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Characterization of Parabolic Fractional Integral and Its Commutators in Parabolic Generalized Orlicz-Morrey Spaces

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Abstract. In this paper, we give necessary and sufficient condition for the Adams type boundedness of parabolic fractional integral and its commutators in parabolic generalized Orlicz-Morrey spaces.

Key Words and Phrases: parabolic generalized Orlicz-Morrey space, parabolic fractional integral, commutator, BMO.

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1. Introduction

The theory of boundedness of classical operators of real analysis, such as maximal operator, fractional maximal operator, Riesz potential, singular integral operator, etc., from one Lebesgue space to another has been well studied. These results have good applications in the theory of partial differential equations. However, in the theory of partial differential equations, along with Lebesgue spaces, Orlicz spaces also play an important role.

For $x \in \mathbb{R}^n$ and $r > 0$, we denote by $B(x,r)$ the open ball centered at x of radius r, and by ${}^{\complement}B(x,r)$ we denote its complement. Let $|B(x,r)|$ be the Lebesgue measure of the ball $B(x,r)$.

Let P be a real $n \times n$ matrix, all of its eigenvalues having positive real part. Let $A_t = t^P \quad (t > 0)$, and set $\gamma = trP$. Then, there exists a quasi-distance ρ associated with P such that

$$(a) \quad \rho(A_t x) = t\rho(x), \quad t > 0, \quad \text{for every} \quad x \in \mathbb{R}^n;$$

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(b) $\rho(0) = 0, \quad \rho(x-y) = \rho(y-x) \geq 0$
and $\quad \rho(x-y) \leq k(\rho(x-z) + \rho(y-z))$;

(c) $dx = \rho^{\gamma-1} d\sigma(w) d\rho, \quad$ where $\quad \rho = \rho(x), w = A_{\rho^{-1}} x$
and $d\sigma(w)$is a C^∞measure on the ellipsoid $\{w : \rho(w) = 1\}$.

Then, $\{\mathbb{R}^n, \rho, dx\}$ becomes a space of homogeneous type in the sense of Coifman-Weiss. Thus $\mathbb{R}^n$, endowed with the metric ρ, defines a homogeneous metric space ([4, 5]). The balls with respect to ρ, centered at x of radius r, are just the ellipsoids $\mathcal{E}(x,r) = \{y \in \mathbb{R}^n : \rho(x-y) < r\}$, with the Lebesgue measure $|\mathcal{E}(x,r)| = v_\rho r^\gamma$, where v_ρ is the volume of the unit ellipsoid in $\mathbb{R}^n$. Let also ${}^{\complement}\mathcal{E}(x,r) = \mathbb{R}^n \setminus \mathcal{E}(x,r)$ be the complement of $\mathcal{E}(x,r)$. If $P = I$, then clearly $\rho(x) = |x|$ and $\mathcal{E}_I(x,r) = \mathcal{E}(x,r)$. Note that in the standard parabolic case $P = (1, \ldots, 1, 2)$ we have

$$\rho(x) = \sqrt{\frac{|x'|^2 + \sqrt{|x'|^4 + x_n^2}}{2}}, \qquad x = (x', x_n).$$

Let $S_\rho = \{w \in \mathbb{R}^n : \rho(w) = 1\}$ be the unit ρ-sphere (ellipsoid) in $\mathbb{R}^n$ ($n \geq 2$) equipped with the normalized Lebesgue surface measure $d\sigma$. The parabolic maximal function $M^P f$ and the parabolic fractional integral $I_\alpha^P f$, $0 < \alpha < \gamma$, of a function $f \in L_1^{\rm loc}(\mathbb{R}^n)$ are defined by

$$M^P f(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |f(y)| dy,$$

$$I_\alpha^P f(x) = \int_{\mathbb{R}^n} \frac{f(y)}{\rho(x-y)^{\gamma-\alpha}} dy.$$

If $P = I$, then $M \equiv M_0^I$ is the Hardy-Littlewood maximal operator and $I_\alpha \equiv I_\alpha^I$ is the fractional integral operator. It is well known that the parabolic fractional integral operators play an important role in harmonic analysis (see [6, 15]).

In this work we present the characterization for parabolic fractional integral operator I_α^P (Theorem 9) and its commutators $[b, I_\alpha^P]$ (Theorem 11) in generalized Orlicz-Morrey spaces.

By $A \lesssim B$ we mean that $A \leq CB$ with some positive constant C independent of related quantities. If $A \lesssim B$ and $B \lesssim A$, we write $A \approx B$ and say that A and B are equivalent.

2. On Young functions and Orlicz spaces

Orlicz space was first introduced by Orlicz in [12, 13] as a generalizations of Lebesgue spaces L^p. Since then, this space has been one of important functional frames in the mathematical analysis, especially in real and harmonic analysis. Orlicz space is also a suitable substitute for L^1 space when the latter does not work.

First, we recall the definition of Young functions.

Definition 1. *A function* $\Phi : [0, \infty) \to [0, \infty]$ *is called a Young function if* Φ *is convex, left-continuous,* $\lim\limits_{r\to+0} \Phi(r) = \Phi(0) = 0$ *and* $\lim\limits_{r\to\infty} \Phi(r) = \infty$*.*

From the convexity and $\Phi(0) = 0$ it follows that any Young function is increasing. If there exists $s \in (0, \infty)$ such that $\Phi(s) = \infty$, then $\Phi(r) = \infty$ for $r \geq s$. The set of Young functions such that

$$0 < \Phi(r) < \infty \qquad \text{for} \qquad 0 < r < \infty$$

will be denoted by $\mathcal{Y}$. If $\Phi \in \mathcal{Y}$, then Φ is absolutely continuous on every closed interval in $[0, \infty)$ and bijective from $[0, \infty)$ to itself.

For a Young function Φ and $0 \leq s \leq \infty$, let

$$\Phi^{-1}(s) = \inf\{r \geq 0 : \Phi(r) > s\}.$$

If $\Phi \in \mathcal{Y}$, then Φ^{-1} is the usual inverse function of Φ. It is well known that

$$r \leq \Phi^{-1}(r)\widetilde{\Phi}^{-1}(r) \leq 2r \qquad \text{for } r \geq 0, \tag{1}$$

where $\widetilde{\Phi}(r)$ is defined by

$$\widetilde{\Phi}(r) = \begin{cases} \sup\{rs - \Phi(s) : s \in [0, \infty)\} & , \quad r \in [0, \infty) \\ \infty & , \quad r = \infty. \end{cases}$$

A Young function Φ is said to satisfy the Δ_2-condition, denoted as $\Phi \in \Delta_2$, if

$$\Phi(2r) \leq C\Phi(r), \qquad r > 0$$

for some $C > 1$. If $\Phi \in \Delta_2$, then $\Phi \in \mathcal{Y}$. A Young function Φ is said to satisfy the ∇_2-condition, denoted by $\Phi \in \nabla_2$, if

$$\Phi(r) \leq \frac{1}{2C}\Phi(Cr), \qquad r \geq 0$$

for some $C > 1$.

Note that by the convexity of Φ and concavity of Φ^{-1} we have the following properties:

$$\begin{cases} \Phi(\alpha t) \leq \alpha\Phi(t), & \text{if} \quad 0 \leq \alpha \leq 1 \\ \Phi(\alpha t) \geq \alpha\Phi(t), & \text{if} \quad \alpha > 1 \end{cases} \text{and} \begin{cases} \Phi^{-1}(\alpha t) \geq \alpha\Phi^{-1}(t), & \text{if} \quad 0 \leq \alpha \leq 1 \\ \Phi^{-1}(\alpha t) \leq \alpha\Phi^{-1}(t), & \text{if} \quad \alpha > 1. \end{cases} \tag{2}$$

The Orlicz space and weak Orlicz space are defined as follows.

Definition 2. *For a Young function Φ,*

$$L_\Phi(\mathbb{R}^n) = \left\{ f \in L_1^{\text{loc}}(\mathbb{R}^n) : \int_{\mathbb{R}^n} \Phi(\epsilon|f(x)|)dx < \infty \textit{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{L_\Phi} = \inf\left\{\lambda > 0 : \int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\lambda}\Big)dx \leq 1\right\},$$

$$WL_\Phi(\mathbb{R}^n) := \left\{ f \in L^1_{\text{loc}}(\mathbb{R}^n) : \sup_{r>0} \Phi(r) d_{\epsilon f}(r) < \infty \textit{ for some } \epsilon > 0 \right\},$$

$$\|f\|_{WL_\Phi} = \inf\left\{\lambda > 0 \ : \ \sup_{t>0} \Phi(\frac{t}{\lambda}) d_f(t) \ \leq 1\right\},$$

where $d_f(t) = |\{x \in \mathbb{R}^n : |f(x)| > t\}|$.

We note that

$$\int_{\mathbb{R}^n} \Phi\Big(\frac{|f(x)|}{\|f\|_{L_\Phi}}\Big)dx \leq 1 \tag{3}$$

and

$$\sup_{t>0} \Phi(\frac{t}{\|f\|_{WL_\Phi}}) d_f(t) \ \leq 1. \tag{4}$$

The following analogue of the Hölder's inequality is well known (see, for example, [14]).

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a measurable set and functions f and g be measurable on Ω. For a Young function Φ and its complementary function $\widetilde{\Phi}$, the following inequality is valid*

$$\int_\Omega |f(x)g(x)|dx \leq 2\|f\|_{L^\Phi(\Omega)}\|g\|_{L^{\widetilde{\Phi}}(\Omega)}.$$

By elementary calculations we have the following property.

Lemma 1. *Let Φ be a Young function and $\mathcal{E}$ be a parabolic ball in $\mathbb{R}^n$. Then*

$$\|\chi_\mathcal{E}\|_{L^\Phi} = \|\chi_\mathcal{E}\|_{WL^\Phi} = \frac{1}{\Phi^{-1}\left(|\mathcal{E}|^{-1}\right)}.$$

By Theorem 1, Lemma 1 and (1) we get the following estimate.

Lemma 2. *For a Young function* Φ *and for the parabolic ball* $\mathcal{E} = \mathcal{E}(x,r)$ *the following inequality is valid:*

$$\int_{\mathcal{E}} |f(y)|dy \le 2|\mathcal{E}|\Phi^{-1}\left(|\mathcal{E}|^{-1}\right) \|f\|_{L^{\Phi}(\mathcal{E})}.$$

3. Parabolic fractional integral and its commutators in Orlicz spaces

In [1] the boundedness of the parabolic maximal operator M^P in Orlicz spaces $L_\Phi(\mathbb{R}^n)$ was obtained.

Theorem 2. *[1] Let* Φ *be any Young function. Then the parabolic maximal operator* M^P *is bounded from* $L_\Phi(\mathbb{R}^n)$ *to* $WL_\Phi(\mathbb{R}^n)$*. Moreover, if* $\Phi \in \nabla_2$ *, then* M^p *is bounded in* $L_\Phi(\mathbb{R}^n)$*.*

We recall that the space $BMO(\mathbb{R}^n) = \{b \in L^1_{\text{loc}}(\mathbb{R}^n) \ : \ \|b\|_* < \infty\}$ is defined by the seminorm

$$\|b\|_* := \sup_{x\in\mathbb{R}^n, r>0} \frac{1}{|\mathcal{E}(x,r)|} \int_{\mathcal{E}(x,r)} |b(y) - b_{\mathcal{E}(x,r)}|dy < \infty,$$

where $b_{\mathcal{E}(x,r)} = \frac{1}{|\mathcal{E}(x,r)|}\int_{\mathcal{E}(x,r)} b(y)dy$. We refer for instance to [9] and [10] for more details on this space and its properties.

Lemma 3. [2] *Let* $b \in BMO(\mathbb{R}^n)$ *and* Φ *be a Young function with* $\Phi \in \Delta_2$*. Then*

$$\|b\|_* \approx \sup_{x\in\mathbb{R}^n, r>0} \Phi^{-1}\big(r^{-\gamma}\big) \left\|b(\cdot) - b_{\mathcal{E}(x,r)}\right\|_{L_\Phi(\mathcal{E}(x,r))}.$$

The commutator generated by $b \in L^1_{\text{loc}}(\mathbb{R}^n)$ and the parabolic maximal operator M^P is defined by

$$M^P_b(f)(x) = \sup_{t>0} |\mathcal{E}(x,t)|^{-1} \int_{\mathcal{E}(x,t)} |b(x) - b(y)||f(y)|dy.$$

The known boundedness statement for the commutator operator M^P_b in Orlicz spaces is given as follows (see [7, Corollary 2.3]).

Theorem 3. *Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$ and $b \in BMO(\mathbb{R}^n)$. Then M_b^P is bounded in $L_\Phi(\mathbb{R}^n)$ and the inequality*

$$\|M_b^P f\|_{L_\Phi} \le C_0 \|b\|_* \|f\|_{L_\Phi}$$

holds with constant C_0 independent of f.

The known boundedness statement for I_α^P in Orlicz spaces:

Theorem 4. *[11] Let $\Phi, \Psi \in \mathcal{Y}$ and*

$$\int_r^\infty t^{\alpha-1}\Phi^{-1}\left(t^{-\gamma}\right) dt \lesssim r^\alpha \Phi^{-1}\left(r^{-\gamma}\right) \qquad \text{for } 0<r<\infty, \tag{5}$$

$$r^\alpha \Phi^{-1}\left(r^{-\gamma}\right) \lesssim \Psi^{-1}\left(r^{-\gamma}\right) \qquad \text{for } 0<r<\infty. \tag{6}$$

Then I_α^P is bounded from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, then I_α^P is bounded from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$.

The following estimate is valid.

Lemma 4. *[2] If $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$, then $r_0^\alpha \lesssim I_\alpha^P \chi_{\mathcal{E}_0}(x)$ for every $x \in \mathcal{E}_0$.*

Theorem 5. *[2] Let $\Phi, \Psi \in \mathcal{Y}$. If (5) holds, then the condition (6) is necessary and sufficient for the boundedness of I_α^P from $L_\Phi(\mathbb{R}^n)$ to $WL_\Psi(\mathbb{R}^n)$. Moreover, if $\Phi \in \nabla_2$, the condition (6) is necessary and sufficient for the boundedness of I_α^P from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$.*

The commutators $[b, I_\alpha^P]$, $|b, I_\alpha^P|$ generated by $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and the operator I_α^P are defined by

$$[b, I_\alpha^P]f(x) = \int_{\mathbb{R}^n} \frac{b(x)-b(y)}{\rho(x-y)^{\gamma-\alpha}} f(y)dy,$$

$$|b, I_\alpha^P|f(x) = \int_{\mathbb{R}^n} \frac{|b(x)-b(y)|}{\rho(x-y)^{\gamma-\alpha}} f(y)dy, \qquad 0<\alpha<\gamma,$$

respectively.

Theorem 6. [2] *Let $0<\alpha<\gamma$, $b \in BMO(\mathbb{R}^n)$ and $\Phi, \Psi \in \mathcal{Y}$.*

1. *If $\Phi \in \nabla_2$ and $\Psi \in \Delta_2$, then the condition*

$$r^\alpha \Phi^{-1}\big(r^{-\gamma}\big) + \int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1} dt \le C\Psi^{-1}\big(r^{-\gamma}\big) \tag{7}$$

for all $r>0$, where $C>0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha^P]$ from $L_\Phi(\mathbb{R}^n)$ to $L_\Psi(\mathbb{R}^n)$.

2. *If* $\Psi \in \Delta_2$, *then the condition* (6) *is necessary for the boundedness of* $|b, I_\alpha^P|$ *from* $L_\Phi(\mathbb{R}^n)$ *to* $L_\Psi(\mathbb{R}^n)$.

3. *Let* $\Phi \in \nabla_2$ *and* $\Psi \in \Delta_2$. *If the condition*

$$\int_r^\infty \Big(1 + \ln \frac{t}{r}\Big) \Phi^{-1}\big(t^{-\gamma}\big)\, t^{\alpha-1} dt \le C r^\alpha \Phi^{-1}\big(r^{-\gamma}\big) \tag{8}$$

holds for all $r > 0$, *where* $C > 0$ *does not depend on* r, *then the condition* (6) *is necessary and sufficient for the boundedness of* $|b, I_\alpha^P|$ *from* $L_\Phi(\mathbb{R}^n)$ *to* $L_\Psi(\mathbb{R}^n)$.

3.1. Parabolic maximal operator and its commutators in parabolic generalized Orlicz-Morrey spaces

The parabolic generalized Orlicz-Morrey spaces and the weak parabolic generalized Orlicz-Morrey spaces are defined as follows.

Definition 3. *Let* $\varphi(r)$ *be a positive measurable function on* $(0,\infty)$ *and* Φ *be any Young function. We denote by* $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ *the generalized Orlicz-Morrey space, the space of all functions* $f \in L_\Phi^{\rm loc}(\mathbb{R}^n)$ *with finite quasinorm*

$$\|f\|_{M_{\Phi,\varphi,P}} \equiv \|f\|_{M_{\Phi,\varphi,P}(\mathbb{R}^n)} = \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)^{-1}\Phi^{-1}(|\mathcal{E}(x,r)|)^{-1})\|f\|_{L_\Phi(\mathcal{E}(x,r))},$$

where $L_\Phi^{\rm loc}(\mathbb{R}^n)$ *is defined as the set of all functions* f *such that* $f\chi_{\mathcal{E}} \in L_\Phi(\mathbb{R}^n)$ *for all ellipsoids* $\mathcal{E} \subset \mathbb{R}^n$.

Also by $WM_{\Phi,\varphi,P}(\mathbb{R}^n)$ *we denote the weak generalized Orlicz-Morrey space of all functions* $f \in WL_\Phi^{\rm loc}(\mathbb{R}^n)$ *for which*

$$\|f\|_{WM_{\Phi,\varphi,P}} \equiv \|f\|_{WM_{\Phi,\varphi,P}(\mathbb{R}^n)} =$$
$$= \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)^{-1}\Phi^{-1}(|\mathcal{E}(x,r)|)^{-1})\|f\|_{WL_\Phi(\mathcal{E}(x,r))} < \infty,$$

where $WL_\Phi^{\rm loc}(\mathbb{R}^n)$ *is defined as the set of all functions* f *such that* $f\chi_{\mathcal{E}} \in WL_\Phi(\mathbb{R}^n)$ *for all balls* $\mathcal{E} \subset \mathbb{R}^n$.

Remark 1. *Thanks to* (2) *we have*

$$\Phi^{-1}(|\mathcal{E}(x,r)|)^{-1}) \approx \Phi^{-1}(r^{-\gamma}).$$

Therefore we can also write

$$\|f\|_{M_{\Phi,\varphi,P}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)^{-1}\Phi^{-1}(r^{-\gamma})\|f\|_{L_\Phi(\mathcal{E}(x,r))},$$

and

$$\|f\|_{WM_{\Phi,\varphi,P}} \equiv \sup_{x\in\mathbb{R}^n, r>0} \varphi(r)^{-1}\Phi^{-1}(r^{-\gamma})\|f\|_{WL_\Phi(\mathcal{E}(x,r))},$$

respectively.

According to this definition, we recover the parabolic generalized Morrey space $M_{p,\varphi,P}(\mathbb{R}^n)$ and weak parabolic generalized Morrey space $WM_{p,\varphi,P}(\mathbb{R}^n)$ by choosing $\Phi(r) = r^p$, $1 \le p < \infty$. If $\Phi(r) = r^p$, $1 \le p < \infty$ and $\varphi(r) = r^{\frac{\lambda-\gamma}{p}}$, $0 \le \lambda \le \gamma$, then $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ and $WM_{\Phi,\varphi,P}(\mathbb{R}^n)$ coincide with $M_{p,\lambda,P}(\mathbb{R}^n)$ and $WM_{p,\lambda,P}(\mathbb{R}^n)$, respectively, and if $\varphi(r) = \Phi^{-1}(r^{-\gamma})$, then $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ and $WM_{\Phi,\varphi,P}(\mathbb{R}^n)$ coincide with the $L_\Phi(\mathbb{R}^n)$ and $WL_\Phi(\mathbb{R}^n)$, respectively.

A function $\varphi : (0,\infty) \to (0,\infty)$ is said to be almost increasing (resp. almost decreasing) if there exists a constant $C > 0$ such that

$$\varphi(r) \le C\varphi(s) \qquad (\text{resp. } \varphi(r) \ge C\varphi(s)) \quad \text{for } r \le s.$$

For a Young function Φ, we denote by $\mathcal{G}_\Phi$ the set of all almost decreasing functions $\varphi : (0,\infty) \to (0,\infty)$ such that $t \in (0,\infty) \mapsto \frac{\varphi(t)}{\Phi^{-1}(t^{-\gamma})}$ is almost increasing.

Lemma 5. *[3] Let $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$. If $\varphi \in \mathcal{G}_\Phi$, then there exists $C > 0$ such that*

$$\frac{1}{\varphi(r_0)} \le \|\chi_{\mathcal{E}_0}\|_{M_{\Phi,\varphi,P}} \le \frac{C}{\varphi(r_0)}.$$

Theorem 7. *[3] Let $\Phi \in \mathcal{Y}$, the functions φ_1, φ_2 and Φ satisfy the condition*

$$\sup_{r<t<\infty} \Phi^{-1}\big(t^{-\gamma}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \le C\, \varphi_2(r), \tag{9}$$

where C does not depend on r. Then the parabolic maximal operator M^P is bounded from $M_{\Phi,\varphi_1,P}(\mathbb{R}^n)$ to $WM_{\Phi,\varphi_2,P}(\mathbb{R}^n)$ and for $\Phi \in \nabla_2$, the operator M^P is bounded from $M_{\Phi,\varphi_1,P}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2,P}(\mathbb{R}^n)$.

Theorem 8. *[3] Let Φ be a Young function with $\Phi \in \Delta_2 \cap \nabla_2$, $b \in BMO(\mathbb{R}^n)$ and the functions φ_1, φ_2 and Φ satisfy the condition*

$$\sup_{r<t<\infty} \Big(1 + \ln\frac{t}{r}\Big) \Phi^{-1}\big(t^{-\gamma}\big) \operatorname*{ess\,inf}_{t<s<\infty} \frac{\varphi_1(s)}{\Phi^{-1}\big(s^{-\gamma}\big)} \le C\, \varphi_2(r), \tag{10}$$

where C does not depend on r. Then the operator M_b^P is bounded from $M_{\Phi,\varphi_1,P}(\mathbb{R}^n)$ to $M_{\Phi,\varphi_2,P}(\mathbb{R}^n)$.

4. Parabolic fractional integral and its commutators in parabolic generalized Orlicz-Morrey spaces

The following theorem is one of our main results.

Theorem 9. *Let* $0<\alpha<\gamma$, $\Phi\in\mathcal{Y}$, $\beta\in(0,1)$, $\eta(t)\equiv\varphi(t)^{\beta}$ *and* $\Psi(t)\equiv\Phi(t^{1/\beta})$.

1. *If* $\Phi\in\nabla_2$ *and* $\varphi(t)$ *satisfies* (9), *then the condition*

$$t^{\alpha}\varphi(t)+\int_t^{\infty} r^{\alpha}\,\varphi(r)\frac{dr}{r}\le C\varphi(t)^{\beta}, \tag{11}$$

for all $t>0$, *where* $C>0$ *does not depend on* t, *is sufficient for boundedness of* I_{α}^{P} *from* $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ *to* $M_{\Psi,\eta,P}(\mathbb{R}^n)$.

2. *If* $\varphi\in\mathcal{G}_{\Phi}$, *then the condition*

$$t^{\alpha}\varphi(t)\le C\varphi(t)^{\beta}, \tag{12}$$

for all $t>0$, *where* $C>0$ *does not depend on* t, *is necessary for boundedness of* I_{α}^{P} *from* $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ *to* $M_{\Psi,\eta,P}(\mathbb{R}^n)$.

3. *Let* $\Phi\in\nabla_2$. *If* $\varphi\in\mathcal{G}_{\Phi}$ *satisfies the regularity condition*

$$\int_t^{\infty} r^{\alpha}\,\varphi(r)\frac{dr}{r}\le Ct^{\alpha}\varphi(t), \tag{13}$$

for all $t>0$, *where* $C>0$ *does not depend on* t, *then the condition* (12) *is necessary and sufficient for boundedness of* I_{α}^{P} *from* $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ *to* $M_{\Psi,\eta,P}(\mathbb{R}^n)$.

Proof. Proof of the first part of the theorem:

For arbitrary parabolic ball $\mathcal{E}=\mathcal{E}(x,t)$ we represent f as

$$f=f_1+f_2,\qquad f_1(y)=f(y)\chi_{\mathcal{E}}(y),\qquad f_2(y)=f(y)\chi_{\complement_{\mathcal{E}}}(y),$$

and have

$$I_{\alpha}^{P}f(x)=I_{\alpha}^{P}f_1(x)+I_{\alpha}^{P}f_2(x).$$

For $I_{\alpha}^{P}f_1(x)$, following Hedberg's trick, we obtain $|I_{\alpha}^{P}f_1(x)|\le C_1t^{\alpha}M^{P}f(x)$. For $I_{\alpha}f_2(x)$, by Lemma 2, we have

$$\begin{aligned}\int_{\complement_{\mathcal{E}}}\frac{|f(y)|}{\rho(x-y)^{\gamma-\alpha}}dy &\approx \int_{\complement_{\mathcal{E}}}|f(y)|\int_{\rho(x-y)}^{\infty}\frac{dr}{r^{\gamma+1-\alpha}}dy\\ &\approx \int_t^{\infty}\int_{\mathcal{E}(x,r)\setminus\mathcal{E}(x,t)}|f(y)|dy\frac{dr}{r^{\gamma+1-\alpha}}\\ &\le C_2\int_t^{\infty}\Phi^{-1}(r^{-\gamma})r^{\alpha-1}\|f\|_{L_{\Phi}(\mathcal{E}(x,r))}dr.\end{aligned}$$

Consequently, we have

$$\begin{aligned}|I_{\alpha}^{P}f(x)| &\lesssim t^{\alpha}M^{P}f(x)+\int_t^{\infty}\Phi^{-1}(r^{-\gamma})r^{\alpha-1}\|f\|_{L_{\Phi}(B(x,r))}dr\\ &\lesssim t^{\alpha}Mf(x)+\|f\|_{M_{\Phi,\varphi,P}}\int_t^{\infty}r^{\alpha}\varphi(r)\frac{dr}{r}.\end{aligned}$$

From (11) we obtain

$$\begin{aligned} |I_\alpha^P f(x)| &\lesssim \min\{\varphi(t)^{\beta-1} M^P f(x), \varphi(t)^\beta \|f\|_{M_{\Phi,\varphi,P}}\} \\ &\lesssim \sup_{s>0} \min\{s^{\beta-1} M^P f(x), s^\beta \|f\|_{M_{\Phi,\varphi,P}}\} \\ &= (M^P f(x))^\beta \, \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta}, \end{aligned}$$

where we have used the fact that the supremum is achieved when the minimum parts are balanced. Hence for every $x \in \mathbb{R}^n$ we have

$$|I_\alpha^P f(x)| \lesssim (M^P f(x))^\beta \, \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta}. \tag{14}$$

By using the inequality (14) we have

$$\|I_\alpha^P f\|_{L_\Psi(\mathcal{E})} \lesssim \|(Mf)^\beta\|_{L_\Psi(\mathcal{E})} \, \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta}.$$

Note that from (3) we get

$$\int_{\mathcal{E}} \Psi\left(\frac{(M^P f(x))^\beta}{\|M^P f\|_{L_\Phi(\mathcal{E})}^\beta}\right) dx = \int_{\mathcal{E}} \Phi\left(\frac{M^P f(x)}{\|M^P f\|_{L_\Phi(\mathcal{E})}}\right) dx \le 1.$$

Thus, $\|(M^P f)^\beta\|_{L_\Psi(\mathcal{E})} \le \|M^P f\|_{L_\Phi(\mathcal{E})}^\beta$. Consequently, by using this inequality we have

$$\|I_\alpha^P f\|_{L_\Psi(\mathcal{E})} \lesssim \|M^P f\|_{L_\Phi(\mathcal{E})}^\beta \, \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta}. \tag{15}$$

From Theorem 7 and (15), we get

$$\begin{aligned} \|I_\alpha^P f\|_{M_{\Psi,\eta,P}} &= \sup_{x\in\mathbb{R}^n, t>0} \eta(t)^{-1} \Psi^{-1}(t^{-\gamma}) \|I_\alpha^P f\|_{L_\Psi(\mathcal{E})} \\ &\lesssim \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta} \sup_{x\in\mathbb{R}^n, t>0} \eta(t)^{-1} \Psi^{-1}(t^{-\gamma}) \|M^P f\|_{L_\Phi(\mathcal{E})}^\beta \\ &= \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta} \left(\sup_{x\in\mathbb{R}^n, t>0} \varphi(t)^{-1} \Phi^{-1}(t^{-\gamma}) \|M^P f\|_{L_\Phi(\mathcal{E})}\right)^\beta \\ &\lesssim \|f\|_{M_{\Phi,\varphi,P}}. \end{aligned}$$

Proof of the second part of the theorem:

Let $\mathcal{E}_0 = \mathcal{E}(x_0, t_0)$ and $x \in \mathcal{E}_0$. By Lemma 4 we have $t_0^\alpha \le C I_\alpha^P \chi_{\mathcal{E}_0}(x)$. Therefore, by Lemma 5 we have

$$t_0^\alpha \le C\Psi^{-1}(|\mathcal{E}_0|^{-1}) \|I_\alpha^P \chi_{\mathcal{E}_0}\|_{L_\Psi(\mathcal{E}_0)} \le C\eta(t_0) \|I_\alpha^P \chi_{\mathcal{E}_0}\|_{M_{\Psi,\eta,P}}$$

$$\leq C\eta(t_0)\|\chi_{\mathcal{E}_0}\|_{M_{\Phi,\varphi,P}} \leq C\frac{\eta(t_0)}{\varphi(t_0)} = C\varphi(t_0)^{\beta-1}.$$

Since this is true for every $t_0 > 0$, we are done.

The third statement of the theorem follows from the first and the second ones. ◀

Remark 2. *Note that in the isotropic case $P = I$ Theorem 9 was proved in [8, Theorem 4.3].*

Theorem 10. *Let $0 < \alpha < \gamma$, $\Phi \in \mathcal{Y}$, $\beta \in (0,1)$, $\eta(t) \equiv \varphi(t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$.*

1. *If $\varphi(t)$ satisfies (9), then the condition (11) is sufficient for boundedness of I_α^P from $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ to $WM_{\Psi,\eta,P}(\mathbb{R}^n)$.*
2. *If $\varphi \in \mathcal{G}_\Phi$, then the condition (12) is necessary for boundedness of I_α^P from $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ to $WM_{\Psi,\eta,P}(\mathbb{R}^n)$.*
3. *If $\varphi \in \mathcal{G}_\Phi$ satisfies the regularity condition (13), then the condition (12) is necessary and sufficient for boundedness of I_α^P from $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ to $WM_{\Psi,\eta,P}(\mathbb{R}^n)$.*

Proof. Proof of the first part of the theorem:

By using the inequality (14) we have

$$\|I_\alpha^P f\|_{WL_\Psi(\mathcal{E})} \lesssim \|(M^P f)^\beta\|_{WL_\Psi(\mathcal{E})}\, \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta},$$

where $\mathcal{E} = \mathcal{E}(x,t)$. Note that from (4) we get

$$\sup_{t>0} \Psi\left(\frac{t^\beta}{\|M^P f\|_{WL_\Phi(\mathcal{E})}^\beta}\right) d_{(M^P f)^\beta}(t^\beta) = \sup_{t>0} \Phi\left(\frac{t}{\|M^P f\|_{WL_\Phi(\mathcal{E})}}\right) d_{M^P f}(t) \leq 1.$$

Thus, $\|(M^P f)^\beta\|_{WL_\Psi(\mathcal{E})} \leq \|M^P f\|_{WL_\Phi(\mathcal{E})}^\beta$. Consequently, by using this inequality we have

$$\|I_\alpha^P f\|_{WL_\Psi(\mathcal{E})} \lesssim \|M^P f\|_{WL_\Phi(\mathcal{E})}^\beta\, \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta}. \tag{16}$$

From Theorem 7 and (16), we get

$$\|I_\alpha^P f\|_{WM_{\Psi,\eta,P}} = \sup_{x\in\mathbb{R}^n, t>0} \eta(t)^{-1}\Psi^{-1}(t^{-\gamma})\|I_\alpha^P f\|_{WL_\Psi(\mathcal{E})}$$
$$\lesssim \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta} \sup_{x\in\mathbb{R}^n, t>0} \eta(t)^{-1}\Psi^{-1}(t^{-\gamma})\|M^P f\|_{WL_\Phi(\mathcal{E})}^\beta$$
$$= \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta} \left(\sup_{x\in\mathbb{R}^n, t>0} \varphi(t)^{-1}\Phi^{-1}(t^{-\gamma})\|M^P f\|_{WL_\Phi(\mathcal{E})}\right)^\beta$$

$$\lesssim \|f\|_{M_{\Phi,\varphi,P}}.$$

Proof of the second part of the theorem: Let $\mathcal{E}_0 = \mathcal{E}(x_0, t_0)$ and $x \in \mathcal{E}_0$. By Lemma 4 we have $t_0^\alpha \leq CI_\alpha^P \chi_{\mathcal{E}_0}(x)$. Therefore, by Lemma 5

$$t_0^\alpha \leq C\Psi^{-1}(|\mathcal{E}_0|^{-1})\|I_\alpha^P \chi_{\mathcal{E}_0}\|_{WL_\Psi(\mathcal{E}_0)} \leq C\eta(t_0)\|I_\alpha^P \chi_{\mathcal{E}_0}\|_{WM_{\Psi,\eta,P}}$$
$$\leq C\eta(t_0)\|\chi_{\mathcal{E}_0}\|_{M_{\Phi,\varphi,P}} \leq C\frac{\eta(t_0)}{\varphi(t_0)} = C\varphi(t_0)^{\beta-1}.$$

Since this is true for every $t_0 > 0$, we are done.

The third statement of the theorem follows from the first and the second ones.
◀

Lemma 6. *If $b \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $\mathcal{E}_0 := \mathcal{E}(x_0, r_0)$, then*

$$r_0^\alpha |b(x) - b_{\mathcal{E}_0}| \lesssim |b, I_\alpha^P| \chi_{\mathcal{E}_0}(x)$$

for every $x \in \mathcal{E}_0$, where $b_{\mathcal{E}_0} = \frac{1}{|\mathcal{E}_0|}\int_{\mathcal{E}_0} b(y)dy$.

Proof. If $x, y \in \mathcal{E}_0$, then $\rho(x-y) \leq k\rho(x-x_0) + k\rho(y-x_0) < 2kr_0$. Since $0 < \alpha < \gamma$, we get $r_0^{\alpha-\gamma} \leq C\rho(x-y)^{\alpha-\gamma}$. Therefore

$$|b, I_\alpha^P|\chi_{\mathcal{E}_0}(x) = \int_{\mathcal{E}_0} |b(x) - b(y)|\rho(x-y)^{\alpha-\gamma}dy \geq Cr_0^{\alpha-\gamma}\int_{\mathcal{E}_0} |b(x) - b(y)|dy$$
$$\geq Cr_0^{\alpha-\gamma}\left|\int_{\mathcal{E}_0} (b(x) - b(y))dy\right| = Cr_0^\alpha |b(x) - b_{\mathcal{E}_0}|.$$

◀

The following theorem is one of our main results.

Theorem 11. *Let $0 < \alpha < \gamma$, $\Phi \in \mathcal{Y}$, $b \in BMO(\mathbb{R}^n)$, $\beta \in (0,1)$, $\eta(t) \equiv \varphi(t)^\beta$ and $\Psi(t) \equiv \Phi(t^{1/\beta})$.*

1. *If $\Phi \in \Delta_2 \cap \nabla_2$ and φ satisfies* (10)*, then the condition*

$$r^\alpha\varphi(r) + \int_r^\infty \Big(1 + \ln\frac{t}{r}\Big)\varphi(t)t^\alpha\frac{dt}{t} \leq C\varphi(r)^\beta, \tag{17}$$

for all $r > 0$, where $C > 0$ does not depend on r, is sufficient for the boundedness of $[b, I_\alpha^P]$ from $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ to $M_{\Psi,\eta,P}(\mathbb{R}^n)$.

2. *If $\Phi \in \Delta_2$ and $\varphi \in \mathcal{G}_\Phi$, then the condition* (12) *is necessary for the boundedness of $|b, I_\alpha^P|$ from $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ to $M_{\Psi,\eta,P}(\mathbb{R}^n)$.*

3. *Let* $\Phi \in \Delta_2 \cap \nabla_2$. *If* $\varphi \in \mathcal{G}_\Phi$ *satisfies the conditions*

$$\sup_{r<t<\infty} \Big(1+\ln\frac{t}{r}\Big)\varphi(t) \le C\,\varphi(r)$$

and

$$\int_r^\infty \Big(1+\ln\frac{t}{r}\Big)\varphi(t)t^\alpha \frac{dt}{t} \le Cr^\alpha\varphi(r)$$

for all $r>0$, *where* $C>0$ *does not depend on* r, *then the condition* (12) *is necessary and sufficient for the boundedness of* $|b, I_\alpha^P|$ *from* $M_{\Phi,\varphi,P}(\mathbb{R}^n)$ *to* $M_{\Psi,\eta,P}(\mathbb{R}^n)$.

Proof. For arbitrary $x_0 \in \mathbb{R}^n$, set $\mathcal{E} = \mathcal{E}(x_0, r)$ for the ball centered at x_0 and of radius r. Write $f = f_1 + f_2$ with $f_1 = f\chi_{2k\mathcal{E}}$ and $f_2 = f\chi_{\complement_{(2k\mathcal{E})}}$, where k is the constant from the triangle inequality.

If we use the same notation and proceed as in the proof of Theorem 6, for $x \in \mathcal{E}$ we have

$$\begin{aligned} J_0(x)+J_1 &\lesssim \|b\|_* r^\alpha M_b^P f(x) + \|b\|_* \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)\|f\|_{L_\Phi(B(x_0,t))}\Phi^{-1}\big(t^{-\gamma}\big)\frac{dt}{t^{1-\alpha}} \\ &\lesssim \|b\|_* \Big(r^\alpha M_b^P f(x) + \|f\|_{M_{\Phi,\varphi,P}} \int_{2r}^\infty \Big(1+\ln\frac{t}{r}\Big)\varphi(t)\frac{dt}{t^{1-\alpha}}\Big). \end{aligned}$$

Thus, by (17) we obtain

$$\begin{aligned} J_0(x)+J_1 &\lesssim \|b\|_* \min\{\varphi(r)^{\beta-1} M_b^P f(x), \varphi(r)^\beta \|f\|_{M_{\Phi,\varphi,P}}\} \\ &\lesssim \|b\|_* \sup_{s>0} \min\{s^{\beta-1} M_b^P f(x), s^\beta \|f\|_{M_{\Phi,\varphi,P}}\} \\ &= \|b\|_* (M_b^P f(x))^\beta \, \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta}. \end{aligned}$$

Consequently, for every $x \in \mathcal{E}$ we have

$$J_0(x)+J_1 \lesssim \|b\|_* (M_b^P f(x))^\beta \, \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta}. \tag{18}$$

By using the inequality (18) we have

$$\|J_0(\cdot)+J_1\|_{L_\Psi(\mathcal{E})} \lesssim \|b\|_* \|(M_b^P f)^\beta\|_{L_\Psi(\mathcal{E})} \, \|f\|_{M_{\Phi,\varphi,P}}^{1-\beta}.$$

Note that from (3) we get

$$\int_{\mathcal{E}} \Psi\left(\frac{(M_b^P f(x))^\beta}{\|M_b^P f\|_{L_\Phi(\mathcal{E})}^\beta}\right)dx = \int_{\mathcal{E}} \Phi\left(\frac{M_b^P f(x)}{\|M_b^P f\|_{L_\Phi(\mathcal{E})}}\right)dx \le 1.$$

Thus, $\|(M_b^P f)^\beta\|_{L_\Psi(\mathcal{E})} \le \|M_b^P f\|^\beta_{L_\Phi(\mathcal{E})}$. Therefore, we have

$$\|J_0(\cdot) + J_1\|_{L_\Psi(\mathcal{E})} \lesssim \|b\|_* \|M_b^P f\|^\beta_{L_\Phi(\mathcal{E})} \|f\|^{1-\beta}_{M_{\Phi,\varphi,P}}.$$

If we again use the same notation and proceed as in the proof of Theorem 6, we get

$$\|J_2\|_{L_\Psi(\mathcal{E})} \lesssim \|b\|_* \frac{1}{\Psi^{-1}\big(r^{-\gamma}\big)} \int_{2r}^{\infty} \|f\|_{L_\Phi(\mathcal{E}(x_0,t))} \Phi^{-1}\big(t^{-\gamma}\big) t^{\alpha-1} dt.$$

From this estimate and condition (17) we have

$$\begin{aligned}\|J_2\|_{L_\Psi(\mathcal{E})} &\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \|f\|_{M_{\Phi,\varphi,P}} \int_{2r}^{\infty} t^\alpha \varphi(t) \frac{dt}{t} \\ &\lesssim \frac{\|b\|_*}{\Psi^{-1}\big(r^{-\gamma}\big)} \|f\|_{M_{\Phi,\varphi,P}} \varphi(r)^\beta.\end{aligned}$$

Consequently, by using Theorem 8, we get

$$\begin{aligned}&\|[b, I_\alpha^P] f\|_{M_{\Psi,\eta,P}} = \sup_{x_0 \in \mathbb{R}^n, r>0} \eta(r)^{-1} \Psi^{-1}(r^{-\gamma}) \|[b, I_\alpha] f\|_{L_\Psi(\mathcal{E})} \\ &\lesssim \|b\|_* \|f\|^{1-\beta}_{M_{\Phi,\varphi,P}} \left(\sup_{x_0 \in \mathbb{R}^n, r>0} \varphi(r)^{-1} \Phi^{-1}(r^{-\gamma}) \|M_b^P f\|_{L_\Phi(\mathcal{E})} \right)^\beta + \|b\|_* \|f\|_{M_{\Phi,\varphi,P}} \\ &\lesssim \|b\|_* \|f\|_{M_{\Phi,\varphi,P}}.\end{aligned}$$

We shall now prove the second part. Let $\mathcal{E}_0 = \mathcal{E}(x_0, r_0)$ and $x \in \mathcal{E}_0$. By Lemma 6 we have $r_0^\alpha |b(x) - b_{\mathcal{E}_0}| \le C|b, I_\alpha^P|\chi_{\mathcal{E}_0}(x)$. Therefore, by Lemma 3 and Lemma 5

$$\begin{aligned}r_0^\alpha &\le C \frac{\||b, I_\alpha^P|\chi_{\mathcal{E}_0}\|_{L_\Psi(\mathcal{E}_0)}}{\|b(\cdot) - b_{\mathcal{E}_0}\|_{L_\Psi(\mathcal{E}_0)}} \le \frac{C}{\|b\|_*} \||b, I_\alpha^P|\chi_{\mathcal{E}_0}\|_{L_\Psi(\mathcal{E}_0)} \Psi^{-1}(r^{-\gamma}) \\ &\le \frac{C}{\|b\|_*} \eta(r_0) \||b, I_\alpha^P|\chi_{\mathcal{E}_0}\|_{M_{\Psi,\eta,P}} \le C\varphi_2(r_0) \|\chi_{\mathcal{E}_0}\|_{M_{\Phi,\varphi,P}} \le C \frac{\eta(r_0)}{\varphi(r_0)} \le C\varphi(r_0)^{\beta-1}.\end{aligned}$$

Since this is true for every $r_0 > 0$, we are done.

The third statement of the theorem follows from the first and the second ones. ◄

Remark 3. *Note that in the isotropic case $P = I$ Theorem 11 was proved in [8, Theorem 6.4].*

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