hermite polynomials of degree n.*

Remark 5. *Taking* $x=0$ *in determinant definitions* (17)-(18) *of* $B_{n,q}(x)$ *and* $E_{n,q}(x)$ *and on expanding the determinants with respect to first row and using suitable notations from Table 1 (I-II), the determinant definitions of the related numbers* $B_{n,q}$ *and* $E_{n,q}$ *can be obtained.*

Remark 6. *Taking* $x=0$ *in determinant definitions* (19)-(23) *of* $B^{[2]}_{n,q}(x)$, $E^{[2]}_{n,q}(x)$, ${}_BE_{n,q}(x)$ *(or* ${}_EB_{n,q}(x)$*),* ${}_BH_{n,q}(x)$ *(or* ${}_HB_{n,q}(x)$*) and* ${}_EH_{n,q}(x)$ *(or* ${}_HE_{n,q}(x)$*) and then using suitable notations from Tables 1, 4 and 5 (I-III), the determinant definitions of the related numbers* $B^{[2]}_{n,q}$, $E^{[2]}_{n,q}$, ${}_BE_{n,q}$ *(or* ${}_EB_{n,q}$*),* ${}_BH_{n,q}$ *(or* ${}_HB_{n,q}$*) and* ${}_EH_{n,q}$ *(or* ${}_HE_{n,q}$*) can be obtained.*

4.3. Identities for the 2-iterated q-Appell polynomials

In order to establish the identities for the 2IqAP $A^{[2]}_{n,q}(x)$, the following identities for the q-Appell polynomials are considered [5].

$$\begin{aligned} A_{n,q}(x) &= \frac{1}{\beta_{0,q}}\Big(x^n - \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}\, A_{k,q}(x)\Big), \quad n=1,2,\cdots \quad, \\ x^n &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}\, A_{k,q}(x), \quad n=1,2,\cdots \quad . \end{aligned}$$

Replacing the powers of x, *i.e.* x^1 and x^n by the corresponding polynomials $A_{1,q}(x)$ and $A_{n,q}(x)$ in above equations and then using equation (9) in the resultant equations, the following identities for the 2IqAP $A^{[2]}_{n,q}(x)$ are obtained:

$$\begin{aligned} A^{[2]}_{n,q}(x) &= \frac{1}{\beta_{0,q}}\Big(A_{n,q}(x) - \sum_{k=0}^{n-1} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}\, A^{[2]}_{k,q}(x)\Big), \quad n=1,2,\cdots \quad, \\ A_{n,q}(x) &= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_q \beta_{n-k,q}\, A^{[2]}_{k,q}(x), \quad n=1,2,\cdots \quad . \end{aligned}$$

The above examples illustrate that the operational correspondence established in this article can be applied to derive the results for the newly introduced q-special polynomials given in Tables 2 and 3 from the results of the corresponding member belonging to the q-Appell family.

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References

[1] W.A. Al-Salam, *q-Appell polynomials* Ann. Mat. Pura Appl., **4(17)**, 1967, 31-45.

[2] W.A. Al-Salam, *q-Bernoulli numbers and polynomials*, Math. Nachr., **17**, 1959, 239-260.

[3] G.E. Andrews, R. Askey, R. Roy, *71th Special Functions of Encyclopaedia of Mathematics and its Applications*, Cambridge University Press, Cambridge, 1999.

[4] T. Ernst, *q-Bernoulli and q-Euler polynomials, An umbral approach*, Int. J. Diff. Equ., **1(1)**, 2006, 31-80.

[5] M.E. Keleshteri, N.I. Mahmudov, *A study on q-Appell polynomials from determinantal point of view*, Appl. Math. Comput., **260**, 2015, 351-369.

[6] M.E. Keleshteri, N.I. Mahmudov, *A q-umbral approach to q-Appell polynomials*, arXiv:1505.05067.

[7] T. Kim, *On the q-extension of Euler and Genocchi numbers*, J. Math. Anal. Appl., **326**, 2007, 1458-1465.

[8] N.I. Mahmudov, *On a class of q-Bernoulli and q-Euler polynomials*, Adv. Difference Equ., **108**, 2013, 1-11.

[9] S. Roman, *The theory of the umbral calculus I*, J. Math. Anal. Appl., **87**, 1982, 58-115.

[10] S. Roman, *The Umbral Calculus*, Academic Press, New York, 1984.

[11] S. Roman, *More on the umbral calculus, with emphasis on the q-umbral calculus*, J. Math. Anal. Appl., **107**, 1985, 222-254.

[12] C.S. Ryoo, *A note on q-Bernoulli numbers and polynomials* Appl. Math. Lett., **20**, 2007, 524-531.

[13] C.S. Ryoo, T. Kim, *A numerical computation of the structure of the roots of q-Bernoulli polynomials*, J. Comput. Appl. Math., **214**, 2008, 319-332.

[14] C.S. Ryoo, T. Kim, R.P. Agarwal, *A numerical investigation of the roots of q-polynomials*, Int. J. Comput. Math., **83(2)**, 2006, 223-234.

[15] W. Wang, *A determinantal approach to Sheffer sequences*, Linear Algebra Appl., **463**, 2014, 228-254.

Mumtaz Riyasat
Department of Mathematics, Aligarh Muslim University, Aligarh, India
E-mail:mumtazrst@gmail.com

Subuhi Khan
Department of Mathematics, Aligarh Muslim University, Aligarh, India

E-mail:subuhi2006@gmail.com

Nazim I. Mahmudov
Department of Mathematics, Eastern Mediterranean University, Gazimagusa, TRNC, Mersin 10, Turkey
E-mail:nazim.mahmudov@emu.edu.tr

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