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Some Properties of Uniformly Starlike and Convex Hypergeometric Functions

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Abstract. The purpose of this paper is to introduce some characterizations of (Gaussian) hypergeometric function to be in various subclasses of uniformly starlike and uniformly convex functions. Operators related to hypergeometric functions are also considered. Some of our results correct previously known results.

Key Words and Phrases: univalent, starlike, convex, uniformly starlike, uniformly convex, hypergeometric functions, convolution invariance.

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1. Introduction

Let A denote the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \tag{1}$$

which are analytic in the open unit disc $U = \{z : z \in C \ \ and \ \ |z| < 1\}$, and let S be the subclass of all functions in A, which are univalent. Let $g(z) \in$ A be given by

$$g(z) = z + \sum_{n=2}^{\infty} g_n z^n. \tag{2}$$

Then the convolution invariance of two power series $f(z)$ and $g(z)$ is given by (see [1])

$$f(z) \oplus_k g(z) = z + \sum_{n=2}^{\infty} \frac{a_n g_n}{C_n(k)} z^n = g(z) \oplus_k f(z), \tag{3}$$

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where

$$C_n(k) = \binom{n+k-1}{k} \left(k \in N_0 = N \bigcup \{0\}, N = \{1, 2, ...\} \right). \tag{4}$$

Let $S^*(\alpha)$ and $\mathrm{K}(\alpha)$ denote the subclasses of starlike and convex functions of order α $(0 \leq \alpha < 1)$, respectively. We will denote $S^*(0) = S^*$ and $\mathrm{K}(0) = \mathrm{K}$ (see, for example, Srivastava and Owa [14]).

Recently, Bharati et al. [3] introduced the classes $UCV(\alpha, \beta)$ and $S_p(\alpha, \beta)$ as follows:

Definition 1. *[3] (i) A function $f(z)$ of the form (1) is said to be in the class $S_p(\alpha, \beta)$, if it satisfies the following condition:*

$$Re\left\{ \frac{zf'(z)}{f(z)} - \alpha \right\} > \beta \left| \frac{zf'(z)}{f(z)} - 1 \right| (-1 \leq \alpha < 1; \beta \geq 0; z \in U). \tag{5}$$

(ii) A function $f(z)$ of the form (1) is said to be in the class $UCV(\alpha, \beta)$ if and only if $zf'(z) \in S_p(\alpha, \beta)$.

Denote by T the subclass of S consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n \ (a_n \geq 0). \tag{6}$$

Also, denote by $T^*(\alpha) = S^*(\alpha) \cap T$, $C(\alpha) = \mathrm{K}(\alpha) \cap T$ the subclasses of starlike and convex functions of order α $(0 \leq \alpha < 1)$ with negative coefficients, which were introduced and studied by Silverman [12]. Also let $UCT(\alpha) = UCV(\alpha) \cap T$, $S_pT(\alpha) = S_p(\alpha) \cap T$, $UCT(\alpha, \beta) = UCV(\alpha, \beta) \cap T$ and $S_pT(\alpha, \beta) = S_p(\alpha, \beta) \cap T$.

Let $S(\lambda, \alpha, \beta)$ $(-1 \leq \alpha < 1, \beta \geq 0 \, and \, 0 \leq \lambda < 1)$ denote the subclass of S consisting of functions of the form (1) and satisfying the analytic criterion

$$Re\left\{ \frac{zf'(z)}{(1-\lambda) f(z) + \lambda z f'(z)} - \alpha \right\} \geq \beta \left| \frac{zf'(z)}{(1-\lambda) f(z) + \lambda z f'(z)} - 1 \right| \ (z \in U), \tag{7}$$

and $C(\lambda, \alpha, \beta)$ $(-1 \leq \alpha < 1, \beta \geq 0 \, and \, 0 \leq \lambda < 1)$ denote the subclass of S consisting of functions of the form (1) and satisfying the analytic criterion

$$Re\left\{ \frac{f'(z) + zf''(z)}{f'(z) + \lambda z f''(z)} - \alpha \right\} \geq \beta \left| \frac{f'(z) + zf''(z)}{f'(z) + \lambda z f''(z)} - 1 \right| \ (z \in U). \tag{8}$$

The classes $S(\lambda, \alpha, \beta)$ and $C(\lambda, \alpha, \beta)$ were introduced and studied by Aouf et al. [2, with $g(z) = \frac{z}{1-z}$ and $g(z) = \frac{z}{1-z}$, respectively]. It follows from (7) and (8) that

$$f(z) \in C(\lambda, \alpha, \beta) \Leftrightarrow zf'(z) \in S(\lambda, \alpha, \beta). \tag{9}$$

Further, we define the classes $TS(\lambda,\alpha,\beta)$ and $TC(\lambda,\alpha,\beta)$ by

$TS(\lambda,\alpha,\beta)=S(\lambda,\alpha,\beta)\cap T$ and $TC(\lambda,\alpha,\beta)=C(\lambda,\alpha,\beta)\cap T$, respectively.

Let ${}_2F_1(a,b;c;z)$ be the (Gaussian) hypergeometric function defined by

$$ {}_2F_1(a,b;c;z)=\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}z^n \qquad (z\in U), $$

where $c\neq 0,-1,-2,\dots$ and

$$ (\lambda)_n=\begin{cases}1 & if\;\; n=0,\\ \lambda(\lambda+1)(\lambda+2)\dots(\lambda+n-1) & if\;\; n\in N.\end{cases} $$

We note that ${}_2F_1(a,b;c;1)$ converges for $Re(c-a-b)>0$ and is related to Gamma functions (see [7, Lemma 6.1.1, pp. 205]) by

$$ {}_2F_1(a,b;c;1)=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}. \tag{10} $$

Define the function

$$ h_\mu(a,b;c;z)=(1-\mu)(z{}_2F_1(a,b;c;z))+\mu z(z{}_2F_1(a,b;c;z))' \;(\mu\geq 0). \tag{11} $$

The mapping properties of the function $h_\mu(a,b;c;z)$ were studied by Shukla and Shukla [15]. Corresponding to the Gaussian hypergeometric function ${}_2F_1(a,b;c;z)$, we define the linear operator $N_{a,b,c}^{\mu,k}:\mathrm{A}\to\mathrm{A}$ by the convolution invariance as follows:

$$ \left[N_{a,b,c}^{\mu,k}(f)\right](z)=h_\mu(a,b;c;z)\otimes_k f(z). \tag{12} $$

Merkes and Scott [8] and Ruscheweyh and Singh [11] used continued fractions to find sufficient conditions for $z{}_2F_1(a,b;c;z)$ to be in the class $S^*(\alpha)$ $(0\leq\alpha<1)$ for various choices of the parameters a,b and c. Carlson and Shaffer [4] showed how some convolution results about the class $S^*(\alpha)$ may be expressed in terms of a linear operator acting on hypergeometric functions. Recently, Silverman [13] gave necessary and sufficient conditions for $z{}_2F_1(a,b;c;z)$ to be in the classes $S^*(\alpha)$and $\mathrm{K}(\alpha)$.

In this paper, we obtain necessary and sufficient conditions for $h_\mu(a,b;c;z)$ to be in the subclasses $TS(\lambda,\alpha,\beta)$ and $TC(\lambda,\alpha,\beta)$. Also, we consider an operator related to hypergeometric function.

2. Main results

Unless otherwise mentioned, we assume throughout this paper that $-1\leq\alpha<1,\beta\geq 0\,,0\leq\lambda<1$ *and* $k\in N_0$.

To establish our results, we need the following lemmas due to Aouf et al. [2].

Lemma 1. *[2, Theorem 1, with $g(z)=\frac{z}{1-z}$] . A sufficient condition for $f(z)$ defined by (1) to be in the class $S(\lambda,\alpha,\beta)$ is*

$$\sum_{n=2}^{\infty}\{n(1+\beta)-(\alpha+\beta)[1+\lambda(n-1)]\}|a_n|\leq 1-\alpha. \tag{13}$$

Lemma 2. *[2, Theorem 2, with $g(z)=\frac{z}{1-z}$]. A necessary and sufficient condition for $f(z)$ defined by (6) to be in the class $TS(\lambda,\alpha,\beta)$ is*

$$\sum_{n=2}^{\infty}\{n(1+\beta)-(\alpha+\beta)[1+\lambda(n-1)]\}a_n\leq 1-\alpha. \tag{14}$$

Lemma 3. *[2, Theorem 1, with $g(z)=\frac{z}{(1-z)^2}$]. A sufficient condition for $f(z)$ defined by (1) to be in the class $C(\lambda,\alpha,\beta)$ is*

$$\sum_{n=2}^{\infty}n\{n(1+\beta)-(\alpha+\beta)[1+\lambda(n-1)]\}|a_n|\leq 1-\alpha. \tag{15}$$

Lemma 4. *[2, Theorem 2, with $g(z)=\frac{z}{(1-z)^2}$]. A necessary and sufficient condition for $f(z)$ defined by (6) to be in the class $TC(\lambda,\alpha,\beta)$ is*

$$\sum_{n=2}^{\infty}n\{n(1+\beta)-(\alpha+\beta)[1+\lambda(n-1)]\}a_n\leq 1-\alpha. \tag{16}$$

Theorem 1. *Let $a,b>0$ and $c>a+b+2$. Then the sufficient condition for $h_\mu(a,b;c;z)$ to be in the class $S(\lambda,\alpha,\beta)$ is*

$$\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\left[1+\frac{ab[(1+\beta)(1+2\mu)-(\alpha+\beta)(\lambda+\mu+\lambda\mu)]}{(1-\alpha)(c-a-b-1)}\right.$$
$$\left.+\frac{(a)_2(b)_2[\mu(1+\beta)(1+2\mu)-\lambda\mu(\alpha+\beta)]}{(1-\alpha)(c-a-b-2)_2}\right]\leq 2. \tag{17}$$

Also, condition (17) is necessary and sufficient for $h^*(a,b;c;z)=z\left(2-\frac{h_\mu(a,b;c;z)}{z}\right)$ to be in the class $TS(\lambda,\alpha,\beta)$.

Proof. Since

$$h_\mu(a,b;c;z)=z+\sum_{n=2}^{\infty}[1+\mu(n-1)]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}z^n,$$

according to Lemma 1, we only need to show that

$$\sum_{n=2}^{\infty}\left\{n\left(1+\beta\right)-\left(\alpha+\beta\right)\left[1+\lambda\left(n-1\right)\right]\right\}\,\left[1+\mu\left(n-1\right)\right]\frac{(a)_{n-1}\,(b)_{n-1}}{(c)_{n-1}\,(1)_{n-1}}\leq 1-\alpha.$$

Thus

$$\begin{aligned}&\textstyle\sum_{n=2}^{\infty}\left\{n\left(1+\beta\right)-\left(\alpha+\beta\right)\left[1+\lambda\left(n-1\right)\right]\right\}\,\left[1+\mu\left(n-1\right)\right]\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}=\\&=\left[\left(1+\beta\right)\left(1+2\mu\right)-\left(\alpha+\beta\right)\left(\lambda+\mu+\lambda\mu\right)\right]\textstyle\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-2}}+\\&+\left[\mu\left(1+\beta\right)-\lambda\mu\left(\alpha+\beta\right)\right]\textstyle\sum_{n=3}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-3}}+\left(1-\alpha\right)\textstyle\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}}{(c)_{n-1}(1)_{n-1}}.\end{aligned}\tag{18}$$

Since $(\lambda)_n=\lambda\left(\lambda+1\right)_{n-1}$, from (10), we may express (18) as

$$\begin{aligned}&\left[\left(1+\beta\right)\left(1+2\mu\right)-\left(\alpha+\beta\right)\left(\lambda+\mu+\lambda\mu\right)\right]\frac{ab}{c}\sum_{n=0}^{\infty}\frac{(a+1)_n\,(b+1)_n}{(c+1)_n\,(1)_n}+\left[\mu\left(1+\beta\right)-\right.\\&\left.\lambda\mu\left(\alpha+\beta\right)\right]\frac{(a)_2\,(b)_2}{(c)_2}\sum_{n=0}^{\infty}\frac{(a+2)_n\,(b+2)_n}{(c+2)_n\,(1)_n}+\left(1-\alpha\right)\left[\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n}{(c)_n\,(1)_n}-1\right]=\\&=\left[\left(1+\beta\right)\left(1+2\mu\right)-\left(\alpha+\beta\right)\left(\lambda+\mu+\lambda\mu\right)\right]\frac{ab}{c}\frac{\Gamma\left(c+1\right)\Gamma\left(c-a-b-1\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}+\\&+\left[\mu\left(1+\beta\right)-\lambda\mu\left(\alpha+\beta\right)\right]\frac{(a)_2\,(b)_2}{(c)_2}\frac{\Gamma\left(c+2\right)\Gamma\left(c-a-b-2\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}+\\&+\left(1-\alpha\right)\left[\frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}-1\right]=\\&=\frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}\left[\frac{ab\left[\left(1+\beta\right)\left(1+2\mu\right)-\left(\alpha+\beta\right)\left(\lambda+\mu+\lambda\mu\right)\right]}{\left(c-a-b-1\right)}+\right.\\&\left.+\frac{(a)_2\,(b)_2\left[\mu\left(1+\beta\right)-\lambda\mu\left(\alpha+\beta\right)\right]}{\left(c-a-b-2\right)_2}+\left(1-\alpha\right)\right]-\left(1-\alpha\right).\end{aligned}$$

This last expression is bounded above by $(1-\alpha)$ if (17) holds. Since

$$h^*\left(a,b;c;z\right)=z-\sum_{n=2}^{\infty}\left[1+\mu\left(n-1\right)\right]\frac{(a)_{n-1}\,(b)_{n-1}}{(c)_{n-1}\,(1)_{n-1}}z^n,$$

the necessity of (17) for $h^*\left(a,b;c;z\right)$ to be in the class $TS\left(\lambda,\alpha,\beta\right)$ follows from Lemma 2. This completes the proof of Theorem 1. ◄

Remark 1. *(i) Putting $\lambda = \alpha = 0$ in Theorem 1, we obtain the result obtained by Ramachandran et al. [10, Theorem 1, with $p = 2$ and $q = 1$];*

(ii) Putting $\lambda = \mu = \beta = 0$ in Theorem 1, we obtain the result obtained by Silverman [13, Theorem 1];

(iii) Putting $\beta = 1$ and $\lambda = \mu = 0$ in Theorem 1, we obtain the result obtained by Cho et al. [5, Theorem 1];

(vi) Putting $\lambda = \mu = 0$ in Theorem 1, we obtain the result obtained by Swaminathan [16, Theorem 1].

Theorem 2. *Let $a, b > -1, ab < 0$ and $c > a + b + 2$. Then the necessary and sufficient condition for $h_\mu(a, b; c; z)$ to be in the class $TS(\lambda, \alpha, \beta)$ is*

$$(a)_2 (b)_2 [\mu (1+\beta) - \lambda\mu (\alpha+\beta)] + ab (c-a-b-2) \times$$

$$\times (1+\beta)(1+2\mu) - (\alpha+\beta)(\lambda+\mu+\lambda\mu) + (1-\alpha)(c-a-b-2)_2 \geq 0. \quad (19)$$

Proof. Since

$$\begin{aligned} h_\mu(a,b;c;z) &= z + \tfrac{ab}{c} \textstyle\sum_{n=2}^{\infty} [1+\mu(n-1)] \tfrac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n = \\ &= z - \left|\tfrac{ab}{c}\right| \textstyle\sum_{n=2}^{\infty} [1+\mu(n-1)] \tfrac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} z^n, \end{aligned}$$

according to Lemma 2, we only need to prove that

$$\sum_{n=2}^{\infty} \{n(1+\beta) - (\alpha+\beta)[1+\lambda(n-1)]\} [1+\mu(n-1)] \frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq$$

$$\leq \left|\frac{c}{ab}\right| (1-\alpha). \quad (20)$$

Thus

$$\begin{aligned} &\textstyle\sum_{n=2}^{\infty} \{n(1+\beta) - (\alpha+\beta)[1+\lambda(n-1)]\} [1+\mu(n-1)] \tfrac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} = \\ &= [(1+\beta)(1+2\mu) - (\alpha+\beta)(\lambda+\mu+\lambda\mu)] \textstyle\sum_{n=2}^{\infty} \tfrac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} + [\mu(1+\beta) - \\ &-\lambda\mu(\alpha+\beta)] \textstyle\sum_{n=3}^{\infty} \tfrac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-3}} + (1-\alpha) \textstyle\sum_{n=2}^{\infty} \tfrac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} = \\ &= [(1+\beta)(1+2\mu) - (\alpha+\beta)(\lambda+\mu+\lambda\mu)] \textstyle\sum_{n=0}^{\infty} \tfrac{(a+1)_n(b+1)_n}{(c+1)_n(1)_n} + \\ &+ [\mu(1+\beta) - \lambda\mu(\alpha+\beta)] \tfrac{(a+1)(b+1)}{(c+1)} \textstyle\sum_{n=0}^{\infty} \tfrac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n} + \\ &+ (1-\alpha) \tfrac{c}{ab} \left[\textstyle\sum_{n=0}^{\infty} \tfrac{(a)_n(b)_n}{(c)_n(1)_n} - 1\right] = \end{aligned}$$

$$\begin{aligned}
&[(1+\beta)(1+2\mu)-(\alpha+\beta)(\lambda+\mu+\lambda\mu)]\tfrac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+\\
&+[\mu(1+\beta)-\lambda\mu(\alpha+\beta)]\tfrac{(a+1)(b+1)}{(c+1)}\tfrac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}+\\
&+(1-\alpha)\tfrac{c}{ab}\left[\tfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]=\\
&=\tfrac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\left\{[(1+\beta)(1+2\mu)-(\alpha+\beta)(\lambda+\mu+\lambda\mu)](c-a-b-2)+\right.\\
&\left.+(a+1)(b+1)[\mu(1+\beta)-\lambda\mu(\alpha+\beta)]+\tfrac{(1-\alpha)}{ab}(c-a-b-2)_2\right\}-(1-\alpha)\tfrac{c}{ab}.
\end{aligned}$$

Hence (20) is equivalent to

$$\begin{aligned}
&\tfrac{\Gamma(c+1)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}\left\{[(1+\beta)(1+2\mu)-(\alpha+\beta)(\lambda+\mu+\lambda\mu)](c-a-b-2)+\right.\\
&\left.+(a+1)(b+1)[\mu(1+\beta)-\lambda\mu(\alpha+\beta)]+\tfrac{(1-\alpha)}{ab}(c-a-b-2)_2\right\}\leq\\
&\leq(1-\alpha)\tfrac{c}{ab}-(1-\alpha)\tfrac{c}{ab}=0\,.
\end{aligned}\tag{21}$$

Thus, from (21), we have

$$(a+1)(b+1)[\mu(1+\beta)-\lambda\mu(\alpha+\beta)]+$$
$$+[(1+\beta)(1+2\mu)-(\alpha+\beta)(\lambda+\mu+\lambda\mu)](c-a-b-2)+$$
$$+\frac{(1-\alpha)}{ab}(c-a-b-2)_2\leq 0,$$

or, equivalently,

$$(a)_2(b)_2[\mu(1+\beta)-\lambda\mu(\alpha+\beta)]+$$
$$+ab[(1+\beta)(1+2\mu)-(\alpha+\beta)(\lambda+\mu+\lambda\mu)](c-a-b-2)+$$
$$+(1-\alpha)(c-a-b-2)_2\geq 0.$$

This completes the proof of Theorem 2. ◄

Putting $\lambda=\alpha=0$ in Theorem 2, we get the correct from of the result obtained by Ramachandran et al. [10, Theorem 2, with $p=2$ and $q=1$].

Corollary 1. *Let $a,b>-1, ab<0$ and $c>a+b+2$. Then the necessary and sufficient condition for $h_\mu(a,b;c;z)$ to be in the class $S_pT(\beta)$ is*

$$\mu(1+\beta)\frac{(a+1)(b+1)}{(c+1)}{}_2F_1(a+2,b+2;c+2;1)+$$

$$[\mu(\beta+2)+\beta+1]\,{}_2F_1(a+1,b+1;c+1;1)+\frac{c}{ab}{}_2F_1(a,b;c;1)\leq 0.$$

Remark 2. *(i) Putting $\lambda = \mu = \beta = 0$ in Theorem 2, we obtain the result obtained by Silverman [13, Theorem 2];*

(ii) Putting $\lambda = \mu = 0$ and $\beta = 1$ in Theorem 2, we obtain the result obtained by Cho et al. [5, Theorem 2];

(iii) Putting $\lambda = \mu = 0$ in Theorem 2, we obtain the result obtained by Swaminathan [16, Theorem 2].

Theorem 3. *Let $a, b > 0$ and $c > a + b + 3$. Then the sufficient condition for $h_\mu (a, b; c; z)$ to be in the class $C (\lambda, \alpha, \beta)$ is*

$$\begin{aligned}&\tfrac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}\Big[1+\tfrac{ab\{(1+\beta)(3+4\mu)-(\alpha+\beta)(1+2\lambda+2\mu+2\lambda\mu)\}}{(1-\alpha)(c-a-b-1)}+\\&+\tfrac{(a)_2(b)_2\{(1+\beta)(1+5\mu)-(\alpha+\beta)(\lambda+\mu+4\lambda\mu)\}}{(1-\alpha)(c-a-b-2)_2}+\\&+\tfrac{(a)_3(b)_3\{\mu(1+\beta)-\lambda\mu(\alpha+\beta)\}}{(1-\alpha)(c-a-b-2)_3}\Big]\le 2.\end{aligned} \tag{22}$$

Also, condition (22) is necessary and sufficient for $h^* (a, b; c; z)$ to be in the class $TC (\lambda, \alpha, \beta)$.

Proof. According to Lemma 3, we only need to prove that

$$\sum_{n=2}^{\infty} n \{n (1 + \beta) - (\alpha + \beta) [1 + \lambda (n - 1)]\} [1 + \mu (n - 1)] \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} \le 1 - \alpha.$$

Hence

$$\sum_{n=2}^{\infty} n \{n (1 + \beta) - (\alpha + \beta) [1 + \lambda (n - 1)]\} \, [1 + \mu (n - 1)] \, \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-1}} =$$

$$= [(1 + \beta) (3 + 4\mu) - (\alpha + \beta) (1 + 2\lambda + 2\mu + 2\lambda\mu)] \sum_{n=2}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-2}} +$$

$$+ [(1 + \beta) (1 + 5\mu) - (\alpha + \beta) (\lambda + \mu + 4\lambda\mu)] \sum_{n=3}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-3}}$$

$$+ [\mu (1 + \beta) - \lambda\mu (\alpha + \beta)] \sum_{n=4}^{\infty} \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (1)_{n-4}} + (1 - \alpha) \sum_{n=2}^{\infty} \frac{(a)_{n-2} (b)_{n-2}}{(c)_{n-1} (1)_{n-1}} =$$

$$= [(1 + \beta) (3 + 4\mu) - (\alpha + \beta) (1 + 2\lambda + 2\mu + 2\mu\lambda)] \frac{ab}{c} \sum_{n=0}^{\infty} \frac{(a + 1)_n (b + 1)_n}{(c + 1)_n (1)_n} +$$

$$+\left[(1+\beta)(1+5\mu)-(\alpha+\beta)(\lambda+\mu+4\lambda\mu)\right]\frac{(a)_2(b)_2}{(c)_2}\sum_{n=0}^{\infty}\frac{(a+2)_n(b+2)_n}{(c+2)_n(1)_n}+$$

$$+\left[\mu(1+\beta)-\lambda\mu(\alpha+\beta)\right]\frac{(a)_3(b)_3}{(c)_3}\sum_{n=0}^{\infty}\frac{(a+3)_n(b+3)_n}{(c+3)_n(1)_n}+(1-\alpha)\times$$

$$\times\left[\sum_{n=0}^{\infty}\frac{(a)_n(b)_n}{(c)_n(1)_n}-1\right]=$$

$$=\left[(1+\beta)(3+4\mu)-(\alpha+\beta)(1+2\lambda+2\mu+2\lambda\mu)\right]\frac{ab}{c}\frac{\Gamma(c+1)\Gamma(c-a-b-1)}{\Gamma(c-a)\Gamma(c-b)}+$$

$$+\left[(1+\beta)(1+5\mu)-(\alpha+\beta)(\lambda+\mu+4\lambda\mu)\right]\frac{(a)_2(b)_2}{(c)_2}\frac{\Gamma(c+2)\Gamma(c-a-b-2)}{\Gamma(c-a)\Gamma(c-b)}+$$

$$+\left[\mu(1+\beta)-\lambda\mu(\alpha+\beta)\right]\frac{(a)_3(b)_3}{(c)_3}\frac{\Gamma(c+3)\Gamma(c-a-b-3)}{\Gamma(c-a)\Gamma(c-b)}+$$

$$(1-\alpha)\left[\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}-1\right]=\frac{\Gamma(c)\Gamma(c-a-b)}{\Gamma(c-a)\Gamma(c-b)}$$

$$=\left\{\left[(1+\beta)(3+4\mu)-(\alpha+\beta)(1+2\lambda+2\mu+2\lambda\mu)\right]\frac{ab}{(c-a-b-1)}+\right.$$

$$+\left[(1+\beta)(1+5\mu)-(\alpha+\beta)(\lambda+\mu+4\lambda\mu)\right]\frac{(a)_2(b)_2}{(c-a-b-2)_2}$$

$$\left.+\left[\mu(1+\beta)-\lambda\mu(\alpha+\beta)\right]\frac{(a)_3(b)_3}{(c-a-b-3)_3}+(1-\alpha)\right\}-(1-\alpha).$$

From the last expression, we get the desired conclusion. Also, the necessity of (22) for $h^*(a,b;c;z)$ to be in the class $TC(\lambda,\alpha,\beta)$ follows from Lemma 4. This completes the proof of Theorem 3. ◄

Remark 3. *(i) Putting $\lambda=\mu=\beta=0$ in Theorem 3, we obtain the result obtained by Silverman [13, Theorem 3];*

(ii) Putting $\beta=1$ and $\lambda=\mu=0$ in Theorem 3, we obtain the result obtained by Cho et al. [5, Theorem 3];

(iii) Putting $\lambda=\mu=0$ in Theorem 3, we obtain the result obtained by Swaminathan [16, Theorem 3].

Putting $\lambda=\alpha=0$ in Theorem 3, we get the correct form of the result obtained by Ramachandran et al. [10, Theorem 3, with p = 2 and q = 1].

Corollary 2. *Let $a, b > 0$ and $c > a + b + 3$. Then the sufficient condition for $h_\mu(a, b; c; z)$ to be in the class $UCT(\beta)$ is*

$$\mu(1+\beta)\frac{(a)_3(b)_3}{(c)_3}\,{}_2F_1(a+3, b+3; c+3; 1) +$$

$$(4\mu\beta + 5\mu + \beta + 1)\frac{(a)_2(b)_2}{(c)_2}\,{}_2F_1(a+2, b+2; c+2; 1) +$$

$$(2\mu\beta + 4\mu + 2\beta + 3)\frac{ab}{c}\,{}_2F_1(a+1, b+1; c+1; 1) + {}_2F_1(a, b; c; z) \leq 2\,. \tag{23}$$

Also, condition (23) is necessary and sufficient for $h^*(a, b; c; z) = z\left(2 - \frac{h_\mu(a,b;c;z)}{z}\right)$ to be in the class $UCT(\beta)$.

Theorem 4. *Let $a, b > -1$; $ab < 0$ and $c > a + b + 3$. Then the necessary and sufficient condition for $h_\mu(a, b; c; z)$ to be in the class $TC(\lambda, \alpha, \beta)$ is*

$$(a)_3(b)_3[\mu(1+\beta) - \lambda\mu(\alpha+\beta)] +$$

$$+(a)_2(b)_2[(1+\beta)(1+5\mu) - (\alpha+\beta)(\lambda+\mu+4\lambda\mu)](c-a-b-3) +$$

$$+ab[(1+\beta)(3+4\mu) - (\alpha+\beta)(1+2\lambda+2\mu+2\lambda\mu)](c-a-b-3)_2 +$$

$$+(1-\alpha)(c-a-b-3)_3 \geq 0. \tag{24}$$

Proof. According to Lemma 4, we only need to prove that

$$\sum_{n=2}^{\infty} n\{n(1+\beta) - (\alpha+\beta)[1+\lambda(n-1)]\}[1+\mu(n-1)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} \leq$$

$$\leq \left|\frac{c}{ab}\right|(1-\alpha)\,. \tag{25}$$

Thus

$$\sum_{n=2}^{\infty} n\{n(1+\beta) - (\alpha+\beta)[1+\lambda(n-1)]\}[1+\mu(n-1)]\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-1}} =$$

$$= [(1+\beta)(3+4\mu) - (\alpha+\beta)(1+2\lambda+2\mu+2\lambda\mu)]\sum_{n=2}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-2}} +$$

$$+[(1+\beta)(1+5\mu) - (\alpha+\beta)(\lambda+\mu+4\lambda\mu)]\sum_{n=3}^{\infty}\frac{(a+1)_{n-2}(b+1)_{n-2}}{(c+1)_{n-2}(1)_{n-3}} +$$

$$+\left[\mu\left(1+\beta\right)-\lambda\mu\left(\alpha+\beta\right)\right]\sum_{n=4}^{\infty}\frac{\left(a+1\right)_{n-2}\left(b+1\right)_{n-2}}{\left(c+1\right)_{n-2}\left(1\right)_{n-4}}+$$

$$+\left(1-\alpha\right)\sum_{n=2}^{\infty}\frac{\left(a+1\right)_{n-2}\left(b+1\right)_{n-2}}{\left(c+1\right)_{n-2}\left(1\right)_{n-1}}=$$

$$=\left[\left(1+\beta\right)\left(3+4\mu\right)-\left(\alpha+\beta\right)\left(1+2\lambda+2\mu+2\lambda\mu\right)\right]\sum_{n=0}^{\infty}\frac{\left(a+1\right)_{n}\left(b+1\right)_{n}}{\left(c+1\right)_{n}\left(1\right)_{n}}+$$

$$+\left[\left(1+\beta\right)\left(1+5\mu\right)-\left(\alpha+\beta\right)\left(\lambda+\mu+4\lambda\mu\right)\right]\frac{\left(a+1\right)\left(b+1\right)}{\left(c+1\right)}\sum_{n=0}^{\infty}\frac{\left(a+2\right)_{n}\left(b+2\right)_{n}}{\left(c+2\right)_{n}\left(1\right)_{n}}+$$

$$+\left[\mu\left(1+\beta\right)-\lambda\mu\left(\alpha+\beta\right)\right]\frac{\left(a+1\right)_{2}\left(b+1\right)_{2}}{\left(c+1\right)_{2}}\sum_{n=0}^{\infty}\frac{\left(a+3\right)_{n}\left(b+3\right)_{n}}{\left(c+3\right)_{n}\left(1\right)_{n}}+$$

$$+\left(1-\alpha\right)\frac{c}{ab}\left[\sum_{n=0}^{\infty}\frac{\left(a\right)_{n}\left(b\right)_{n}}{\left(c\right)_{n}\left(1\right)_{n}}-1\right]$$

$$=\left[\left(1+\beta\right)\left(3+4\mu\right)-\left(\alpha+\beta\right)\left(1+2\lambda+2\mu+2\lambda\mu\right)\right]\frac{\Gamma\left(c+1\right)\Gamma\left(c-a-b-1\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}+$$

$$+\left[\left(1+\beta\right)\left(1+5\mu\right)-\left(\alpha+\beta\right)\left(\lambda+\mu+4\lambda\mu\right)\right]\times$$

$$\times\frac{\left(a+1\right)\left(b+1\right)}{\left(c+1\right)}\frac{\Gamma\left(c+2\right)\Gamma\left(c-a-b-2\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}+$$

$$+\left[\mu\left(1+\beta\right)-\lambda\mu\left(\alpha+\beta\right)\right]\frac{\left(a+1\right)_{2}\left(b+1\right)_{2}}{\left(c+1\right)_{2}}\frac{\Gamma\left(c+3\right)\Gamma\left(c-a-b-3\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}+$$

$$+\left(1-\alpha\right)\frac{c}{ab}\left[\frac{\Gamma\left(c\right)\Gamma\left(c-a-b\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}-1\right]=$$

$$=\frac{\Gamma\left(c+1\right)\Gamma\left(c-a-b-3\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}\times$$

$$\times\left\{\left[\left(1+\beta\right)\left(3+4\mu\right)-\left(\alpha+\beta\right)\left(1+2\lambda+2\mu+2\lambda\mu\right)\right]\left(c-a-b-3\right)_{2}+\right.$$

$$+\left(a+1\right)\left(b+1\right)\left[\left(1+\beta\right)\left(1+5\mu\right)-\left(\alpha+\beta\right)\left(\lambda+\mu+4\lambda\mu\right)\right]\left(c-a-b-3\right)+$$

$$\left.+\left(a+1\right)_{2}\left(b+1\right)_{2}\left[\mu\left(1+\beta\right)-\lambda\mu\left(\alpha+\beta\right)\right]+\frac{\left(1-\alpha\right)}{ab}\left(c-a-b-3\right)_{3}\right\}-\left(1-\alpha\right)\frac{c}{ab}.$$

Hence (25) is equivalent to

$$\frac{\Gamma\left(c+1\right)\Gamma\left(c-a-b-3\right)}{\Gamma\left(c-a\right)\Gamma\left(c-b\right)}\times$$

$$\times\{[(1+\beta)(3+4\mu)-(\alpha+\beta)(1+2\lambda+2\mu+2\lambda\mu)](c-a-b-3)_2+$$
$$+(a+1)(b+1)[(1+\beta)(1+5\mu)-(\alpha+\beta)(\lambda+\mu+4\lambda\mu)]+$$
$$+(a+1)_2(b+1)_2[\mu(1+\beta)-\lambda\mu(\alpha+\beta)]+\frac{(1-\alpha)}{ab}(c-a-b-3)_3\}$$
$$\leq(1-\alpha)\frac{c}{ab}-(1-\alpha)\frac{c}{ab}=0. \tag{26}$$

Thus, from (26), we have

$$(a+1)_2(b+1)_2[\mu(1+\beta)-\lambda\mu(\alpha+\beta)]+$$
$$+(a+1)(b+1)[(1+\beta)(1+5\mu)-(\alpha+\beta)(\lambda+\mu+4\lambda\mu)](c-a-b-3)+$$
$$+[(1+\beta)(3+4\mu)-(\alpha+\beta)(1+2\lambda+2\mu+2\lambda\mu)](c-a-b-3)_2+$$
$$+\frac{(1-\alpha)}{ab}(c-a-b-3)_3\leq 0,$$

or, equivalently,

$$(a)_3(b)_3[\mu(1+\beta)-\lambda\mu(\alpha+\beta)]$$
$$+(a)_2(b)_2[(1+\beta)(1+5\mu)-(\alpha+\beta)(\lambda+\mu+4\lambda\mu)](c-a-b-3)+$$
$$+ab[(1+\beta)(3+4\mu)-(\alpha+\beta)(1+2\lambda+2\mu+2\lambda\mu)](c-a-b-3)_2+$$
$$+(1-\alpha)(c-a-b-3)_3\geq 0.$$

This completes the proof of Theorem 4. ◄

Putting $\lambda=\alpha=0$ in Theorem 4, we correct the result obtained by Ramachandran et al. [10, Theorem 4, with $p=2$ and $q=1$].

Corollary 3. *Let $a,b>0$ and $c>a+b+3$. Then the sufficient condition for $h_\mu(a,b;c;z)$ to be in the class $UCT(\beta)$ is*

$$\mu(1+\beta)\frac{(a+1)_2(b+1)_2}{(c+1)_2}\,{}_2F_1(a+3,b+3;c+3;1)+$$
$$+(4\mu\beta+5\mu+\beta+1)\frac{(a+1)(b+1)}{(c+1)}\,{}_2F_1(a+2,b+2;c+2;1)+$$
$$+(2\mu\beta+4\mu+2\beta+3)\,{}_2F_1(a+1,b+1,c+1;1)+\frac{c}{ab}F_1(a,b;c;1)\leq 0.$$

Putting $\lambda=\mu=\beta=0$ in Theorem 4, we obtain the following corollary which corrects the result obtained by Silverman [13, Theorem 4].

Corollary 4. *Let $a, b > -1$, $ab < 0$ and $c > a + b + 2$. Then the necessary and sufficient condition for $g(a, b; c; z)$ to be in the class $C(\alpha)$ is*

$$(a)_2 (b)_2 + (3-\alpha)(c-a-b-2)\, ab + (1-\alpha)(c-a-b-2)_2 \geq 0.$$

Putting $\lambda = \mu = 0$ and $\beta = 1$ in Theorem 4, we obtain the following corollary which corrects the result obtained by Cho et al. [5,Theorem 4].

Corollary 5. *Let $a, b > -1$, $ab < 0$ and $c > a + b + 2$. Then the necessary and sufficient condition for $g(a, b; c; z)$ to be in the class $UCT(\alpha)$ is that*

$$2(a)_2 (b)_2 + (5-\alpha)(c-a-b-2) + (1-\alpha)(c-a-b-2)_2 \geq 0.$$

Putting $\lambda = \mu = 0$ in Theorem 4, we obtain the following corollary which corrects the result obtained by Swaminathan [16, Theorem 4].

Corollary 6. *Let $a, b > -1$, $ab < 0$ and $c > a + b + 2$. Then the necessary and sufficient condition for $g(a, b; c; z)$ to be in the class $UCT(\alpha, \beta)$ is*

$$(1+\beta)(a)_2 (b)_2 + (3+2\beta-\alpha)\, ab\, (c-a-b-2) + (1-\alpha)(c-a-b-2)_2 \geq 0.$$

Theorem 5. *If the inequality*

$$\frac{6(a)_3 (b)_3}{(c)_3 (k+1)_3} \left[\frac{\mu(1+\beta) - \lambda\mu(\alpha+\beta)}{(1-\alpha)}\right]_3 F_2(a+3, b+3, 4; c+3, k+4; 1) +$$
$$\frac{2(a)_2 (b)_2}{(c)_2 (k+1)_2} \left[\frac{(1+5\mu)(1+\beta) - (\lambda+\mu+4\lambda\mu)(\alpha+\beta)}{(1-\alpha)}\right] {}_3F_2(a+2, b+2, 3;$$
$$c+2, k+3; 1) + \frac{ab}{c(k+1)} \left[\frac{(3+4\mu)(1+\beta) - (1+2\lambda+2\mu+2\lambda\mu)(\alpha+\beta)}{(1-\alpha)}\right].$$
$${}_3F_2(a+1, b+1, 2; c+1, k+2; 1) + {}_3F_2(a, b, 1; c, k+1; 1) \leq 2\,, \tag{27}$$

holds, then $\left[\mathrm{N}_{a,b,c}^{\mu,k}(f)\right](z)$ maps the class S (or S^*) to the class $S(\lambda, \alpha, \beta)$.

Proof. Since

$$\left[\mathrm{N}_{a,b,c}^{\mu,k}(f)\right](z) = z + \sum_{n=2}^{\infty} [1+\mu(n-1)] \frac{(a)_{n-1} (b)_{n-1}}{(c)_{n-1} (k+1)_{n-1}} a_n z^n\,, \tag{28}$$

according to Lemma 1, we only need to prove that

$$\sum_{n=2}^{\infty} \{n(1+\beta) - (\alpha+\beta)[1+\lambda(n-1)]\} [1+\mu(n-1)] \times$$

$$\times\frac{(a)_{n-1}\,(b)_{n-1}}{(c)_{n-1}\,(k+1)_{n-1}}\,|a_n|\le 1-\alpha\ . \tag{29}$$

The left-hand side of (29), by the fact that $|a_n|\le n$ for $f(z)\in S(\text{or } S^*)$ (see [5]), is less than or equal to

$$\sum_{n=2}^{\infty} n\left\{n(1+\beta)-(\alpha+\beta)\left[1+\lambda(n-1)\right]\right\}\left[1+\mu(n-1)\right]\frac{(a)_{n-1}\,(b)_{n-1}}{(c)_{n-1}\,(k+1)_{n-1}}=T_1\ .$$

Thus

$$T_1=\left[\mu(1+\beta)-\lambda\mu(\alpha+\beta)\right]\textstyle\sum_{n=4}^{\infty}\frac{(a)_{n-1}(b)_{n-1}(1)_{n-1}}{(c)_{n-1}(k+1)_{n-1}(1)_{n-4}}+$$
$$\left[(1+5\mu)(1+\beta)-(\lambda+\mu+\lambda\mu)(\alpha+\beta)\right]\textstyle\sum_{n=3}^{\infty}\frac{(a)_{n-1}(b)_{n-1}(1)_{n-1}}{(c)_{n-1}(k+1)_{n-1}(1)_{n-3}}+$$
$$\left[(3+4\mu)(1+\beta)-(1+2\lambda+2\mu+2\lambda\mu)(\alpha+\beta)\right]\textstyle\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}(1)_{n-1}}{(c)_{n-1}(k+1)_{n-1}(1)_{n-2}}+$$
$$(1-\alpha)\textstyle\sum_{n=2}^{\infty}\frac{(a)_{n-1}(b)_{n-1}(1)_{n-1}}{(c)_{n-1}(k+1)_{n-1}(1)_{n-1}}=$$

$$=6\left[\mu(1+\beta)-\lambda\mu(\alpha+\beta)\right]\frac{(a)_3\,(b)_3}{(c)_3\,(k+1)_3}\sum_{n=0}^{\infty}\frac{(a+3)_n\,(b+3)_n\,(4)_n}{(c+3)_n\,(k+4)_n\,(1)_n}+$$

$$+2\left[(1+5\mu)(1+\beta)-(\lambda+\mu+\lambda\mu)(\alpha+\beta)\right]\frac{(a)_2\,(b)_2}{(c)_2\,(k+1)_2}\times$$

$$\times\sum_{n=0}^{\infty}\frac{(a+2)_n\,(b+2)_n\,(3)_n}{(c+2)_n\,(k+3)_n\,(1)_n}+$$

$$+\left[(3+4\mu)(1+\beta)-(1+2\lambda+2\mu+2\lambda\mu)(\alpha+\beta)\right]\frac{ab}{c\,(k+1)}\times$$

$$\times\sum_{n=0}^{\infty}\frac{(a+1)_n\,(b+1)_n\,(2)_n}{(c+1)_n\,(k+2)_n\,(1)_n}+$$

$$+(1-\alpha)\left[\sum_{n=0}^{\infty}\frac{(a)_n\,(b)_n\,(1)_n}{(c)_n\,(k+1)_n\,(1)_n}-1\right].$$

From the last expression we get the assertion of the theorem. This completes the proof of Theorem 5. ◄

Using the same arguments as in the proof of the above theorems, we obtain the following theorem.

Theorem 6. *If the inequality*

$$24\mu\left(\frac{1+\beta}{1-\alpha}\right)\left(\frac{(a)_4\,(b)_4}{(c)_4\,(k+1)_4}\right)_3 F_2\,(a+4,b+4,5;c+4,k+5;1)+$$

$$+\frac{6\,(a)_3\,(b)_3}{(c)_3\,(k+1)_3}\left[\frac{(1+9\mu)\,(1+\beta)-(\lambda+\mu+\lambda\mu)\,(\alpha+\beta)}{(1-\alpha)}\right].$$

$${}_3F_2\,(a+3,b+3,4;c+3,k+4;1)+$$

$$+\frac{2\,(a)_2\,(b)_2}{(c)_2\,(k+1)_2}\left[\frac{(6+19\mu)\,(1+\beta)-(1+5\lambda+5\mu+4\lambda\mu)\,(\alpha+\beta)}{(1-\alpha)}\right]$$

$${}_3F_2\,(a+2,b+2,3;c+2,k+3;1)+$$

$$+\frac{ab}{c\,(k+1)}\left[\frac{(7+8\mu)\,(1+\beta)-(3+4\lambda+4\mu+2\lambda\mu)\,(\alpha+\beta)}{(1-\alpha)}\right].$$

$${}_3F_2\,(a+1,b+1,2;c+1,k+2;1)+{}_3F_2\,(a,b,1;c,k+1;1)\leq 2, \tag{30}$$

is true , then $\left[N_{a,b,c}^{\mu,k}\,(f)\right](z)$ *maps the class* S *(or* S^* *) to the class* $C\,(\lambda,\alpha,\beta)$.

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