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On Completeness of Root Vectors of Fourth Order Operator Pencil Corresponding to Eigenvalues of Quarter Plane

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Abstract. In this paper, we find sufficient conditions for the existence and uniqueness of a regularly holomorphic solution of boundary value problem for a class of fourth-order operator differential equations. Moreover, for an operator pencil associated with the boundary value problem under consideration, we prove the completeness of its root vectors, corresponding to eigenvalues from the sector $\tilde{S}_{\frac{\pi}{4}} = \left\{\lambda : |\arg\lambda - \pi| < \frac{\pi}{4}\right\}$. We also establish the completeness of elementary regularly holomorphic solutions of the considered operator differential equation.

Key Words and Phrases: operator pencil, root vectors, holomorphic solution, completeness of the system.

2010 Mathematics Subject Classifications: 34G10, 35P10, 46E40, 47L75

In memory of M. G. Gasymov on his 80th birthday

Let us consider a fourth order polynomial operator pencil

$$P(\lambda) = \lambda^4 E - A^4 + \lambda^3 A_1 + \lambda^2 A_2 + \lambda A_3 \tag{1}$$

in a separable Hilbert space H, where E is an identity operator, λ is a spectral parameter, and the remaining coefficients satisfy the following conditions:

1. A is a positive definite self-adjoint operator with completely continuous inverse A^{-1};

2. the operators $B_j = A_j A^{-j}, j = \overline{1, 3}$, are bounded in H.

Obviously, the domain of the operator A^{γ} $(\gamma \geq 0)$ is the Hilbert space H_{γ} with respect to scalar product $(x, y)_{\gamma} = (A^{\gamma}x, A^{\gamma}y)$, $x, y \in D(A^{\gamma})$; for $\gamma = 0$ we assume that $H_0 = H$.

In this paper, we find the conditions on the coefficients of the operator pencil (1) that provide the completeness of its eigen- and adjoint vectors (root vectors) corresponding to the eigenvalues from the sector

$$\tilde{S}_{\frac{\pi}{4}} = \left\{ \lambda : \, |\arg \lambda - \pi| < \frac{\pi}{4} \right\}.$$

Note that the completeness (multiple completeness) of eigenvectors and adjoint vectors corresponding to eigenvalues from the left half-plane was studied by M.G. Gasymov [1, 2, 3], G.V. Radzievskii [4], A.A. Shkalikov [5], S.S. Mirzoev [6, 7], A.R. Aliev and A.A. Gasymov [8], A.R. Aliev and A.S. Mohamed [9] (see also the references therein).

The completeness of the eigenvectors and adjoined vectors with a defect, corresponding to eigenvalues from some sector, was considered by M.G. Gasymov [3], G.V. Radzievskii [4] and others.

Denote

$$S_{\frac{\pi}{4}} = \left\{ \lambda : \, |\arg \lambda| < \frac{\pi}{4} \right\}.$$

Definition 1. *If the equation* $P(\lambda_n)\, x_{0,n,j} = 0$ *has a non-trivial solution* $x_{0,n,j}$, *then* λ_n *is called an eigenvalue, and* $x_{0,n,j}$ *the corresponding eigenvector of the operator pencil* $P(\lambda)$, *corresponding to* λ_n. *If the vectors* $x_{0,n,j}, x_{1,n,j}, ..., x_{h,n,j}$, $h = \overline{0, m_{n,j}}$, $j = \overline{1, q_n}$, *satisfy the equations*

$$\sum_{q=1}^{h} \frac{d^q P(\lambda)}{q! d\lambda^q} x_{h-q,n,j} = 0,$$

then $x_{0,n,j}, x_{1,n,j}, ..., x_{h,n,j}$ *are called eigen- and adjoint vectors of the pencil* $P(\lambda)$, *corresponding to the eigenvalue* λ_n.

Let's denote by $L_2(R_+;\, H)$ the Hilbert space of functions $f(t)$ defined almost everywhere on $R_+ = (0, +\infty)$, with values in H, for which

$$\|f\|_{L_2(R_+;\, H)} = \left(\int_0^{+\infty} \|f(t)\|^2 \, dt \right)^{\frac{1}{2}} < +\infty.$$

Further, following the monograph [10], we define the Hilbert space

$$W_2^4(R_+;\, H) = \left\{ u(t) : \, u^{(4)} \in L_2(R_+;\, H),\, A^4 u \in L_2(R_+;\, H) \right\}$$

with the norm

$$\|u\|_{W_2^4(R_+;\,H)} = \left(\left\|A^4u\right\|^2_{L_2(R_+;\,H)} + \left\|u^{(4)}\right\|_{L_2(R_+;\,H)}\right)^{\frac{1}{2}}.$$

Here the derivatives are understood in the sense of the theory of distributions [10].

Following M.G. Gasymov [3], we denote by $H_{4,\,\frac{\pi}{4}}$ a linear set of functions $f\,(z)$ with values in H, which are holomorphic in the sector $S_{\frac{\pi}{4}}$, satisfy $f(te^{i\alpha}) \equiv f_\alpha\,(t) \in L_2\,(R_+;\,H)$ for each $\alpha \in \left(-\frac{\pi}{4},\,\frac{\pi}{4}\right)$, and

$$\sup_{|\alpha|<\frac{\pi}{4}} \int_0^{+\infty} \left\|f(te^{i\alpha})\right\|^2 dt < +\infty.$$

The functions $f(z) \in H_{4,\,\frac{\pi}{4}}$ have boundary values (almost everywhere or in $L_2\,(R_+;\,H)$) $f_\pm\,(t) \in L_2\,(R_+;\,H)$ on the rays $\Gamma_{\pm\frac{\pi}{4}} = \left\{\lambda = te^{\pm i\frac{\pi}{4}},\, t > 0\right\}$. $H_{4,\,\frac{\pi}{4}}$ becomes a Hilbert space with the norm

$$\|f\|_{H_{4,\,\frac{\pi}{4}}} = \frac{1}{\sqrt{2}}\left(\|f_+(t)\|^2_{L_2(R_+;\,H)} + \|f_-(t)\|^2_{L_2(R_+;\,H)}\right)^{\frac{1}{2}}.$$

We introduce the Hilbert space

$$W^4_{2,\,\frac{\pi}{4}} = \left\{u(z) :\, u^{(4)}(z) \in H_{4,\,\frac{\pi}{4}},\; A^4u(z) \in H_{4,\,\frac{\pi}{4}}\right\}$$

with the norm

$$\|u\|_{W^4_{2,\,\frac{\pi}{4}}} = \left(\left\|u^{(4)}(z)\right\|^2_{H_{4,\,\frac{\pi}{4}}} + \left\|A^4u(z)\right\|^2_{H_{4,\,\frac{\pi}{4}}}\right)^{\frac{1}{2}}$$

and the subspace $\overset{\circ}{W}{}^4_{2,\frac{\pi}{4}}$ of the space $W^4_{2,\frac{\pi}{4}}$:

$$\overset{\circ}{W}{}^4_{2,\frac{\pi}{4}} = \left\{u(z) :\, u(z) \in W^4_{2,\frac{\pi}{4}},\, u(0) = 0\right\}.$$

Note that the subspace $\overset{\circ}{W}{}^4_{2,\frac{\pi}{4}}$ is defined correctly, as there are analogues of theorems on intermediate derivatives and traces for the functions $u(z) \in W^4_{2,\frac{\pi}{4}}$, i.e. if $u(z) \in W^4_{2,\frac{\pi}{4}}$, then

$$A^{4-j}u^{(j)}(z) \in H_{4,\,\frac{\pi}{4}},\, j = \overline{0,\,3}, u^{(j)}\,(0) \in H_{4-j-\frac{1}{2}},\, j = \overline{0,\,3},$$

and

$$\left\| A^{4-j} u^{(j)} \right\|_{H_{4,\,\frac{\pi}{4}}} \le const\ \|u\|_{W^4_{2,\frac{\pi}{4}}}\ ,\ j = \overline{0,\,3},$$

$$\left\| u^{(j)}\,(0) \right\|_{4-j-\frac{1}{2}} \le const\ \|u\|_{W^4_{2,\frac{\pi}{4}}}\ ,\ j = \overline{0,\,3}.$$

If $u(z) \in W^4_{2,\frac{\pi}{4}}$, then $u_{\pm}(t) \in W^4_2\,(R_+;\,H)$. Further, we note that if e^{-tA} is a semigroup of bounded operators generated by the operator $-A$, then $e^{-zA}\varphi$ belongs to $W^4_{2,\frac{\pi}{4}}$ if and only if $\varphi \in H_{7/2}$, moreover

$$\left\| e^{-zA}\varphi \right\|_{W^4_{2,\frac{\pi}{4}}} \le const\, \|\varphi\|_{7/2}\,.$$

Now we associate the pencil (1) with the boundary value problem

$$P\left(\frac{d}{dz}\right) u(z) = 0, \quad z \in S_{\frac{\pi}{4}}, \tag{2}$$

$$u(0) = \varphi, \quad \varphi \in H_{\frac{7}{2}}. \tag{3}$$

Here the derivatives are understood in the sense of complex analysis in H.

Definition 2. *If for any $\varphi \in H_{\frac{7}{2}}$ there is a function $u(z) \in W^4_{2,\frac{\pi}{4}}$ that satisfies the equation* (2) *identically in $S_{\frac{\pi}{4}}$, the boundary condition* (3) *in the sense of convergence*

$$\lim_{\substack{z \to 0 \\ |\arg z| < \frac{\pi}{4}}} \|u(z) - \varphi\|_{\frac{7}{2}} = 0$$

and the estimate

$$\|u\|_{W^4_{2,\frac{\pi}{4}}} \le const\ \|\varphi\|_{\frac{7}{2}}\ ,$$

then the problem (2), (3) *is called regularly solvable, and $u(z)$ is called a regular holomorphic solution of this problem.*

It is evident that if $\lambda_n \in \tilde{S}_{\frac{\pi}{4}}$ and $\{x_{h,\,n,j}\}$ is the system of eigen- and adjoint vectors of the pencil $P\,(\lambda)$ corresponding to the eigenvalue λ_n, then

$$u_{h,n,j}\,(z) = e^{\lambda_n z}\left(\frac{z^h}{h!}x_{0,n,j} + \frac{z^{h-1}}{(h-1)!}x_{1,n,j} + ... + x_{h,n,j}\right),\ h = \overline{0,\,m_{n,j}},\ j = \overline{1,\,q_n},$$

belong to the space $W^4_{2,\frac{\pi}{4}}$, satisfy the equation (2), and are called elementary holomorphic solutions of equation (2) in the sector $S_{\frac{\pi}{4}}$. Conversely, if all the functions $u_{h,n,j}\,(z)$ belong to $W^4_{2,\frac{\pi}{4}}$ and satisfy equation (2), then the vectors

$x_{0,n,j},\ x_{1,n,j}, ..., x_{h,n,j}$ are eigen- and adjoint vectors of the pencil $P(\lambda)$, corresponding to the eigenvalue $\lambda_n \in \tilde{S}_{\frac{\pi}{4}}$. Denote this system by $K\left(\frac{\pi}{4}\right)$.

The goal of this paper is to find conditions that provide the completeness of the system $K\left(\frac{\pi}{4}\right)$ in the space of traces of regular holomorphic solutions of the equation (2), i.e. in $H_{\frac{7}{2}}$, and the completeness of the system of holomorphic elementary solutions $\{u_{h,n,j}(z)\}_{n=1}^{\infty}$, $h = \overline{0,\ m_{n,j}}$, $j = \overline{1, q_n}$, in the space of holomorphic solutions of the boundary value problem (2), (3).

Note that the completeness of the system $K\left(\frac{\pi}{4}\right)$ with a finite-dimensional defect in H with $B_j \in \sigma_\infty$, $j = \overline{1,3}$, was proved in [3].

Now we prove some statements.

Let's introduce the notations

$$P_0 u \equiv P_0\left(\frac{d}{dz}\right) u(z) = u^{(4)}(z) - A^4(z), \quad P_1 u = \sum_{j=1}^{3} A_{4-j} u^{(j)}(z), \quad u \in \overset{0}{W}{}^4_{2,\frac{\pi}{4}},$$

and

$$P_0(\lambda) = \lambda^4 E - A^4,\ \ P_1(\lambda) = \sum_{j=1}^{3} \lambda^j A_{4-j}.$$

First we investigate some analytical properties of the resolvent $P^{-1}(\lambda)$.

Lemma 1. *Let the conditions 1), 2) and the inequality*

$$\sum_{j=1}^{3} d_{4,j} \|B_{4-j}\| < 1$$

hold, where

$$d_{4,j} = \left(\frac{j}{4}\right)^{\frac{j}{4}} \left(\frac{4-j}{4}\right)^{\frac{4-j}{4}},\ \ j = \overline{1,\ 3}. \tag{4}$$

Then the estimates

$$\left\|\lambda^s A^{4-s} P^{-1}(\lambda)\right\| \le const,\ \ s \in [0, 4],$$

hold on the rays $\gamma_\pm = \left\{\lambda : \lambda = te^{\pm i\frac{3\pi}{4}},\ t \ge 0\right\}$.

Proof. Let $\lambda \in \gamma_+$, i.e. $\lambda = te^{i\frac{3\pi}{4}}$. Then the operator pencil $P_0(\lambda) = \lambda^4 E - A^4 = -t^4 E - A^4$ is invertible in H and, from the representation

$$P(\lambda) = P_0(\lambda) + P_1(\lambda) = \left(E + P_1(\lambda) P_0^{-1}(\lambda)\right) P_0(\lambda),\ \ \lambda \in \gamma_+,$$

we see that if $\left\|P_1(\lambda) P_0^{-1}(\lambda)\right\| < \theta < 1$ for $\lambda \in \gamma_+$, then the pencil $P(\lambda)$ is also invertible in H. As the inequality

$$\left\|P_1(\lambda) P_0^{-1}(\lambda)\right\| \le \sum_{j=1}^{3} \|B_{4-j}\| \ \left\|\lambda^j A^{4-j} P_0^{-1}(\lambda)\right\|$$

holds for $\lambda \in \gamma_+$, it follows from the spectral decomposition of operator A that

$$\left\|\lambda^s A^{4-s} P_0^{-1}(\lambda)\right\| = \sup_{\mu \in \sigma(A)} \left| t^s \mu^{4-s} \left(t^4 + \mu^4\right)^{-1} \right| \le \sup_{\tau > 0} \left| \tau^s \left(\tau^4 + 1\right)^{-1} \right| = d_{4,s},$$

where $d_{4,s} = \left(\frac{s}{4}\right)^{\frac{s}{4}} \left(\frac{4-s}{4}\right)^{\frac{4-s}{4}}$, $s = \overline{1,3}$, $d_{4,s} = 1$, for $s = 0$ and $s = 4$.

Thus

$$\left\|P_1(\lambda) P_0^{-1}(\lambda)\right\| \le \sum_{j=1}^{3} d_{4,j} \|B_{4-j}\| < 1.$$

Then

$$P^{-1}(\lambda) = P_0^{-1}(\lambda) \left(E + P_1(\lambda) P_0^{-1}(\lambda)\right)^{-1}$$

and for $s \in [0,4]$ the inequality

$$\left\|\lambda^s A^{4-s} P^{-1}(\lambda)\right\| \le \left\|\lambda^s A^{4-s} P_0^{-1}(\lambda)\right\| \cdot \left\|E + P_1(\lambda) P_0^{-1}(\lambda)\right\| \le const.$$

holds. ◄

Theorem 1. *Let the condition 1) be satisfied. Then the operator* $P_0 = P_0\left(\frac{d}{dz}\right)$ *isomorphically maps the space* $\overset{\circ}{W}{}^4_{2,\frac{\pi}{4}}$ *onto the space* $H_{2,\frac{\pi}{4}}$.

Proof. For $x \in H_{\frac{7}{2}}$ the function $u_0(z) = e^{-zA}x$ is a general solution of the equation $P_0\left(\frac{d}{dz}\right) u(z) = 0$ from the space $W^4_{2,\frac{\pi}{4}}$. From the condition $u(z) \in \overset{\circ}{W}{}^4_{2,\frac{\pi}{4}}$ it follows that $e^{-zA}x = 0$, i.e., $x = 0$. Therefore, $Ker\, P_0 = \{0\}$. On the other hand, it is easy to see that the vector function

$$\omega(z) = \frac{1}{2\pi i} \sum_{k=1}^{2} \int_{\Gamma_k} (-1)^k P_0^{-1}(\lambda) \hat{f}(\lambda)\, d\lambda, \ \Gamma_1 = \gamma_+, \ \Gamma_2 = \gamma_-,$$

where $\hat{f}(\lambda)$ is the Laplace transform of the function $f(z) \in H_{2,\frac{\pi}{4}}$, is a particular solution of the equation $P_0\left(\frac{d}{dz}\right) u(z) = f(z)$. As $f(z) \in H_{2,\frac{\pi}{4}}$, $\hat{f}(\lambda)$ is holomorphic in the sector $\mathbb{C} \backslash \tilde{S}_{\frac{\pi}{4}}$, $\left\|\hat{f}(\lambda)\right\| \to 0$, $\lambda \to \infty$ in this sector, and $\hat{f}(\lambda)$ has boundary values on γ_+ and γ_-. Lemma 1, in particular, implies that $\omega(z) \in W^4_{2,\frac{\pi}{4}}$.

Then the general solution of the equation $P_0\left(\frac{d}{dz}\right)u(z) = f(z)$ from the space $W^4_{2,\frac{\pi}{4}}$ has the form $u(z) = \omega(z) + e^{-zA}x,\ x \in H_{\frac{7}{2}}$. From the condition $u(0) = 0$ it follows that $x = -\omega(0) \in H_{\frac{7}{2}}$, besides $u\,(z) \in \overset{\circ}{W}{}^4_{2,\frac{\pi}{4}}$. Thus, $ImP_0 = W^4_{2,\frac{\pi}{4}}$. Further, given that $\|P_0u\|_{H_{2,\frac{\pi}{4}}} = \left\|u^{(4)} - A^4u\right\|_{H_{2,\frac{\pi}{4}}} \le const\,\|u\|_{W^4_{2,\frac{\pi}{4}}}$, from Banach's inverse operator theorem we get the validity of the theorem. ◀

Now we prove the following theorem.

Theorem 2. *Let the conditions 1), 2) and the inequality*

$$\sum_{j=1}^{3} N_j \,\|B_{4-j}\| \, < 1$$

holds, where

$$N_j = \sup_{0 \ne u \in \overset{\circ}{W}{}^4_{2,\frac{\pi}{4}}} \left\|A^{4-j}u^{(j)}\right\|_{H_{2,\frac{\pi}{4}}} \|P_0u\|^{-1}_{H_{2,\frac{\pi}{4}}}\,, \quad j = 1,\,2,3.$$

Then the problem (2), (3) *is regularly solvable.*

Proof. Replacing $u(z) = \omega(z) + e^{-zA}\varphi,\ \varphi \in H_{\frac{7}{2}}$, from equation (2) we obtain

$$P_0\left(\frac{d}{dz}\right)\omega(z) + P_1\left(\frac{d}{dz}\right)\omega(z) = -P_1\left(\frac{d}{dz}\right)e^{-zA}\varphi.$$

It's obvious that

$$\|g(z)\|_{H_{2,\frac{\pi}{4}}} \equiv \left\|P_1\left(\frac{d}{dt}\right)e^{-zA}\varphi\right\|_{H_{2,\frac{\pi}{4}}} \le \sum_{j=1}^{3}\|B_{4-j}\|\ \left\|A^{4-j}A^je^{-zA}\varphi\right\|_{H_{2,\frac{\pi}{4}}} \le$$

$$\le \sum_{j=1}^{3}\|B_{4-j}\|\ \left\|A^4e^{-zA}\varphi\right\|_{H_{2,\frac{\pi}{4}}} \le const\,\|\varphi\|_{7/2}\,,$$

i.e. $g(z) = -P_1\left(\frac{d}{dz}\right)e^{-zA}\varphi \in H_{2,\frac{\pi}{4}}$. So, given that $\omega\,(0) = 0$, we obtain the following equation for ω :

$$P\omega = P_0\omega + P_1\omega = g,\, \omega \in \overset{\circ}{W}{}^4_{2,\frac{\pi}{4}}, g \in H_{2,\,\frac{\pi}{4}}.$$

Since P_0 is an isomorphism, after replacement $\upsilon = P_0\omega$ we obtain the equation $\left(E + P_1P_0^{-1}\right)\upsilon = g$ in the space $H_{2,\,\frac{\pi}{4}}$. As

$$\left\|P_1P_0^{-1}\upsilon\right\|_{H_{2,\,\frac{\pi}{4}}} = \|P_1\omega\|_{H_{2,\,\frac{\pi}{4}}} \le \sum_{j=1}^{3}\|B_{4-j}\|\ \left\|A^{4-j}\omega^{(j)}\right\|_{H_{2,\frac{\pi}{4}}} \le$$

$$\leq \sum_{j=1}^{3} N_j \left\| B_{4-j} \right\| \left\| P_0 \omega \right\|_{H_{2,\, \frac{\pi}{4}}} < \left\| \upsilon \right\|_{H_{2,\, \frac{\pi}{4}}} ,$$

we have

$$\upsilon = \left(E + P_1 P_0^{-1} \right)^{-1} g,$$

and

$$\omega = P_0^{-1} \left(E + P_1 P_0^{-1} \right)^{-1} g.$$

It easily follows that

$$\left\| \omega \right\|_{W^4_{2, \frac{\pi}{4}}} \leq const \left\| g \right\|_{H_{2, \frac{\pi}{4}}} \leq const \left\| \varphi \right\|_{\frac{7}{2}} .$$

Thus, $u(z) = \omega(z) + e^{-zA}\varphi$ is a regular solution to the problem (2), (3), and

$$\left\| u \right\|_{W^4_{2, \frac{\pi}{4}}} \leq const \left\| \varphi \right\|_{\frac{7}{2}} . \quad ◀$$

From this theorem it is clear that in order to find the solvability conditions for the boundary value problem (2), (3), it is necessary to find the exact values N_j, $j = 1, 2, 3$, or estimate them from above. To this end, we prove some statements.

Lemma 2. *For* $\beta \in \left[0, d_{4,j}^{-2}\right)$*, where the numbers* $d_{4,j}$ *are defined by the equalities* (4)*, polynomial operator pencils*

$$P_j(\lambda;\ \beta;\ A) = (i\lambda)^8 E + 2 (i\lambda)^4 A^4 + A^8 - \beta (i\lambda)^{2j} A^{8-2j}, \ \ j = 1, 2, 3,$$

are represented as

$$P_j(\lambda;\ \beta;\ A) = \phi_j(\lambda;\ \beta;\ A)\, \phi_j(-\lambda;\ \beta;\ A), \tag{5}$$

where the operator pencil

$$\phi_j(\lambda;\ \beta;\ A) = \lambda^4 E + A^4 + \alpha_{1,j}(\beta)\, \lambda A^3 + \alpha_{2,j}(\beta)\, \lambda^2 A^2 + \alpha_{3,j}(\beta)\, \lambda^3 A,$$

$\phi_j(\lambda;\ \beta;\ A)$ *has a spectrum only from the left half-plane, and the numbers* $\alpha_{l,j}(\beta)$, $l = 1, 2, 3$, *satisfy the relations*

a) for $j = 1$*:* $\alpha_{1,1}^2(\beta) - 2\alpha_{2,1}(\beta) = -\beta$, $\alpha_{2,1}^2(\beta) = 2\alpha_{1,1}(\beta)\, \alpha_{3,1}(\beta)$, $\alpha_{3,1}^2(\beta) = 2\alpha_{2,1}(\beta)$;

b) for $j = 2$*:* $\alpha_{1,2}^2(\beta) = 2\alpha_{2,2}(\beta)$, $\alpha_{2,2}^2(\beta) - 2\alpha_{1,2}(\beta)\, \alpha_{3,2}(\beta) = -\beta$, $\alpha_{3,2}^2(\beta) = 2\alpha_{2,2}(\beta)$;

c) for $j = 3$*:* $\alpha_{1,3}^2(\beta) = 2\alpha_{2,3}(\beta)$, $\alpha_{2,3}^2(\beta) = 2\alpha_{1,3}(\beta)\, \alpha_{3,3}(\beta)$, $\alpha_{3,3}^2(\beta) - 2\alpha_{2,3}(\beta) = -\beta$.

Proof. It is evident that for $\mu \in \sigma(A)$ the polynomial $P_j(\lambda;\ \beta;\ \mu)$ has no roots on the imaginary axis if $\beta \in \left[0,\ d_{4,j}^{-2}\right)$. Therefore, it can be represented as

$$P_j(\lambda;\ \beta;\ \mu) = \phi_j(\lambda;\ \beta;\ \mu)\,\phi_j(-\lambda;\ \beta;\ \mu),$$

where

$$\phi_j(\lambda;\ \beta;\ \mu) = (\lambda - \mu\omega_{1,j}(\beta))\,(\lambda - \mu\omega_{2,j}(\beta))\,(\lambda - \mu\omega_{3,j}(\beta))\,(\lambda - \mu\omega_{4,j}(\beta)),$$

and $Re\omega_{k,j}(\beta) < 0,\ k = \overline{1,4}$. Using the spectral decomposition of the operator A, we obtain the representation (5), and the relations a), b), and c) are obtained by comparing identical powers of λ in the representation (5). ◄

This lemma implies

Lemma 3. *For any* $u \in W_2^4(R_+; H)$ *the equality*

$$\left\|\phi_j\left(\frac{d}{dt}; \beta; A\right)u\right\|^2_{L_2(R_+;H)} + (S_j(\beta; A)\,\tilde{\varphi}, \tilde{\varphi})_{H^4} =$$

$$= \left\|u^{(4)} + A^4 u\right\|^2_{L_2(R_+;H)} - \beta\left\|A^{4-j}u^{(j)}\right\|^2_{L_2(R_+;H)},\ j = 1,\ 2, 3,$$

is true, where

$$\tilde{\varphi} = \left(\varphi_\nu = A^{4-\nu-\frac{1}{2}}u^{(\nu)}(0)\right)^3_{\nu=0},$$

and

$$S_j(\beta) = \begin{pmatrix} \alpha_{1,j} & \alpha_{2,j} & \alpha_{3,j} & 0 \\ \alpha_{2,j} & \alpha_{1,j}\alpha_{2,j} - \alpha_{3,j} & \alpha_{1,j}\alpha_{3,j} & \alpha_{1,j} \\ \alpha_{3,j} & \alpha_{3,j}\alpha_{1,j} & \alpha_{3,j}\alpha_{2,j} - \alpha_{1,j} & \alpha_{2,j} \\ 0 & \alpha_{1,j} & \alpha_{2,j} & \alpha_{3,j} \end{pmatrix}.$$

Proof. Integrating by parts and taking into account the relations a), b), and c), we obtain

$$\left\|\phi_j\left(\frac{d}{dt}; \beta; A\right)u\right\|^2_{L_2(R_+;H)} = \left\|u^{(4)}\right\|^2_{L_2(R_+;H)} +$$

$$+2\left\|A^2 u''\right\|^2_{L_2(R_+;H)} + \left\|A^4 u\right\|^2_{L_2(R_+;H)} - (Q_j(\beta)\,\tilde{\varphi}, \tilde{\varphi})_{H^4},$$

where all elements of the matrix $Q_j(\beta)$ are equal to the elements of the matrix $S_j(\beta)$, except

$$q_{4,1} = 1,\ q_{3,2} = \alpha_{1,j}\alpha_{2,j} - 1,\ q_{2,3} = \alpha_{3,j}\alpha_{2,j} - 1,\ q_{1,4} = 1.$$

On the other hand,

$$\left\|u^{(4)}+A^4u\right\|^2_{L_2(R_+;H)}=\left\|u^{(4)}\right\|^2_{L_2(R_+;H)}+2\left\|A^2u''\right\|^2_{L_2(R_+;H)}+$$

$$+\left\|A^4u\right\|^2_{L_2(R_+;H)}-(Q_0\tilde{\varphi},\tilde{\varphi})_{H^4},$$

where all elements of the matrix Q_0 are zero except for $\overset{0}{q}_{4,1}=1,\ \overset{0}{q}_{3,2}=-1,\ \overset{0}{q}_{2,3}=-1, \overset{0}{q}_{1,4}=1$.

Thus,

$$\left\|\phi_j\left(\frac{d}{dt};\beta;A\right)u\right\|^2_{L_2(R_+;H)}+\left(\left(Q_j(\beta)-Q_0\right)\tilde{\varphi},\tilde{\varphi}\right)_{H^4}=$$

$$=\left\|u^{(4)}+A^4u\right\|^2_{L_2(R_+;H)}-\beta\left\|A^{4-j}u^{(j)}\right\|^2_{L_2(R_+;H)}.$$

This implies the assertion of the lemma. ◄

Theorem 3. *Let* $\beta\in\left[0,\,d_{4,j}^{-2}\right)$ *and* $u\in\overset{\circ}{W}{}^4_{2,\frac{\pi}{4}}$. *Then the following equality is true:*

$$\left\|P_0\left(\frac{d}{dz}\right)u\right\|^2_{H_{2,\frac{\pi}{4}}}-\beta\left\|A^{4-j}u^{(j)}(z)\right\|^2_{H_{2,\frac{\pi}{4}}}=\frac{1}{2}\left\|\phi_j\left(\frac{d}{dt};\beta;A\right)u_+\right\|^2_{L_2(R_+;H)}+$$

$$+\frac{1}{2}\left\|\phi_j\left(\frac{d}{dt};\beta;A\right)u_-\right\|^2_{L_2(R_+;H)}+\left(\overset{0}{S}_j(\beta)\,\tilde{\varphi}_{\frac{\pi}{4}},\tilde{\varphi}_{\frac{\pi}{4}}\right)_{H^3},\ j=1,\,2,\,3, \qquad (6)$$

where

$$\tilde{\varphi}_{\frac{\pi}{4}}=\left(\varphi_\nu=A^{4-\nu-\frac{1}{2}}u^{(\nu)}(0)\right)^3_{\nu=1},\ u_\pm=u\left(te^{\pm i\,\frac{\pi}{4}}\right),$$

and

$$\overset{0}{S}_j(\beta)=\begin{pmatrix}\alpha_{1,j}\alpha_{2,j}-\alpha_{3,j} & \frac{1}{\sqrt{2}}\alpha_{1,j}\alpha_{3,j} & 0\\ \frac{1}{\sqrt{2}}\alpha_{1,j}\alpha_{3,j} & \alpha_{3,j}\alpha_{2,j}-\alpha_{1,j} & \frac{1}{\sqrt{2}}\alpha_{2,j}\\ 0 & \frac{1}{\sqrt{2}}\alpha_{2,j} & \alpha_{3,j}\end{pmatrix}.$$

Proof. We have:

$$\left\|P_0\left(\frac{d}{dz}\right)u(z)\right\|^2_{H_{2,\frac{\pi}{4}}}=\left\|u^{(4)}(z)-A^4u(z)\right\|^2_{H_{2,\frac{\pi}{4}}}=$$

$$=\frac{1}{2}\left\|u_+^{(4)}e^{-4\cdot\frac{i\pi}{4}}-A^4u_+\right\|^2_{L_2(R_+;H)}+$$

$$+\frac{1}{2}\left\|u_-^{(4)}e^{4\cdot i\cdot\frac{\pi}{4}}-A^4u_-\right\|^2_{L_2(R_+;H)}=$$

$$=\frac{1}{2}\left\|u_+^{(4)}+A^4u_+\right\|^2_{L_2(R_+;H)}+\frac{1}{2}\left\|u_-^{(4)}+A^4u_-\right\|^2_{L_2(R_+;H)}.$$

Then from Lemma 3 and from the relation $\frac{\partial^l}{\partial t^l}u\left(te^{i\varphi}\right)=e^{il\varphi}\frac{d^l}{dt^l}u\left(te^{i\varphi}\right)$ $\left(z=te^{i\varphi}\right)$, we obtain $\tilde{\varphi}_{\frac{\pi}{4}}=\tilde{U}\tilde{\varphi}$, $\tilde{\varphi}_{-\frac{\pi}{4}}=\tilde{U}^{-1}\tilde{\varphi}$, where $\tilde{U}=diag\left(e^{i\frac{\pi}{4}},e^{2i\frac{\pi}{4}},e^{3i\frac{\pi}{4}}\right)$. Theorem is proved. ◄

Theorem 4. *The following estimates are true:*

$$N_1\leq 2^{-\frac{3}{4}},\ \ N_2\leq\left(\frac{18}{35+\sqrt{73}}\right)^{\frac{1}{2}},\ N_3\leq 3^{-\frac{1}{4}}.$$

Proof. Obviously, when $\beta=0$, $\alpha_{1,j}(0)=\alpha_{3,j}(0)=2\sqrt{2}$, $\alpha_{2,j}(0)=4$. Then it is easy to see that $\overset{0}{S}_j(0)>0$. Therefore, the first eigenvalue of the matrix $\overset{0}{S}_j(\beta)>0$ for small $\beta>0$. Further, if $N_j>d_{4,j}$, then $N_j^{-2}\in\left(0,\ d_{4,j}^{-2}\right)$. In this case, for $\beta\in\left(N_j^{-2},\ d_{4,j}^{-2}\right)$, by the definition of N_j, there is a function $u_\beta(z)\in\overset{\circ}{W}{}^4_{2,\frac{\pi}{4}}$ such that $\|P_0u_\beta\|^2_{H_{2,\frac{\pi}{4}}}<\beta\ \left\|A^{4-j}u^{(j)}\right\|^2_{H_{2,\frac{\pi}{4}}}$. Then from equality (6) we find that the first eigenvalue of the matrix $\overset{0}{S}_j(\beta)$ is less than zero for $\beta\in\left(N_j^{-2},\ d_{4,j}^{-2}\right)$. Thus, the first eigenvalue $\lambda_j^{(1)}(\beta)$ vanishes at some points in the interval $\left(0,\ d_{4,j}^{-2}\right)$. Consequently, the equation $\det\overset{0}{S}_j(\beta)=0$ has a solution from the interval $\left(0,\ d_{4,j}^{-2}\right)$ and it is obvious that $N_j^{-2}\geq\mu_j(0)$, where $\mu_j(0)$ is the smallest root of $\det\overset{0}{S}_j(\beta)=0$ in the interval $\left(0,\ d_{4,j}^{-2}\right)$. But if $N_j\leq d_{4,j}$ and $\mu_j(0)$ does exist, then it is evident that again $N_j\leq\mu_j^{\frac{1}{2}}(0)$. It follows from the above that we must solve the equation $\det\overset{0}{S}_j(\beta)=0$ with regard to the equalities a), b), or c). For example, for $j=1$, from the equation $\det\overset{0}{S}_j(\beta)=0$ with regard to condition a), we obtain $\alpha_{2,1}=\sqrt{8}$, $\alpha_{3,1}=2\sqrt[4]{8}$, $\alpha_{1,1}=\sqrt[4]{8}$. Therefore, $\beta=2\alpha_{2,1}-\alpha_{1,1}^2=\sqrt{8}$, i.e., $\mu_1(0)=\sqrt{8}\in\left(0,\ d_{4,j}^{-2}\right)$. Then $N_1\leq\left(\sqrt{8}\right)^{-\frac{1}{2}}=2^{-\frac{3}{4}}$. Similarly, for $j=2$ we have $\mu_2(0)=\frac{35+\sqrt{73}}{18}$, and for $j=3$, $\mu_j(0)=\sqrt{3}$. Therefore, $N_2\leq\left(\frac{18}{35+\sqrt{73}}\right)^{\frac{1}{2}}$, $N_3\leq 3^{-\frac{1}{4}}$. ◄

Thus, we obtain the theorem on the solvability of the boundary value problem (2), (3).

Theorem 5. *Let the conditions 1, 2) and the inequality*

$$\tau = 2^{-\frac{3}{4}} \|B_3\| + \left(\frac{18}{35+\sqrt{73}}\right)^{\frac{1}{2}} \|B_2\| + 3^{-\frac{1}{4}} \|B_1\| < 1, \tag{7}$$

hold. Then the boundary value problem (2), (3) *is regularly solvable.*

The following theorem is valid.

Theorem 6. *Let the conditions 1) and 2) $A^{-1} \in \sigma_\rho \, (0 < \rho < \infty)$ and one of the following conditions be satisfied: a) $\tau < 1$ for $0 < \rho \le 2$, $\tau < \sin\frac{\pi}{\rho}$ for $2 \le \rho < \infty$; b) $\tau < 1$, the operators $B_j = A_j A^{-j}$, $j = 1, 2, 3$, are completely continuous in H, where the number τ is determined from the inequality* (7)*. Then the system $K\left(\frac{\pi}{4}\right)$ is complete in $H_{\frac{7}{2}}$.*

Proof. From the results obtained in [1, 2, 4, 11] it follows that if $A^{-1} \in \sigma_\rho \, (0 < \rho < \infty)$, then $A^4 P^{-1}(\lambda)$ is represented as a ratio of two integer functions of order not higher than ρ and of minimal type with order ρ. Under the conditions of this theorem, the boundary value problem (2), (3) is regularly solvable. Then for any $\varphi \in H_{\frac{7}{2}}$ there is a solution to the problem $u(z) \in W^4_{2,\frac{\pi}{4}}$. If $K\left(\frac{\pi}{4}\right)$ is not complete, then there is a vector $\varphi \in H_{\frac{7}{2}}$ orthogonal to the system $K\left(\frac{\pi}{4}\right)$ in space $H_{\frac{7}{2}}$. Then, using the form of the regular solution we have

$$u(z) = \frac{1}{2\pi i} \sum_{k=1}^{2} (-1)^k \int_{\Gamma_k} \hat{u}(\lambda) \, e^{\lambda z} d\lambda,$$

where $\hat{u}(\lambda) = P^{-1}(\lambda) \cdot r(\lambda)$, $r(\lambda)$ are third order vector functions with respect to λ. In this case, for $z \in S_{\frac{\pi}{4}}$

$$(u(z), \varphi)_{\frac{7}{2}} = \left(A^{\frac{7}{2}} u(z), \; A^{\frac{7}{2}} \varphi\right) =$$

$$= \frac{1}{2\pi i} \sum_{k=1}^{2} (-1)^k \int_{\Gamma_k} \left(A^{\frac{7}{2}} P^{-1}(\lambda) \, r(\lambda), \, A^{\frac{7}{2}} \varphi\right) e^{\lambda z} d\lambda =$$

$$= \frac{1}{2\pi i} \sum_{k=1}^{2} (-1)^k \int_{\Gamma_k} \left(r(\lambda), \left(A^{\frac{7}{2}} P^{-1}(\lambda)\right)^* A^{\frac{7}{2}} \varphi\right) e^{\lambda z} d\lambda.$$

As $\left(A^{\frac{7}{2}}P^{-1}(\lambda)\right)^* A^{\frac{7}{2}}\varphi$ is an integer function of order not higher than ρ, $S(\lambda) = \left(r(\lambda), \left(A^{\frac{7}{2}}P^{-1}(\lambda)\right)^* A^{\frac{7}{2}}\varphi\right)$ is also an integer function of order not higher than ρ. It is evident that the function $S(\lambda) \to 0$ as $\lambda \to 0,\ \lambda \in \mathbb{C}\backslash\tilde{S}_{\frac{\pi}{4}}$ and has boundary values on the rays $\Gamma_k,\ k = 1, 2$. On the other hand, in case a), for $0 < \rho \le 2$ on the rays $\Gamma_k,\ k = 1, 2$, we have $\|S(\lambda)\| \le const\,|\lambda|^4$. Then, by the Phragmén–Lindelöf theorem, this estimate holds on the entire complex domain. It follows that the polynomial $S(\lambda) = \sum_{k=0}^{4} b_k\lambda^k$. Then $(u(z), \varphi)_{\frac{7}{2}} = 0$ for $z \in S_{\frac{\pi}{4}}$. Hence, when $z \to 0,\ z \in S_{\frac{\pi}{4}}$ we obtain $\|\varphi\|^2 = 0$, i.e., $\varphi = 0$. If, in case a), $2 \le \rho < \infty$, it is obvious that on the rays $\Gamma_{\pm\rho} = \left\{\lambda :\ \arg\lambda = \pi \pm \frac{\pi}{2\rho}\right\}$ we have

$$\left\|P_1(\lambda)\,P_0^{-1}(\lambda)\right\| \le \sum_{j=1}^{3}\|B_{4-j}\|\ \left\|\lambda^j A^{4-j}P_0^{-1}(\lambda)\right\|$$

and

$$\left\|\lambda^j A^{4-j}P_0^{-1}(\lambda)\right\| = \sup_{\mu\in\sigma(A)}\left|r^j\mu^{4-j}\left(r^8+\mu^8-2r^4\mu^4\cos\tfrac{2\pi}{\rho}\right)^{-\frac{1}{2}}\right| \le$$

$$\le \sup_{\mu\in\sigma(A)}\left|r^j\mu^{4-j}\left(r^4+\mu^4\right)^{-1}\left(1-2r^4\mu^4\left(r^4+\mu^4\right)^{-2}\left(1+\cos\tfrac{2\pi}{\rho}\right)\right)^{-\frac{1}{2}}\right| \le$$

$$\le d_{4,j}\left(1-\cos^2\frac{\pi}{\rho}\right)^{-\frac{1}{2}} = d_{4,j}\sin^{-1}\frac{\pi}{\rho}.$$

Therefore, when $\tau < \sin\frac{\pi}{\rho}$, on the rays $\Gamma_{\pm\rho}$ there exists

$$P^{-1}(\lambda) = P_0^{-1}(\lambda)\left(E + P_1(\lambda)\,P_0^{-1}(\lambda)\right)^{-1},$$

and again, repeating the same reasoning, we obtain $\varphi = 0$. In case b), from the existence of a solution of the boundary value problem (2), (3) for $\tau < 1$ and from the Keldysh lemma [11], it follows that for $0 < \rho < \infty$ we can apply the Phragmén–Lindelöf theorem and obtain $\varphi = 0$. ◄

Theorem 7. *Let the conditions of Theorem 6 hold. Then the system of elementary holomorphic solutions of equation* (2) *is complete in the space of regular holomorphic solutions of the boundary value problem* (2)*,* (3)*.*

Proof. Obviously, the space of regularly holomorphic solutions of the boundary value problem (2), (3) is closed. Then from the theorems on traces and uniqueness of regularly holomorphic solutions it follows that

$$c_1 \left\| \varphi \right\|_{\frac{7}{2}} \le \left\| u \right\|_{W^4_{2,\frac{\pi}{4}}} \le c_2 \left\| \varphi \right\|_{\frac{7}{2}} .$$

As the system $K\left(\frac{\pi}{4}\right)$ is complete in $H_{\frac{7}{2}}$, for any $\varepsilon > 0$ we can find numbers $c^N_{h,n,j}(\varepsilon)$ such that

$$\left\| \varphi - \sum_{n=1}^{N} \sum_{(h,j)} c^N_{h,n,j}(\varepsilon)\, x_{h,n,j} \right\|_{\frac{7}{2}} < \frac{\varepsilon}{c_2}.$$

Then, given that $\varphi = u(0)$, $x_{h,n,j} = u_{h,n,j}(0)$, it follows

$$\left\| u - \sum_{n=1}^{N} \sum_{(h,j)} c^N_{h,n,j}(\varepsilon)\, u_{h,n,j} \right\| < \varepsilon. \quad ◀$$

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