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On the Maximal Regularity Property for Evolution Equations

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Abstract. This article is a survey of recent results concerning nonautonomous first order operator-differential equations. We briefly describe the autonomous case as well and expose the main results related to the maximal regularity property which can be called fundamentals of the general theory. The main attention is paid to the Cauchy problem.

Key Words and Phrases: maximal regularity, analytic semigroup, Sobolev space, evolution equation, Cauchy problem.

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1. Introduction

Let $\{A(t)\}_{t\in J=[0,T]}$ $(0 < T \leq \infty)$ be a family of closed linear operators in a Banach space X. We consider the Cauchy problem

$$L(t)u = u_t - A(t)u - B(t)u = f, \tag{1}$$

$$u(0) = u_0, \tag{2}$$

where the family of operators $B(t) : X \to X$, $t \in [0,T]$ is subordinate in a certain sense to the family $A(t)$ and $f : J \to X$ is a given function. Assume first that the operators $\{A(t)\}$ are independent of t and $u_0 = 0$, i.e., we deal with the autonomous problem

$$Lu = u_t - Au = f(t), \quad u(0) = 0. \tag{3}$$

By definition, this problem has the property of maximal L_p-regularity, if, for every $f \in L_p(J;X)$, there exists a unique function u satisfying (3) almost everywhere and such that $u_t, Au \in L_p(J;X)$. In other words, the maximal L_p-regularity means that each summand in (3) is well-defined and has the same regularity as the right-hand side. If the problem (3) has the property of maximal regularity, then we say that $A \in M_p(X)$. Similarly we can define the maximal regularity property, for instance, in the Hölder space $C^\alpha(J;X)$. It is not difficult to show that the L_p-maximal regularity for a closed densely defined operator A implies that A is a generator of a bounded analytic semigroup in X (see, for example, Coulhon and Lamberton [22], Dore [26], Hieber and Prüss [41] or Prüss [59, Sect. 10]). The maximal L_p-regularity provides the following important estimate

$$\|u_t\|_{L_p(J;X)} + \|u\|_{L_p(J;X)} + \|Au\|_{L_p(J,X)} \leq c\|f\|_{L_p(J;X)}, \tag{4}$$

which is a consequence of the closed graph theorem. The maximal regularity is essential when we study nonlinear problems (see [7, 47]), inverse and control problems [58]. Note that the Cauchy problem has been studied in numerous articles and monographs (see [49, 52, 57, 28, 56, 74, 35, 11, 32, 48]). The main results there rely on the classical semigroup theory and some other classical approaches which are not discussed here. In the present article the main attention is paid to the maximal regularity property.

The article is structured as follows. The next section contains some definitions. Section 3 is dedicated to some results in the autonomous case. The last section provides some basic results in the most general nonautonomous case. We consider only parabolic problems, i.e., the operator $A(t)$ for each t is a generator of an analytic semigroup.

2. Preliminaries

Let X, Y be Banach spaces. The symbol $L(X,Y)$ stands for the space of linear continuous operators defined on X with values in Y. If $X = Y$ then we use the notation $L(X)$. Let $A : X \to X$ be a closed linear operator in X with a dense domain $D(A)$. The symbol $R(A)$ stands for the range of A. Denote by $\sigma(A)$, $\rho(A)$ the spectrum and the resolvent set of A, respectively. Let $\mathbb{C}^- = \{z \in \mathbb{C} : \ Re\, z < 0\}$ ($C^+ = \{z \in \mathbb{C} : \ Re\, z > 0\}$) and $\Sigma_\theta = \{z \in \mathbb{C} : \ |\arg z| < \theta\}$. Given a measurable function $\varphi(t)$ positive almost everywhere, define the space $L_{p,\varphi(t)}(0,T;X)$ (X is a Banach space) as the space of strongly measurable functions, defined on $[0,T]$ with values in X such that $\int_0^T \varphi^p(t)\|u(t)\|_X^p dt < \infty$. This space for $\varphi(t) \equiv 1$ is denoted as $L_p(0,T;H)$. We use also the Sobolev spaces $W_p^s(0,T;X)$ (see the definition, for instance, in [38, 70]). The space of bounded continuous functions defined on $[0,T]$ with values in X is denoted by $C([0,T];X)$ and, the corresponding space of m-times continuously differentiable functions is denoted by $C^m([0,T];X)$. The latter space is endowed with the norm $\|u(t)\|_{C^m([0,T];X)} = \sum_{i=0}^m \|u^{(i)}\|_{C([0,T])}$. The Hölder space $C^\alpha([0,T];X)$ (see the definitions in [47, Subsect. 0.1, 0.2]) is endowed with the norm $\|u\|_{C^\alpha([0,T];X)} = \sup_{t,s\in[0,T],t\neq s} \frac{\|u^{(k)}(t)-u^{(k)}(s)\|}{|t-s|^{\alpha-k}} + \|u(t)\|_{C^k([0,T];X)}$ $(k=[\alpha])$. An operator A is called sectorial if

$$\overline{D(A)} = X,\ \overline{R(A)} = X,\ (-\infty,0) \subset \rho(A),\quad \|t(t+A)^{-1}\|_{L(X)} < M\ \forall t > 0,$$

where $M > 0$ is some constant. The class of sectorial operators in X is denoted by $S(X)$. It makes sense to define the spectral angle of $A \in S(X)$ as follows: $\varphi_A = \inf\{\theta :\ \rho(-A) \supset \Sigma_{\pi-\theta},\ \sup_{\lambda\in\Sigma_{\pi-\varphi}} \|\lambda(A+\lambda)^{-1}\|_{L(X)} < \infty\}$. Let $A \in S(X)$. Put $H_k = D(A^k)$ (the latter space is endowed with the graph norm). We can also define the spaces H_k for $k < 0$ (see [37, Sect. 5], more general definitions in the case of $0 \notin \rho(A)$ can be found in [40]). The space H_k can be defined as the completion of X with respect to the norm $\|u\|_{H_k} = \|(A-\lambda I)^{-k}u\|$, where $\lambda \in \rho(A)$. This definition does not agree with that in [37, Sect. 5] but the spaces obtained are the same. In the reflexive case the space H_k $(k < 0)$ coincides with the dual to $D((L^*)^{-k})$ and the norm can be defined by the equality

$$\|u\|_{H_k} = \sup_{v\in D((L^*)^{-k})} \frac{|< u, v >|}{\|v\|_{D((L^*)^{-k})}},$$

where the brackets $< \cdot,\cdot >$ denote the duality relation between X and the dual space X^*. By the real interpolation method (see [69, 40]) we can construct $B^s_q = (H_m, H_k)_{\theta,q}$, with $1 \le q \le \infty$, $k < s < m$, and $\theta = \frac{m-s}{m+k}$.

For convenience, we present some properties of these spaces. Assume for simplicity that $0 \in \rho(A)$.

Lemma 1. *The definition of the spaces B^s_q is independent of m,k. The space H_k for $k > s$ is dense in B^s_q and H_l for $l < k$. Moreover,*

$$(B^{s_0}_{q_0}, B^{s_1}_{q_1})_{\theta,q} = B^s_q,\quad (B^{s_0}_{q_0}, H_{s_1})_{\theta,q} = B^s_q, \tag{5}$$

where $\theta \in (0,1)$, $s = (1-\theta)s_0 + \theta s_1$, and $\frac{1}{q} = \frac{1-\theta}{q_0} + \frac{\theta}{q_1}$ $(1 \le q_i \le \infty,\ \ i = 0,\ 1)$. The operator A is isomorphism of B^s_q onto B^{s-1}_q and is sectorial in H_k and B^s_q with the domain H_{k+1} and B^{s+1}_q, respectively (see [37, Sect. 5], Sect. 1.14, Sect. 1.15.4 in [69], Prop. 1 in [63]). *The norm in the space B^s_q is equivalent to the norm (see* [69, Subsect. 1.14.3]*)*

$$\|a\|_s = \|t^{\theta(k-m)-r}(A(A+t)^{-1})^l A^{m+r} a\|_{L_{q,t^{-1/p}}(0,\infty;X)},\ s = m+\theta(k-m),\ m < s < k,$$

where r, l, m, k are integers, $0 \leq r < \theta(k-m)$, $l > \theta(k-m) - r$.

Below we will use the operators A being the generators of analytic semigroups. For definiteness, in this case we assume below that $-A$ is sectorial with $\varphi_A < \pi/2$ (see Sect. 1 in [53]) and $0 \in \rho(A)$.

A Banach space X is called a UMD space (the other names are $\zeta - convex$ and HT-spaces) if the Hilbert transform $Pf = \lim\limits_{\varepsilon \to 0} \int\limits_{|t-y|>\varepsilon} \frac{f(t)}{t-y}\, dt$ extends to bounded operator on $L_p(R, X)$ for some (or equivalently, for each) $p \in (1, \infty)$. All subspaces and quotient spaces of $L_q(G, \mu)$ for $1 < q < \infty$ have the UMD property. We can say that Sobolev spaces, Hardy spaces and other well known spaces of analysis are UMD if they are reflexive.

A collection of operators $\tau \subset L(X, Y)$ (X, Y are Banach spaces) is called R-bounded if there exists a constant C_p such that (see [25])

$$\Big(\sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N \in \{-1,1\}} \Big\| \sum_{j=1}^{N} \varepsilon_j T_j x_j \Big\|^p \Big)^{\frac{1}{p}} \leq C_p \Big(\sum_{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N \in \{-1,1\}} \Big\| \sum_{j=1}^{N} \varepsilon_j x_j \Big\|^p \Big)^{\frac{1}{p}},$$

for all N, $T_1, T_2, \ldots, T_N \in \tau$ and $x_1, x_2, \ldots, x_N \in X$. The least constant C_p in this inequality is denoted by $R(\tau)$ and is called the R-bound of the family τ (see equivalent definitions in [53, 23, 24, 60]). Note that this definition is independent of p.

3. Autonomous case

This section is dedicated to the maximal regularity of solutions to the Cauchy problem.

For detailed information on regularity properties of a solution u to the problem (3) within the framework of continuous functions we refer to the monographs [75, 47]. First results on the maximal L_p-regularity property were obtained in the Hilbert space case by de Simon [66] who proved in 1964 that there is maximal L_p-regularity provided X is a Hilbert space. De Simon's proof uses Plancherel's theorem which is known to be valid only in the Hilbert space case. Shortly after this result it was Sobolevskii [67] who showed that the maximal regularity property is independent of p (see also Coulton and Lamberton [22], Cannarsa and Vespri [20]). The first most essential results were obtained in the articles by Grisvard R. [37]-[39] who established the maximal L_p-regularity property in real interpolation spaces. We state now some of his results and their generalizations. We assume for simplicity that $p \in (1, \infty)$. We consider the problem (3), where the homogeneous initial condition is replaced with the inhomogeneous one, i.e., we have the problem

$$Lu = u_t - Au = f(t), \tag{6}$$

$$u(0) = u_0, \tag{7}$$

where $A : X \to X$ is a generator of an analytic semigroup.

Theorem 1. ([37, Sect. 6], [63], [65]) *Let* $f \in L_p(0, T; B_p^s)$ $(s \in \mathbb{R})$ *and* $u_0 \in B_p^{s+1-1/p}$ *Then there exists a unique solution* $u(t) \in L_p(0, T; X)$ *to the Cauchy problem* (6), (7) *such that* $u_t(t), Au(t) \in L_p(0, T; B_p^s)$*. This solution satisfies the estimate*

$$\|u\|_{W_q^1(0,T;B_q^s)} + \|Au\|_{L_q(0,T;B_q^s)} \le C(\|f\|_{L_q(0,T;B_q^s)} + \|u_0\|_{B_q^{s+1-1/q}}), \tag{8}$$

Denote by $\tilde{W}_p^s(0, T; X)$ $(s \ge 0)$ the subspace of the Besov space $B_{p,p}^s(0, T; X)$ comprising the functions $u(t)$ such that $\partial_t^k u(0) = 0$ for $k < s - 1/p$, $\partial_t^k u \in L_{p,t^{-1/p}}(0, T; X)$ for $s = k + 1/p$ (see [37]). For $s < 0$ the space $\tilde{W}_p^s(0, T; X)$ can be defined as the completion of $L_p(0, T)$ with respect to the norm $\|(\partial_t - \lambda I)^{-l} u\|_{\tilde{W}_p^{s+l}(0,T;X)}$ $(s + l > 0)$. The other definitions use duality arguments. The following theorem follows from the results obtained in [39, Sect. 6].

Theorem 2. ([39, Sect. 6]) *Let* $f \in \tilde{W}_p^s(0, T; X)$ $(s \in \mathbb{R})$ *and* $u_0 = 0$*. Then there exists a unique solution* $u(t) \in L_p(0, T; X)$ *of the Cauchy problem* (6), (7) *such that* $u_t(t), Au(t) \in \tilde{W}_p^s(0, T; X)$.

Consider an equation with parameter

$$u_t - Au + \gamma u = f, \quad u|_{t=0} = 0, \quad \gamma > 0. \tag{9}$$

The following theorem is useful when we treat nonlinear problems. It is a consequence of Theorem 1.

Theorem 3. *Let* $f \in L_q(0, T; B_q^s)$*,* $q \in (1, \infty)$*. Then the solution* $u(t)$ *to the Cauchy problem* (9) *satisfies the estimate*

$$\|u\|_{W_q^1(0,T;B_q^s)} + \|Au\|_{L_q(0,T;B_q^s)} + \gamma\|u\|_{L_q(0,T;B_q^s)} \le C\|f\|_{L_q(0,T;B_q^s)}, \tag{10}$$

where the constant C *is independent of* γ.

The assertion of this theorem follows from the fact that the above estimate (8) for a solution involves the estimates for the resolvent of the operator ∂_t which are the same as those for the resolvent of $\partial_t + \gamma$ (see, for instance, the proof of Theorem 2.7 in [38]).

Next, we should note the article by Da Prato and Grisvard (1975) [55], where essential results were obtained in the spaces defined by real interpolation method even in the nonautonomous case but under rather stringent constraints on the operators involved.

By means of the sum method, Dore and Venni [26, 27] prove the maximal L_p-regularity provided X is a UMD Banach space and A admits bounded imaginary powers A^{is} with power angle $\theta_A < \pi/2$. Note that this property of the operator A allows us to describe the domain of fractional powers of A (see [69, Subsect. 1.15.3]). Prüss and Sohr slightly improved Dore and Venni's results in [61]. One more important result was obtained by Lamberton [46] who proved the maximal L_p-regularity for $X = L_q(G)$, $1 < p, q < \infty$, provided A generates a bounded analytic C_0-semigroup which acts on all spaces $L_r(G)$ as a contraction, $1 < r < \infty$. The special case $X = L_q(G)$ was also considered by Hieber and Prüss [41]. In this article, the maximal L_p-regularity is shown in the case where the semigroup e^{At} generated by A admits a suitable heat kernel bound. The proof relies on Calderon-Zygmund theory, interpolation, and the theorem of Benedek, Calderon and Panzone. Kalton and Lancien [50] provided a fairly complete description of the spaces X in which the maximal regularity property holds for every generator of an analytic semigroup. They demonstrated that this property, up to an isomorphism, characterizes Hilbert spaces among spaces with an unconditional basis or (more generally) separable Banach lattices. The complete answer to the question of maximal L_p-regularity was given recently by Weis [73]. He obtained a characterization of the class $M_p(X)$ in the case of UMD space X in terms of R-boundedness. It was shown that $A \in M_p(X)$ if and only if the set $\{i\rho(i\rho + A)^{-1} : \rho \in \mathbb{R}\}$ is R-bounded. The proof of this result depends heavily on a recent theorem on operator-valued Fourier multipliers valid in spaces of UMD class, also due to him. The notion of R-boundedness goes back to Bourgain [16] (see also [15, 21]). A further approach to maximal regularity is due to Kalton and Weis [51]. They extended the scalar H^∞-calculus to the case of R-bounded families of operators and obtained a new maximal regularity result.

We now describe in more details Weis's results and some their generalizations. We assume that A is a generator of analytic semigroup and

(A) a family $\tau = \{\lambda(A - \lambda I)^{-1} : \ \lambda \in \overline{\Sigma_{\theta_0}}\}$ is R-bounded for some $\theta_0 > \pi/2$.

Denote the R-bound of this family by M_A.

Theorem 4. *Let X be a UMD space and let the condition* (A) *hold. Then, for every $f \in L_q(0,T;X)$ $(q \in (1,\infty))$ and $u_0 \in B_q^{1-1/q}$, there exits a unique solution to the problem* (6), (7) *such that $u \in L_q(0,T;D(L))$, $u_t \in L_q(0,T;X)$ and the estimate*

$$\|u_t\|_{L_q(0,T;X)} + \|Au\|_{L_q(0,T;X)} \le C(\|f\|_{L_q(0,T;X)} + \|u_0\|_{B_q^{1-1/q}})$$

holds. The constant C depends on the constant M_A, X, and q and is bounded for *bounded constants M_A.*

Proof. We can refer, for instance to [60, Theorem 3.2], [23, Theorem 4.4, Theorem 3.19]. ◄

In the following theorems we replace the problem (6), (7) with the problem

$$u_t - Au + \gamma u = f, \quad \gamma > 0 \tag{11}$$

$$u|_{t=0} = 0, \tag{12}$$

where $\gamma > 0$ is a parameter and $A : X \to X$ is a generator of an analytic semigroup.

Theorem 5. *Let X be a UMD space and let the condition* (A) *hold. Then, for every $f \in L_q(0,T;X)$ ($q \in (1,\infty)$), there exits a unique solution to the problem* (11), (12) *such that $u \in L_q(0,T;D(A))$, $u_t \in L_q(0,T;X)$ and the estimate*

$$\|u_t\|_{L_q(0,T;X)} + \|Au\|_{L_q(0,T;X)} + \gamma\|u\|_{L_q(0,T;X)} \leq C\|f\|_{L_q(0,T;X)} \tag{13}$$

holds, where the constant C is independent of γ. It depends on the constant M_A, q, and the space X.

Proof. We consider the operator $L - \gamma I$ instead of L and use the estimate in Theorem 4. In order to prove the assertion, we should estimate the quantity $R\{\lambda(L-\lambda-\gamma)^{-1},\ \lambda \in \Sigma_{\theta_0}\}$ by a constant independent of γ and use Theorem 4. First, we can say that $R\{i\xi(L-i\xi-\gamma)^{-1},\ \xi \in \mathbb{R}\} \leq R\{(i\xi+\gamma)(L-i\xi-\gamma)^{-1},\ \xi \in \mathbb{R}\} \leq 2M_L$ in view of Kahane's contraction principle (see Remark 2.3 in [25] and Lemma 3.5 in [23]) and the definition of R-boundedness. Next, we refer to the inequality $R\{\lambda(L-\lambda-\gamma)^{-1},\ Re\lambda \geq 0\} \leq R\{i\xi(L-i\xi-\gamma)^{-1},\ \xi \in \mathbb{R}\} \leq 2M_L$ whose proof is presented in Theorem 4.4 in [23]. The estimate for $R\{\lambda(L-\lambda-\gamma)^{-1},\ \lambda \in \Sigma_{\theta_0}\}$ for some $\theta_0 > \pi/2$ easily follows (see the proof of Theorem 4.4 in [23]). ◄

The above theorems have numerous applications. Moreover, there are a lot of close results. We present here some of them.

Theorem 6. [60, Theorem 3.2] *Let X be a UMD space, and let condition* (A) *be satisfied. Then for arbitrary functions $f \in L_{q,t^{1-\mu}}(0,T;X)$ ($T \leq \infty$, $q \in (1,\infty)$) and $u_0 \in B_q^{\mu-1/q}$ ($\mu \in (1/q,1]$, there exists a unique solution to the Cauchy problem* (6), (7) *such that $u \in L_{q,t^{1-\mu}}(0,T;D(L))$ and $u_t \in L_{q,t^{1-\mu}}(0,T;X)$.*

Theorem 7. [63, Theorem 5] *Let X be a UMD space and let the condition* (A) *hold. If $f \in L_q(0,T;B_q^s) \cap L_{q,t^{-s}}(0,T;X)$ ($s \leq 0, s \neq 1/q-1$) ($q \in (1,\infty)$) and $u_0 \in B_q^{s+1-1/q}$ ($1 < q < \infty$), then there exists a unique solution $u \in W_q^1(0,T;B_q^s) \cap L_q(0,T;B_q^{s+1})$ to the problem* (6), (7) *such that $u_t, Au \in L_{q,t^{-s}}(0,T;X)$.*

Some theorems similar to Theorem 7 can be found in [65].

Recent results and the bibliography dedicated to nonlocal problems can be found in [5, 6, 13, 18, 19, 33]. We present here the results by Uvarova [71, 72]. Consider the nonlocal conditions

$$u(0) = Ru + u_0 \tag{14}$$

$$\int_0^T u(\tau)\, d\sigma(\tau) = u_0, \tag{15}$$

where σ is a function of bounded variation.

Let $H = \{u \in L_{q,t^\delta}(0,T;D(A)) :\ u_t \in L_{q,t^{\delta_1}}(0,T;X), ut^{\delta_2} \in C([0,T];B_q^{1-1/q})\}$.

Theorem 8. [71, Theorem 8] *Assume that $R \in L(H, B_q^{1-1/q})$ with $\delta, \delta_2 > 0$, $q \in (1,\infty)$, and $\delta_1 > 1 - 1/q$, X is a UMD space, and the condition* (A) *holds. Then there exists a number γ_0 such that for every $f \in L_q(0,T;X)$, $u_0 \in L_q(0,T;B_q^{1-1/q})$, and $\gamma \geq \gamma_0$ the problem* (11), (14) *has a unique solution $u \in L_q(0,T;D(A)) \cap W_q^1(0,T;X)$.*

Introduce the function $\varphi(\lambda) = \int_0^T e^{\lambda\tau}\, d\sigma(\tau)$ and assume that

$$\exists \delta_0 > 0\ , \beta \leq 0 :\ |\varphi(\lambda)| \geq \delta_0(1+|\lambda|)^\beta\ \forall \lambda \in \mathbb{C} \setminus \Sigma_{\theta_1}, \tag{16}$$

where $\pi/2 < \theta_1 < \theta_0$ and θ_0 is a parameter in (A).

Theorem 9. [72, Theorem 5] *Let X be a UMD space and let the condition* (A) *hold. Assume also that $f \in L_{q,t^{1-\mu}}(0,T;X)$, $u_0 \in B_q^{\mu-1/q}$ $(1 < q < \infty$, $\mu \in (1/q, 1])$, and the condition* (16) *holds. If $\beta < 0$, then there exists a unique solution $u \in L_{q,t^{1-\mu}}(0,T;B_q^{\beta+1})$ with $u_t \in L_{q,t^{1-\mu}}(0,T;B_q^\beta)$ to the problem* (6), (15). *If $\beta = 0$, then there exists a unique solution $u \in L_{q,t^{1-\mu}}(0,T;D(A))$ with $u_t \in L_{q,t^{1-\mu}}(0,T;X)$ to the problem* (6), (15).

Remark 1. *The condition* (16) *in* [72] *is written in a different form. But it is easy to see that this correction does not influence on the proof and the assertion of Theorem 5 in* [72] *remains valid.*

4. Nonautonomous case

As we have already noted, the first essential results concerning nonautonomous case were obtained by G. Da Prato and P. Grisvard in [55]. They prove the maximal L_p-regularity property in the spaces constructed by the real interpolation method under rather stringent conditions on the resolvent of $A(t)$. In particular, its differentiability with respect to t is required. Weaker conditions were later used in the articles by Acquistapace P. and Terreni B. [2, 3, 1, 4]) in which the maximal regularity property is proven in the spaces of continuous or Hölder continuous functions. Note that the monograph [47] also contains some results of this type.

The main assumptions on the operator family $A(t)$ in these articles (see, for instance, [2, 3]) are the so-called Acquistapace-Terreni conditions related to the behavior of the resolvent and the Hölder continuity of the family $\{A(t)\}$. In particular, a survey of these and some other results is presented in [1]. Further developments of this method can be found in [42, 43, 34]. Some results are also presented in [68, Sect. 6.8]. Let us state the Asquispace-Terreni condition and some of their results (see [2]. Note that it is not assumed in [2] that $D(A(t))$ is dense in X.

The condition (I). The operator $A(t)$ is a generator of an analytic semigroup for every $t \in [0,T]$ $(T < \infty)$, $\Sigma_{\theta_0} \subset \rho(A(t))$ for some $\theta_0 > \pi/2$ and the resolvent $R_{A(t)}(\lambda) = (A(t) - \lambda)^{-1}$ of A satisfies the estimate $\|R_\lambda(A(t))\|_{L(X)} < M/(1 + |\lambda|)$ $\forall \lambda \in \overline{\Sigma_{\theta_0}}$, $\forall t \in [0,T]$.

The condition (II). There exist $B > 0$, α_i, β_i, $k > 0$ such that $0 \le \beta_i < \alpha_i \le 2$ and

$$\|A(t)R_{A(t)}(\lambda)(A^{-1}(t) - A^{-1}(s))\|_{L(X)} \le$$
$$\le B \sum_{i=1}^{k} (t-s)^{\alpha_i} |\lambda|^{\beta_i - 1}, \ \forall \lambda \in \overline{\Sigma_{\theta_0}} \setminus \{0\}, \forall 0 \le s \le t,$$

where $\delta = \min_i(\alpha_i - \beta_i) \in (0,1)$.

The condition (II) called the Asquispace-Terreni condition is often met in literature. It allows to state a lot of existence results for the problem (1), (2). Below we state one of the results of this type (see [2, Theorem 6.1]). Denote by $Z(0,\beta)$ the space of functions $u(t) \in C([0,T];X)$ such that $u \in C^\beta([a,T];X)$ for every $a \in (0,T)$.

Theorem 10. *Let X be a Banach space and let the conditions* (I), (II) *hold. Assume also that $B(t) = 0$ in (1), $f \in Z(0,\beta)$ $(\beta \in (0,\delta))$, $A(0)u_0 + f(0) \in \overline{D(A(0))}$. Then there exists a unique solution $u \in C([0,T];X)$ to the problem (1), (2) such that $u_t, A(t)u(t) \in Z(0,\beta)$ and $\sup_{t\in[a,T]} \|u(t)\|_{B^\beta_\infty} < \infty$ for every $a \in (0,T)$.*

Next, we present the corresponding results in the spaces L_p [34, Theorem 2.2].

Condition (III). X is a UMD space for some $\theta_0 > \pi/2$ and every $t \in [0,T]$ $(T < \infty)$, $\overline{\Sigma_{\theta_0}} \subset \rho(A(t))$ and the family $\tau = \{\lambda(-A(t) + \lambda I)^{-1} : \ \lambda \in \overline{\Sigma_{\theta_0}}\}$ $(\theta_0 > \pi/2)$ is R-bounded and $R(\tau) \le M$, where the constant M is independent of $t \in [0,T]$.

Theorem 11. *Let X be a Banach space and let the conditions* (III), (II) *hold. Assume also that $B(t) = 0$ in* (1), *$f \in L_q(0,T;X)$ $(T < \infty)$, $u_0 \in (D(A(0)), X)_{1/q,q}$. Then there exists a unique solution $u \in C([0,T];X)$ to the problem* (1), (2) *such that $u_t, A(t)u(t) \in L_q(0,T;X)$.*

Another series of results dedicated to the nonautonomous equations deals with the assumption that the domain $D(A(t))$ is independent of t. In this case the different maximal regularity results were obtained. The maximal L_p-regularity is studied, for instance, in [8, 9, 44, 62, 17, 10]. It is typical to require that the operator family $A(t)$ is continuous (belongs to the class $C([0,T];L(D,X))$ with $D = D(A(t))$) and the operator family $B(t)$ is subordinate in a certain sense to the family $A(t)$. We now state one of these theorems assuming this condition fulfilled. The next result is actually a consequence of those in [8, Theorem 7.1]. Introduce the space $H_q^{1,1}(0,T)$ of functions $u \in L_q(0,T;D)$ such that $u_t \in L_q(0,T;X)$. Endow this space with the norm

$$\|u\|^q_{H_q^{1,1}(0,T)} = \int_0^T \|u_t(t)\|^q_X + \|u(t)\|^q_D \, dt.$$

The space $H_q(a,T)$ $(a \in [0,T))$ consists of functions $u \in H_q^{1,1}(a,T)$ such that $u(t) = 0$ for $t < a$ in the case of $a > 0$ and $u(0) = 0$ for $a = 0$. The norm in this space coincide with that in $H_q^{1,1}(0,T)$. Define also the space $H_q(a,b)$ $(0 \le a < b < T)$ as the restriction of functions in $u \in H_q(a,T)$ to the segment $[0,b]$. We endow the space $H_q(a,b)$ with the norm $\inf \|\tilde{u}\|_{H_q(0,T)}$, where the infimum is taken over all extensions $\tilde{u}$ of $u \in H_q^{1,1}(a,b)$ to the whole segment $[0,T]$.

Condition (IV). $B(t) \in L_1(0,T;L(D,X))$ and there exists a continuous function $\beta(\xi) : [0,+\infty) \to \mathbb{R}^+$ such that $\beta(0) = 0$ and $\|B(t)u(t)\|_{L_q(a,b;X)} \le \beta(b - a)\|u(t)\|_{H_q(a,b)}$ for all $u \in H_q(a,b)$ and $0 \le a < b \le T$.

Theorem 12. *Let X be a Banach space and let the conditions* (III), (IV) *hold. Assume also that $f \in L_q(0,T;X)$ $(1 < q < \infty)$, $u_0 \in B_q^{1-1/q}$. Then there exists a unique solution $u \in L_q(0,T;X)$ to the problem* (1), (2) *such that $u_t, A(t)u(t) \in L_q(0,T;X)$.*

Concerning maximal regularity and existence theorems for the Cauchy problem, we of course should refer to the book [7, Ch.4], where the reader can find relevant results as well as the bibliography.

Next series of results is presented in the book [75, Ch. 3], where the maximal regularity property is studied with the use of Hölder continuity of the family $\{A(t)\}$. The main spaces are weighted Hölder spaces. Let us state the conditions on the family $A(t)$.

Condition (V). There exist constants $\mu, \nu \in (0,1])$, $N > 0$ such that $\mu+\nu > 1$,

$$\|A^{\nu}(t)(A^{-1}(t) - A^{-1}(s))\|_{L(X)} \le N|t-s|^{\mu},\ D(A(s)) \subset D(A^{\nu}(t)),\ \forall s,t \in [0,T].$$

Define the space $F^{\beta,\sigma}((0,T];X)$ $(0 < \beta < \sigma \le 1)$ as the space of functions $f(t)$ continuous on $(0,T]$ such that

$$\exists \lim_{t\to a} t^{1-\mu} f(t),\quad \sup_{a\le s<t\le T} \frac{(s-a)^{1-\beta+\sigma}\|f(t)-f(s)\|}{|t-s|^{\sigma}} < \infty,$$

$$\lim_{t\to a} \sup_{s\in(a,t)} \frac{(s-a)^{1-\beta+\sigma}\|f(t)-f(s)\|}{|t-s|^{\sigma}} = 0,$$

Theorem 13. *Let X be a Banach space and let the conditions* (I), (V) *hold. Assume also that $B(t) = 0$ in* (1), *$f \in F^{\beta,\sigma}((0,T];X)$ $(0 < \sigma < \min(\beta, \mu+\nu-1)$, $\beta \in (0,\delta))$, $u_0 \in D(A^{\beta}(0))$ $(0 < \beta \le 1)$. Then there exists a unique solution $u \in C([0,T];X)$ to the problem* (1), (2) *such that $A^{\beta}(t)u(t) \in C([0,T];X)$, $u_t, A(t)u(t) \in F^{\beta,\sigma}((0,T];X)$ and $\sup_{t\in[a,T]} \|u(t)\|_{B^{\beta}_{\infty}} < \infty$ for every $a \in (0,T)$.*

Note that the case of $B(t) \not\equiv 0$ is also considered in [75].

The Hilbert space results dedicated to the problem (1), (2) are often based on the Lax-Milgram theorem and the study of corresponding sesquilinear forms (see [45, 12, 54, 31]).

Next, we present author's recent results (see [64]). The approach of [64] is similar to that described in [7, Ch.4, Sect. 3] where the problem (1), (2) is reduced to an abstract initial-boundary value problem. This approach (see, for instance, [36]) is often used in the study of abstract boundary control problems (see [29, 30] and the references therein). Some recent results on control problems are described in the survey [14].

First, we assume that there exists a Banach spaces $D \subset X$ and Y and a family of linear operators $Q(t) : D \to Y$ such that

Condition (VI). $A(t) \in C([0,T]; L(D,X))$, $Q(t) \in C([0,T]; L(D,Y))$, the operators $A_t = A(t)|_{\ker Q(t)}) : X \to X$ are the generators of analytic semigroups for every $t \in [0,T]$;

Put $B^{s}_{q} = (D,X)_{1-s,q}$, $H^{s,r}_{q}(\alpha,\beta) = W^{s}_{q}(\alpha,\beta;X) \cap L_q(\alpha,\beta;B^{r}_{q})$.

Given a function $g(t)$, define the function $g_{\varepsilon}(t) = \begin{cases} g(t-\varepsilon),\ t \in [\varepsilon,T], \\ 0,\ t \in [0,\varepsilon). \end{cases}$
where $\varepsilon \in (0,T)$. Next, we impose some additional conditions on the mapping Q. We assume that there exists a Banach space $Z \subset L_q(0,T;Y)$ such that

Condition (VII). The mappings $Q : u(t) \to Q(t)u(t)$, $Q_{\tau} : u(t) \to Q(\tau)u(t)$ $(\tau \in [0,T])$ belong to the class $L(H^{1,1}_{q}(0,T), Z)$, the norms $\|Q_{\tau}\|_{L(H^{1,1}_{q}(0,T),Z)}$ are uniformly bounded and the mapping Q_{τ} is surjective for every $\tau \in [0,T]$;

Condition (VIII). For every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|(Q_{\tau_1} - Q_{\tau_2})u\|_Z \le \varepsilon \|u\|_{H_q^{1,1}(0,T)}$$

for all $u \in H_q(0,T)$ and $\tau_1, \tau_2 \in [0,T]$ such that $|\tau_2 - \tau_1| < \delta$;

$$\|(Q - Q_\tau)u\|_Z \le \varepsilon \|u\|_{H_q^{1,1}(0,T)}$$

for all $u \in H_q(\tau,T)$ and τ, b such that $\operatorname{supp} u \subset [\tau, b]$, $0 \le \tau < b \le T$, $b - \tau < \delta$;

$$((Q - Q_0)v)_{\varepsilon_0} \in Z, \ \|((Q - Q_0)u)_{\varepsilon_0}\|_Z \le \varepsilon \|u(t)\|_{H_q^{1,1}(0,T)}$$

for all $v \in H_q^{1,1}(0,T)$, some $\varepsilon_0 \in (0,T)$, and every $u \in H_q(0,T)$ such that $\operatorname{supp} u \subset [0,b]$ with $b < \delta$.

Next, we specify some additional function spaces and describe their properties. Let $g(t) \in Z$. Fix $\varepsilon \in (0,T)$ and define the space $Z_q(0,T)$ as the subspace of functions $g \in Z$ such that there exists $\varepsilon > 0$ such that $g_\varepsilon \in Z$. It is possible to show that if $g_\varepsilon \in Z$ for some $\varepsilon > 0$, then $g_\varepsilon \in Z$ for all $\varepsilon > 0$. So it is natural to fix $\varepsilon_0 > 0$ and introduce the norm $\|g(t)\|_{Z_q(0,T)} = \|g(t)\|_Z + \|g_{\varepsilon_0}(t)\|_Z$.

First, we consider an initial-boundary value problem. In addition to the initial condition (2) we consider the boundary condition

$$Q(t)u(t) = g(t). \tag{17}$$

Clearly, the problem (1), (2), (17) has no solutions for arbitrary g, u_0. So we have the natural consistency condition

$$g(t) - Qv(t) \in Z_q(0,T), \quad g(t) \in Z, \tag{18}$$

where $v(t) \in H_q^{1,1}(0,T)$ is an arbitrary function such that $v(0) = u_0$. We assume that

$$u_0 \in B_q^{1-1/q} = (D, X)_{1/q,q}. \tag{19}$$

In this case there exists a function $v \in H_q^{1,1}(0,T)$ such that $v(0) = u_0$ (Theorem 1.8.3 in [69]). Note that the condition (18) does not depend on this function v. Moreover, note that the condition (18) is equivalent to the condition

$$g(t) - Q_0 v(t) \in Z_q(0,T), \quad g(t) \in Z. \tag{20}$$

Theorem 14. *Assume that $f \in L_q(0,T;X)$ and the conditions* (18), (19), (IV), (VI)-(VIII) *together with the condition* (III), *where the operator $A(t)$ is replaced* *with A_t, hold. Then there exists a unique solution $u \in H_q^{1,1}(0,T)$ to the problem* (1), (2), (17). *This solution satisfies the estimate*

$$\|u\|_{H_q^{1,1}(0,T)} \leq c(\|g - Qv\|_{Z_q(0,T)} + \|u_0\|_{B_q^{1-1/q}} + \|v\|_{H_q^{1,1}(0,T)} + \|f\|_{L_q(0,T;X)}), \tag{21}$$

where the constant c is independent of g, u_0, and f, and $v \in H_q^{1,1}(0,T)$ is an arbitrary function such that $v(0) = u_0$.

As a consequence of Theorem 14 we have the following theorem.

Theorem 15. *Assume that $f \in L_q(0,T;X)$, $u_0 \in B_{0q}^{1-1/q} = (D(A_0),X)_{1/q,q}$, and the conditions* (IV), (VI)-(VIII), *and* (III) (*where the operator $A(t)$ is replaced with A_t*) *hold. Then there exists a unique solution $u \in H_q^{1,1}(0,T)$ to the problem* (1)-(2) *such that $u(t) \in D(A(t))$ for a.a. $t \in [0,T]$. This solution satisfies the estimate*

$$\|u(t)\|_{H_q^{1,1}(0,T)} \leq c(\|u_0\|_{B_{0q}^{1-1/q}} + \|f\|_{L_q(0,T;X)}), \tag{22}$$

where the constant c is independent of g, u_0.

Remark 2. *The results of Theorem* 14 *allows us to consider the classical parabolic problems of the form*

$$u_t - A(t,x,D)u = f(t,x), \quad x \in G \subset \mathbb{R}^n, \ t \in (0,T), \tag{23}$$

$$B_j(t,x,D)u = g_j(t,x) \ (j = 1,...,m), \ x \in \Gamma = \partial G, \ t \in (0,T), \tag{24}$$

$$u(0,x) = u_0(x), x \in G. \tag{25}$$

Here G is a bounded domain in $\mathbb{R}^n$ with boundary $\Gamma \in C^{2m}$, $A(t,x,D) = \sum_{|\alpha|\leq 2m} a_\alpha(x,t)D^\alpha u$, $B_j(t,x,D) = \sum_{|\alpha|\leq m_j} b_{j\alpha}(x,t)D^\alpha u(x,t)$, where a_α and $b_{j\alpha}$ are $L(X)$-valued variable coefficients, $m_j < 2m$, and X is a UMD space. The results of this type and the bibliography can be found in [24] *(see also* [23]*).*

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