Azerbaijan Journal of Mathematics
V. 9, No 1, 2019, January
ISSN 2218-6816

On Deficiency Indices of Singular Differential Operator of Odd Order in Degenerate Case

E.A. Nazirova, Y.T. Sultanaev*

Abstract. Asymptotic formulas for the fundamental system of solutions of a singular differential equation of the 5th order with complex-valued coefficients are obtained. Using the obtained formulas, the deficiency indices of the minimal differential operator generated by the corresponding differential expression are investigated.

Key Words and Phrases: spectral theory, deficiency indices, asymptotic behavior.

2010 Mathematics Subject Classifications: 47E05, 34E05, 34L05

1. Introduction

The following equation is considered:

$$(-1)^n 2iy^{(2n+1)} + \sum_{k=l}^{n} (-1)^k \left(p_k(x)y^{(k)}\right)^{(k)} +$$

$$+i\sum_{j=m}^{n-1} (-1)^j \left[\left(q_j(x)y^{(j)}\right)^{(j+1)} + \left(q_j(x)y^{(j+1)}\right)^{(j)}\right] = i\sigma y.$$

Note that the situation when the equation does not contain a term with the sought function $y(x)$ and its derivatives of lower orders is called degenerate. The asymptotics for the solutions of degenerate equations was studied earlier in [6, 9]. The asymptotics of solutions of a singular differential equation of odd order was studied in [7, 1, 8]. However, the degenerate case for the odd-order equation has not been yet considered.

The aim of this paper is to study the asymptotic behavior of a fundamental system of solutions of a fifth-order equation with complex coefficients, when the contribution of the coefficients to the asymptotic formulas are not the same.

---

*Corresponding author.

As the model case ($n=2$) we consider the equation

$$ly = 2iy^{(5)} + (p_2(x)y'')'' - i[(q_1(x)y')'' + (q_1(x)y'')'] - (p_1(x)y')' = i\sigma y, \qquad (1)$$

where $\sigma \neq 0$ and the coefficients $p_1(x)$, $p_2(x)$, $q_1(x)$ are twice continuously differentiable functions on $(0,\infty)$.

Consider the function $F(x,\sigma,\mu)$, which we will call the characteristic function for differential expression ly and characteristic equation:

$$F(x,\sigma,\mu) = 2i\mu^5 + p_2(x)\mu^4 - 2iq_1(x)\mu^3 - p_1(x)\mu^2 - i\sigma. \qquad (2)$$

Let the following conditions be satisfied:

1. $|p_1(x)| \to +\infty \ x \to \infty$,
2. $p_1'(x)$ does not change sign for sufficiently large $x \geq x_0$ and:
$$\frac{p_1'(x)}{p_1^{1/2}(x)} \to 0, \quad x \to \infty, \quad \left| \int_{x_0}^{\infty} \frac{p_1'^2(x)}{p_1^{3/2}(x)} dx \right| < +\infty,$$
3. for sufficiently large x: $|p_2'| \leq A_2 p_1' p_1$, $|q_1'| \leq A_1 p_1' p_1^{1/2}$, where $A_{1,2}$ are positive constants,
4. for sufficiently large x : $|p_1''| \leq B_0 p_1'^2 p_1^{-1}$ $p_2'' \leq B_2 p_1'^2$, $q_1'' \leq B_1 p_1'^2 p_1^{-1/2}$, where $B_{0,1,2}$ are positive constants,
5. for $i,j = \overline{3,5}$ there exist positive constants A, B, such that
$$A \leq \left| \frac{\mu_i(x,\sigma)}{\mu_j(x,\sigma)} \right| \leq B,$$
6. for sufficiently large x, $Re(\mu_j(x,\sigma) - \mu_i(x,\sigma))$ for $i,j = \overline{3,5}$, does not change sign.

Let's comment on the conditions (1-5). As for conditions (1,2), these are the restrictions on the growth of the function $p_1(x)$. For example, $p_1(x)$ may be the function $x^{\alpha} ln^{\beta} x, \quad 0 < \alpha < 2,\ \beta > 0$.

The conditions (3,4) are the conditions of regular growth (Titcmarsh-Levitan). This conditions mean that the coefficients of equation (1) cannot have oscillating growth at infinity. Condition (5) means that the roots of (2) $\mu_3(x,\sigma)$, $\mu_4(x,\sigma)$, $\mu_5(x,\sigma)$ have the same growth at infinity.

Obviously, if conditions (1-5) are satisfied, then the equation $F(x,\sigma,\mu) = 0$ has two decreasing roots at infinity $\mu_1(x,\sigma), \mu_2(x,\sigma)$ and three increasing ones $\mu_3(x,\sigma), \mu_4(x,\sigma), \mu_5(x,\sigma)$.

2. The transformation of equation (1)

Consider equation $F(x,\sigma,\mu)=0$. We note that using the Newton diagram method (see, for example, [3]), it is possible to investigate the asymptotic behavior of the roots of this equation in both the main term and the first corrections of the asymptotic expansion.

By virtue of condition (5) and Viet theorem, the following estimates are true:

$$B_1 \le \left|\frac{q_1}{p_1^{2/3}}\right| \le A_1,\ B_2 \le \left|\frac{p_2}{p_1^{1/3}}\right| \le A_2,$$

where $A_{1,2}$, $B_{1,2}$ are positive constants.

For simplicity, assume

$$\frac{q_1}{p_1^{2/3}} \sim a, \qquad \frac{p_2}{p_1^{1/3}} \sim b,$$

where $a,b\in\mathbb{R}$ are constants.

Let $\mu=\tau p_1^{-1/3}$ in (2) and denote $\varepsilon=|p_1|^{-5/6}$. Then

$$G(\tau) := 2i\tau^5 + b\tau^4 - 2ia\tau^3 - \tau^2 = i\sigma sign(p_1(x))\varepsilon^2. \tag{3}$$

Carrying out the analysis of the Newton diagram for the given equation, we obtain two series of roots as $x\to\infty$:

$$\tau_j = \pm\sqrt{\frac{|\sigma|}{2}}(1-(-1)^{sign(\sigma p_1)}i)\varepsilon + o(\varepsilon),\ \ j=1,2$$

$$\tau_l = k_l + o(1),\ \ l=3,4,5, \tag{4}$$

where k_l are different roots of the equation

$$2ik^3 + bk^2 - 2iak - 1 = 0. \tag{5}$$

Following the Newton diagram method, we obtain the first correction of our asymptotic expansions:

$$\tau_j = \pm\sqrt{\frac{|\sigma|}{2}}(1-(-1)^{sign(\sigma p_1)}i)\varepsilon - a\sigma\varepsilon^2 + o(\varepsilon^2),\ \ j=1,2,$$

$$\tau_l = k_l + s_l\varepsilon^2 + o(\varepsilon^2),\ \ l=3,4,5,$$

where

$$s_l = \frac{i\sigma}{G'(k_l)},\ \ l=3,4,5.$$

Then for $\mu_j(x,\sigma)$ we get:

$$\mu_j(x,\sigma) = \pm\sqrt{\frac{|\sigma|}{2}}(1-(-1)^{sign(\sigma p_1)}i)|p_1(x)|^{-1/2} - a\sigma p_1^{-4/3}(x) + o(p_1^{-4/3}(x)), \quad j=1,2,$$

$$\mu_l(x,\sigma) = k_l p_1^{1/3}(x) + s_l p_1^{-4/3}(x) + o(p_1^{-4/3}(x)), \quad l=3,4,5. \tag{6}$$

Consider colon vector $Y = col(y^{[0]},\dots,y^{[4]})$, where $y^{[i]}$ are quasi-derivatives of order i, defined by formulas

$$y^{[0]} = y, \; y^{[1]} = \frac{d}{dx}y^{[0]},$$

$$y^{[2]} = \sqrt{2}\frac{d}{dx}y^{[1]},$$

$$y^{[3]} = \sqrt{2}\frac{d}{dx}y^{[2]} - q_1(x)y^{[1]} - i\frac{p_2(x)}{\sqrt{2}}y^{[2]},$$

$$y^{[4]} = \frac{d}{dx}y^{[3]} + ip_1(x)y^{[1]} - \frac{q_1(x)}{\sqrt{2}}y^{[2]}.$$

The differential expression ly, defined by (1), will become

$$ly = \frac{d}{dx}y^{[4]} - i\sigma y$$

and equation (1) may be rewritten as a system:

$$Y' = A(x,\sigma)Y, \tag{7}$$

where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\sqrt{2}} & 0 & 0 \\ 0 & \frac{q_1}{\sqrt{2}} & i\frac{p_2}{2} & \frac{1}{\sqrt{2}} & 0 \\ 0 & -ip_1 & \frac{q_1}{\sqrt{2}} & 0 & 1 \\ \sigma & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Transform the matrix A to diagonal form. Find the eigenvectors and eigenvalues. We have to check that the eigenvalues of matrix A coincide with $\mu_j(x,\sigma)$, $j=\overline{1,5}$. Consider the equation for eigenvectors:

$$A\overline{t_i} = \mu_i\overline{t_i}, \quad i=\overline{1,5}.$$

Let the first component of eigenvector be equal to any scalar function $t_{i,1} = \alpha_i(x)$, $i=\overline{1,5}$. Then the remaining components of eigenvector will become:

$$t_{i,2} = \alpha_i\mu_i,$$

$$t_{i,3} = \alpha_i\sqrt{2}\mu_i^2,$$
$$t_{i,4} = \alpha_i\sqrt{2}(2\mu_i^3 - ip_2\mu_i^2 - q_1\mu_i),$$
$$t_{i,5} = \alpha_i(2\mu_i^4 - ip_2\mu_i^3 - 2q_1\mu_i^2 + ip_1\mu_i). \qquad (8)$$

Denote $C = T^{-1}T'$. Let's show that $\alpha_i(x)$ may be chosen so that the following conditions hold:

$$c_{ii} = 0, \qquad i = \overline{1,5}. \qquad (9)$$

To this end, find the inverse matrix $T^{-1}(x,\sigma) =: B(x,\sigma)$ from the equality

$$BA = \Lambda B,$$

where $\Lambda = diag\{\mu_1, \dots, \mu_5\}$. We get

$$\sigma b_{i,5} = \mu_i b_{i,1}$$

$$b_{i,1} + \frac{q_1}{\sqrt{2}} b_{i,3} - ip_1 b_{i,4} = \mu_i b_{i,2}$$

$$\frac{1}{\sqrt{2}} b_{i,2} + i\frac{p_2}{2} b_{i,3} + \frac{q_1}{\sqrt{2}} b_{i,4} = \mu_i b_{i,3}$$

$$\frac{1}{\sqrt{2}} b_{i,3} = \mu_i b_{i,4}$$

$$b_{i,4} = \mu_i b_{i,5}.$$

Put $b_{i,5}(x,\sigma) = \beta_i(x), \quad i = \overline{1,5}$. Then from (9) we obtain that every functions α_i satisfies the following differential equation (see more [2]):

$$\beta_i \left(\alpha_i \frac{d}{dx} \frac{dF(x,\sigma,\mu_i)}{d\mu} + 2\alpha_i' \frac{dF(x,\sigma,\mu_i)}{d\mu} \right) = 0.$$

Then

$$\alpha_i = \left[\frac{dF(x,\sigma,\mu_i)}{d\mu} \right]^{-1/2}. \qquad (10)$$

Make a substitution in (7):

$$Y = TU. \qquad (11)$$

We obtain

$$U' = \Lambda U - CU. \qquad (12)$$

Unfortunately, the system (12) is not L-diagonal [2]. Let

$$U = (I + G)Z, \qquad (13)$$

where I is an identity matrix and the matrix G has the following form:

$$G = \begin{pmatrix} G_1 & O \\ O & G_2 \end{pmatrix},$$

where G_1 and G_2 are the solutions of the following equations:

$$G_1\Lambda_1 - \Lambda_1 G_1 = C_1, \qquad (14)$$

$$G_2\Lambda_2 - \Lambda_2 G_2 = C_2. \qquad (15)$$

Then we arrive at the system

$$Z' = (I+G)^{-1}(\Lambda - C)(I+G)Z - (I+G)^{-1}G'Z.$$

Suppose the matrices Λ and C have the following forms:

$$\Lambda = \begin{pmatrix} \Lambda_1 & 0 \\ 0 & \Lambda_2 \end{pmatrix}; \qquad C = \begin{pmatrix} C_1 & 0 \\ 0 & C_2 \end{pmatrix} + C_3.$$

Substituting these representations into the system and considering the properties of G_1 and G_2, we obtain the system

$$Z' = \Lambda Z + \begin{pmatrix} (I+G_1)^{-1}C_1G_1 & O \\ O & (I+G_2)^{-1}C_2G_2 \end{pmatrix} Z +$$
$$+ \left((I+G)^{-1}C_3(I+G) - (I+G)^{-1}G'\right) Z. \qquad (16)$$

3. Asymptotic formulas for the fundamental system of solutions of equation (1)

Theorem 1. *The system (16) is L-diagonal.*

Proof. Consider the asymptotic behavior of the elements of matrices C_1, C_2, C_3:

$$c_{i,j} = \frac{\alpha_i \alpha_j}{\mu_i - \mu_j}\left[-p_2'(\mu_i\mu_j)^2 - iq_1'(\mu_i^2\mu_j + \mu_i\mu_j^2) + p_1'\mu_i\mu_j\right]. \qquad (17)$$

First we investigate the asymptotic behavior of the functions $\alpha_i(x,\sigma)$, $i = \overline{1,5}$:

$$\frac{\partial F(x,\sigma,\mu_i)}{\partial \mu} = 10\mu_i^4 - 4ip_2\mu_i^3 - 6q_1\mu_i^2 + 2ip_1\mu_i. \qquad (18)$$

Note that if $j = 1, 2$, then

$$\frac{\partial F(x,\sigma,\mu_j)}{\partial \mu} \sim 2ip_1\mu_j, \quad x \to +\infty.$$

In the case $j = 3, 4, 5$,

$$\frac{\partial F(x, \sigma, \mu_j)}{\partial \mu} \sim const p_1 \mu_j, \quad x \to +\infty.$$

Using (6) and (10), we obtain

$$\alpha_{1,2}(x, \sigma) \sim const p_1^{-1/4},$$

$$\alpha_{3,4,5}(x, \sigma) \sim const p_1^{-2/3}. \tag{19}$$

Using the obtained asymptotic formulas for α_i and μ_i and conditions (3,4), we get

$$c_{ij} \sim const \frac{p_1^{-2/4}}{p_1^{-1/2}} p_1' p_1^{-1} = const \frac{p_1'}{p_1}, \quad i, j = 1, 2, \tag{20}$$

$$c_{ij} \sim const \frac{p_1^{-4/3} p_1^{1/3} p_1'}{p_1^{2/3}} = const \frac{p_1'}{p_1}, \quad i, j = 3, 4, 5, \tag{21}$$

$$c_{ij} \sim const \frac{p_1^{-2/3} p_1^{-1/4} p_1^{1/3} p_1'}{p_1^{1/3} p_1^{-1/2}} = const \frac{p_1'}{p_1^{17/12}}, \quad i = 1, 2, \ \ j = 3, 4, 5. \tag{22}$$

Formulas (22) are also true for $i = 3, 4, 5, \ \ j = 1, 2$. So, as the function $p_1(x)$ is increasing for $x \to \infty$ the elements of matrix $C_{1,2}$ are not summable and following elements of matrix C_3 are summable on $[x_0, \infty)$. From the formulas (6,20,22) we get the following expression for the elements of matrix G for $i, j = 1, 2$, $x \to \infty$:

$$g_{ij} = \frac{(C_1)_{ij}(x, \sigma)}{\mu_i(x, \sigma) - \mu_j(x, \sigma)} \sim const \frac{p_1'(x)}{p_1(x)} \frac{1}{p_1^{-1/2}(x)} = const \frac{p_1'(x)}{p_1^{1/2}(x)} \to 0,$$

and for $i, j = 3, 4, 5$, $x \to \infty$:

$$g_{ij}(x) = \frac{(C_2)_{ij}(x, \sigma)}{\mu_i(x, \sigma) - \mu_j(x, \sigma)} \sim const \frac{p_1'(x)}{p_1(x)} \frac{1}{p_1^{1/3}(x)} =$$

$$= const \frac{p_1'(x)}{p_1^{2/3}(x)} = o\left(\frac{p_1'(x)}{p_1^{1/2}(x)}\right) \to 0.$$

Consider $\mu_i'(x, \sigma)$, $c_{ij}'(x, \sigma)$, $\alpha_i'(x, \sigma)$, $i, j = \overline{1, 5}$. From the identity $F(x, \sigma, \mu_i(x, \sigma)) = 0$ we have

$$\frac{\partial F}{\partial \mu} d\mu + \frac{\partial F}{\partial x} dx = 0.$$

We have μ_i':

$$\frac{d\mu_i(x,\sigma)}{dx} = -\frac{\partial F(x,\sigma,\mu_i)/\partial x}{\partial F(x,\sigma,\mu_i)/\partial \mu}. \tag{23}$$

Consider

$$\frac{\partial F(x,\sigma,\mu_i)}{\partial x} = -ip_2'\mu_i^4 - 2q_1'\mu_i^3 + ip_1'\mu_i^2.$$

Given the conditions 3) and (18) :

$$\mu_i' = \frac{-ip_2'\mu_i^4 - 2q_1'\mu_i^3 + ip_1'\mu_i^2}{10\mu_i^4 - 4ip_2\mu_i^3 - 6q_1\mu_i^2 + 2ip_1\mu_i} = \mu_i\frac{p_1'}{p_1}, \quad i = \overline{1,5}. \tag{24}$$

Consider the functions $\alpha_i'(x,\sigma)$. From (10) it follows

$$\alpha_i'(x,\sigma) = -1/2\left[\frac{\partial F(x,\sigma,\mu_i)}{\partial \mu}\right]^{-3/2}\frac{\partial}{\partial x}\frac{\partial}{\partial \mu}F(x,\sigma,\mu_i),$$

$$\frac{\partial}{\partial x}\frac{\partial}{\partial \mu}F(x,\sigma,\mu_i) = 2i\mu_i p_1'(-2\frac{p_2'}{p_1'}\mu_i^2 - 6\frac{q_1'}{p_1'}\mu_i + 1) = 2i\mu_i p_1'(1+o(1)),$$

therefore

$$\alpha_i'(x,\sigma) \sim -ip_1'\alpha_i^{-3}\mu_i, \quad i = \overline{1,5}. \tag{25}$$

Investigate the asymptotics of the function $(C_{1,2}')_{ij}$. Consider just the element c_{12}' (for others are treated similarly).

$$c_{12}' = \frac{\alpha_1'\alpha_2 + \alpha_1'\alpha_2}{\mu_1 - \mu_2}\left[-p_2'(\mu_1\mu_2)^2 - iq_1'(\mu_1^2\mu_2 + \mu_1\mu_2^2) + p_1'\mu_1\mu_2\right] +$$

$$+\frac{\alpha_1\alpha_2(\mu_1' - \mu_2')}{(\mu_1-\mu_2)^2}\left[-p_2'(\mu_1\mu_2)^2 - iq_1'(\mu_1^2\mu_2 + \mu_1\mu_2^2) + p_1'\mu_1\mu_2\right] +$$

$$+\frac{\alpha_1\alpha_2}{\mu_1 - \mu_2}\left[-p_2''(\mu_1\mu_2)^2 - iq_1''(\mu_1^2\mu_2 + \mu_1\mu_2^2) + p_1''\mu_1\mu_2\right] +$$

$$+\frac{\alpha_1\alpha_2}{\mu_1 - \mu_2}\left[-2p_2'(\mu_1^2\mu_1^2)' - iq_1'(\mu_1^2\mu_2 + \mu_1\mu_2^2)' + p_1'(\mu_1\mu_2)'\right].$$

Substituting the obtained asymptotics (6,19-22,24,25) in last formula, we get

$$c_{12}' = const\frac{p_1'^2}{p_1^2}(1+o(1)). \tag{26}$$

The similar formulas are true for $c_{ij}, \quad i,j = 1,2$, and $i,j = 3,4,5$.

$$g_{ij}'(x) = \left[\frac{(C_{1,2})_{ij}(x,\sigma)}{\mu_i(x,\sigma) - \mu_j(x,\sigma)}\right]' = \frac{c_{ij}'}{\mu_i - \mu_j} - \frac{c_{ij}(\mu_i' - \mu_j')}{(\mu_i - \mu_j)^2} =$$

$$= const \frac{p_1'^2}{p_1^2 p_1^{1/2}}.$$

It follows from condition 2) that $G' \in L[x_0, \infty)$. Also $(I + G_1)^{-1}C_1G_1$, $(I + G_2)^{-1}C_2G_2$, $(I+G)^{-1}C_3G$ and $(I+G_1)^{-1}G_1'$, $(I+G_2)^{-1}G_2'$ belongs to $L[x_0, \infty)$.

We rewrite the system (16) in the form

$$Z' = diag\{\mu_1, .., \mu_5\}Z + \Theta(x, \sigma)Z.$$

We make a substitution

$$Z = W exp \int_{x_0}^{x} \mu_i(t, \sigma)dt. \quad (27)$$

Then

$$W_k' = (\mu_k - \mu_i)W_k + \sum_{m=1}^{5} \Theta_{km} W_m, \quad k = \overline{1, 5}. \quad (28)$$

According to condition (6) and formulas (6), $Re(\mu_i - \mu_j)$ does not change sign for sufficiently large x for all i, j: $i \neq j$, $i, j = \overline{1, 5}$. All this means that for system (28) all the conditions of Lemma 1 of [2, p. 288-292], are satisfied. Applying this Lemma and getting back to the old variable Y, we obtain asymptotic formulas for a system of linearly independent solutions.

Thus, we have proved

Theorem 2. *Suppose that conditions (1-6) are satisfied. Then equation (1) has 5 linearly independent solutions $y_1(x, \sigma)$ such that the following asymptotic formulas hold as $x \to \infty$:*

$$y_j(x, \sigma) = \alpha_j(x, \sigma) exp \int_{x_0}^{x} \mu_i(t, \sigma)dt(1 + o(1)), \quad i = \overline{1, 5}. \quad (29)$$

To derive the asymptotic formulas for solutions $y_1(x, \sigma)$ and $y_2(x, \sigma)$, we used the method proposed in [4] and [5].

Asymptotic formulas (29) enable us in some cases to solve the problem of deficiency indices of the minimal differential operator L_0 generated in $L^2[x_0, \infty)$ by the differential expression (1).

4. On the deficiency indices of the minimal differential operator L_0, generated by the expression ly

Consider the following example. Let

$$p_1(x) = cx^{\alpha}, \quad p_2 = bx^{\alpha/3}, \quad q_1(x) = ax^{2\alpha/3}, \quad c > 0, \quad 0 < \alpha < 2.$$

We note that for such coefficients the conditions (1-5) are satisfied. We study the question of the deficiency indices of the corresponding minimal differential operator L_0.

We first consider a series of solutions (1) corresponding to decreasing roots of the characteristic equation (2). From (6), (19) and (29) we have

$$\mu_{1,2}(x,\sigma) = \pm\sqrt{\frac{|\sigma|}{2}}(1-(-1)^{sign(\sigma)}i)x^{-\alpha/2} + o(x^{-\alpha/2}),$$

$$\alpha_{1,2}(x,\sigma) \sim const x^{-\alpha/4},$$

$$y^2_{1,2}(x,\sigma) = const x^{-\alpha/2} exp \int_{x_0}^{x} \mu_{1,2}(t,\sigma)dt(1+o(1)).$$

The integral

$$\int_{x_0}^{\infty} (t^{-\alpha/2} + o(t^{-\alpha/2}))dt$$

in this case diverges ($\alpha < 2$). Note that for any sign of σ, the real part of the principal term of the asymptotics for $\mu_{1,2}$ is positive, while for the other, it is negative. Thus, one of the solutions $y_{1,2}$ belongs to $L^2[x_0,\infty)$, and the second does not.

We next consider a series of solutions of (1) corresponding to the growing roots of equation (2). From (6), (19) and (29) for $j = 3,4,5$:

$$\mu_j(x,\sigma) = k_j x^{\alpha/3}(x) + s_j x^{-4\alpha/3}(x) + o(x^{-4\alpha/3}),$$

$$\alpha_j(x,\sigma) \sim const x^{-2\alpha/3},$$

$$y^2_j(x,\sigma) = const x^{-4\alpha/3} exp \int_{x_0}^{x} \mu_j(t,\sigma)dt(1+o(1)).$$

We note that for $3/4 < \alpha < 2$ the integral

$$\int_{x_0}^{\infty} (t^{-4\alpha/3} + o(t^{-4\alpha/3}))dt$$

converges and the integral

$$\int_{x_0}^{\infty} t^{\alpha/3} dt$$

diverges.

Let us first consider the case $3/4 < \alpha < 2$. The constants k_j are determined from the equation (5):

$$2ik^3 + bk^2 - 2iak - 1 = 0.$$

Let's make the substitution $k = ir$:

$$2r^3 - br^2 + 2ar - 1 = 0. \tag{30}$$

The last equation is the one with real coefficients, which can have one real and two complex conjugate roots or three real ones (the case of multiple roots is chosen in view of condition (6)). First, let the equation (30) have three real roots $r_j,\ j = 3, 4, 5$. Then $k_j = ip_j$ and

$$y_j^2(x, \sigma) \sim constx^{-4\alpha/3} \in L[x_0, \infty), \quad j = 3, 4, 5,$$

so the deficiency indices of the operator L_0 in this case are **(4,4)**.

If the equation (30) has one real root r_3 and two complex conjugate roots $r_{4,5} = \delta \pm i\gamma$, then

$$k_3 = ir_3, \quad y_3^2(x, \sigma) \sim constx^{-4\alpha/3} \in L[x_0, \infty)$$

$$k_{4,5} = \mp\gamma + i\delta, \quad y_4^2(x, \sigma) \in L[x_0, \infty), \quad y_5^2(x, \sigma) \notin L[x_0, \infty).$$

Obviously, in this case the deficiency indices of the operator L_0 are **(3,3)**.

We proceed further to the second case, where $0 < \alpha \le 3/4$. In this case the integral

$$\int_{x_0}^{\infty} (t^{-4\alpha/3} + o(t^{-4\alpha/3}))dt$$

diverges. The defect indices of the operator L_0 will depend on the real parts of the constants $s_j,\ j = 3, 4, 5$. We recall the formula for $s_j,\ j = 3, 4, 5$:

$$s_j = \frac{i\sigma}{G'(k_j)}, \quad j = 3, 4, 5. \tag{31}$$

Suppose that the equation (30) has three real roots. Then the constants $k_j,\ j = 3, 4, 5$ are imaginary, and the s_j's, $j = 3, 4, 5$ are real. We note that from the formula (31) and the properties of the derivative of a polynomial at its zeros it follows that either two constants $s_j,\ j = 3, 4, 5$ are positive and one is negative, or one positive and two negative. This means that the defect indices of L_0 in this case are **(2,3)** or **(3,2)**.

Acknowledgements

This work was supported by RFBR grants N18-01-00250-A, N18-51-06002 Az-a.

This work was also supported by Ministry of Education and Science of the Republic of Kazakhstan, Project No. AP05133397.

References

[1] M.F. Fedoryuk, *Asymptotic methods for linear ODEs*, Moscow, Nauka, 1983.

[2] M.A. Naimark, *Linear differential operators*, Moscow, 1969.

[3] M.M. Vainberg, V.A. Trenogin, *The branching theory for solutions of nonlinear equations*, Moscow, 1969.

[4] M.S.P. Eastham, C.G.M. Grudniewicz, *Asymptotic theory and deficiency indices for higher-order self-adjoint differential equations*, Department of Mathematics, Chelsea College, University of London, Manresa Road, London SW3 6LX, pp.255-271.

[5] W.N. Everitt, A. Zettl, *Generalized symmetric ordinary differential expressions 1. The general theory* ‖, Niew archief voor wisKunde, **3**, 1979, 363-397.

[6] Ya.T. Sultanaev, *Asymptotics of solutions of ordinary differential equations in the degenerate case*, Trudy sem. I.G. Petrovskii, **3**, 1988, 36-55.

[7] V.R. Mukimov, Ya.T. Sultanaev, *The deficiency indices of a symmetric third-order differential operator*, Math. Notes, **58:3**, 1995, 468-471.

[8] E.A. Nazirova, *On Deficiency Indices of Singular Operators of Odd Order*, Math. Notes, **74:5**, 2003, 792-796.

[9] E.A. Nazirova, Ya.T. Sultanaev, *On the Deficiency Indices of Singular Differential Operators in the Degenerate Case*, Math. Notes, **71:1**, 2002, 131-135.

Elvira A. Nazirova
Bashkir State University, Russia, Ufa
*E-mail:*ellkid@gmail.com

Yaudat T. Sultanaev
Bashkir State Pedagogical University named after M. Akmulla, Russia, Ufa
E-mail: sultanaevyt@gmail.com

Received 27 May 2018
Accepted 03 August 2018