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Inclusions and Noninclusions of Spaces of Multipliers of Some Wiener Amalgam Spaces

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Abstract. The main purpose of this paper is to study inclusions and noninclusions among the spaces of multipliers of the Wiener amalgam spaces. M.G. Cowling and J. J.F. Fournier in [5], L. Hörmander in [22] and G. I. Gaudry in [15] , have worked on the space $M_G(L^p, L^q)$, the space of convolution multipliers from L^p into L^q, and studied inclusions and noninclusions among these spaces. In this paper, we consider much larger classes of spaces than L^p and L^q: we consider the Wiener amalgam spaces $W(L^p, L^q)$ and weighted Wiener amalgam spaces $W(L^p, L^q_\omega)$. Firstly, we work on inclusions between the spaces of multipliers of Wiener amalgam spaces. Later by using the Rudin-Shapiro measures, we investigate noninclusions among the spaces of multipliers of Wiener amalgam spaces.

Key Words and Phrases: multipliers, weighted Lebesgue space, Wiener amalgam spaces.

2010 Mathematics Subject Classifications: 42A45, 43A22

1. Introduction

In this paper we consider the Wiener amalgam spaces $W(L^p, L^q)$ and $W(L^p, L^q_\omega)$, where ω is the weight function. The idea goes back to N. Wiener 1926. He first defined the amalgam spaces $W(L^1, L^2)$, $W(L^2, L^1)$, $W(L^1, L^\infty)$ and $W(L^\infty, L^1)$ [26] . Other special cases were considered in [20] , [21] . In the next few years, there appeared several independent studies of amalgam spaces. H.G. Feichtinger gave a generalization of these spaces in [9] . In his definition, he takes Banach spaces B and C satisfying certain conditions as local and global components and defines the Wiener's amalgam space $W(B, C)$. He also studied in [10] and [11] the interpolation and the Fourier transform in amalgam spaces, respectively. Lastly, A.T. Gürkanlı and İsmail Aydın in [2] and [19] and A.T. Gürkanlı in [18] , defined the variable exponent Wiener amalgam space and worked on some properties of these spaces.

In [22] , L. Hörmander established a large number of results for convolution multipliers from L^p to L^q. Later, many authors worked on multipliers of some functional spaces. For example, in [7] , [14] and [25] the authors studied the multipliers of Lebesgue spaces, weighted Lebesgue spaces and measures; in [4] and [27] , the authors worked on the multipliers of Segal and weighted Segal algebras; in [1] and [8] , the authors investigated the multipliers in Lorentz space and weighted Lorentz space; in [12] and [16] , the authors dealt with the multipliers of the Banach ideals. Finally, in [17] , the author considered the multipliers of modulation spaces.

The main purpose of this paper is to study the inclusions and noninclusions among the spaces of convolution multipliers of the Wiener amalgam spaces. M.G. Cowling and J.J.F. Fournier in [5], L. Hörmander in [22] , and G.I. Gaudry in [15] , worked on the space $M_G(L^p, L^q)$, the space of convolution multipliers from L^p into L^q, and discussed inclusions and noninclusions among these spaces. In this paper, we consider much larger classes of spaces than L^p and L^q: we consider the Wiener amalgam spaces $W(L^p, L^q)$ and weighted Wiener amalgam spaces $W(L^p, L^q_\omega)$. Our paper is organized as follows. In Section 2 we introduce the notations. In Section 3 we treat inclusions between the spaces of multipliers of Wiener amalgam spaces.We investigate non-inclusions among the spaces of multipliers in Wiener amalgam spaces in Section 4. In this section, we use Rudin-Shapiro measures as in [5] .

2. Notation

Let G be a locally compact Abelian group (non-compact and non-discrete) with Haar measure dx. In this paper $C_c(G)$ denotes the space of continuous, complex valued functions on G with compact support. The translation and modulation operators are given by

$$T_x f(t) = f(t-x),\ M_\xi f(t) = e^{2\pi \xi t} f(t),\ t,\ x,\ \xi \in G.$$

For $1 \leq p \leq \infty$, we write $L^p(G)$ to denote the usual Lebesgue space. We shall write $\hat{f}$ for Fourier transform of the function $f \in L^p$. Let ω be a weight function on G, that is a continuous function satisfying $\omega(x) \geq 1$ and $\omega(x+y) \leq \omega(x)\omega(y)$ for $x, y \in G$. Let ω_1, ω_2 be two weight functions. We say that $\omega_1 \preceq \omega_2$ if and only if there exists $C > 0$ such that $\omega_1(x) \leq C\omega_2(x)$ for all $x \in G$. The weighted $L^p(G)$ space $L^p_\omega(G)$ is the set

$$L^p_\omega(G) = \{f : f\omega \in L^p(G)\}, 1 \leq p \leq \infty.$$

It is known that $L^p_\omega(G)$ is a Banach space under the norm

$$\|f\|_{p,\omega} = \|f\omega\|_p, 1 \leq p < \infty,$$

or

$$\|f\|_{\infty,\omega} = \|f\omega\|_{\infty} = \operatorname*{esssup}_{x\in\mathbb{R}^n} |f(x)\,\omega(x)|\,,\; p=\infty$$

[13]. For $1 \le p \le q \le \infty$, the space $M_G(L^p, L^q)$ of convolution multipliers of (p,q) type is defined as follows. It is the space of bounded linear transformations A from L^p to L^q which commute with translation : $AT_a = T_a A$ for all $a \in G$, [5], [22], [23], [25]. Let ω be a weight function and let $1 \le p, q \le \infty$. Take any fixed compact subset $Q \subset G$ with non empty interior. Then the Wiener amalgam space $W(L^p, L^q_\omega)$ consists of all functions (equivalent classes) $f : G \to \mathbb{C}$ such that $f\chi_K \in L^p$ for each compact $K \subset G$, and the control function

$$F_f(x) = F_f^Q(x) = \|f.\chi_{Q+x}\|_p = \|f.T_x\chi_Q\|_p\,,\; x \in G,$$

lies in L^q_ω. The norm on $W(L^p, L^q_\omega)$ is

$$\|f\|_{W\left(L^p, L^q_\omega\right)} = \|F_f\|_{q,\omega} = \left\|\|f.\chi_{Q+x}\|_p\right\|_{q,\omega},$$

[9], [10]. Another equivalent but discrete definition of $W(L^p, L^q)$ is given by using the uniform partition of unity (for short BUPU), that is a sequence of non-negative functions $(\psi_i)_{i\in I}$ on G corresponding to a sequence (y_i) in G such that

a. $\sum_{i\in I} \psi_i \equiv 1,$

b. there exists a compact set U such that $\sup p\psi_i \subset y_i + U$ for all i,

c. for each compact $K \subset G$,

$$\sup_{x\in G} \natural\{i : x \in K + y_i\} = \sup \natural\{j \in I : K + y_i \cap K + y_j \ne \phi\} < \infty,$$

d. $\sup_{x\in I} \|\psi_i\|_{L^\infty} < \infty.$

By using such a BUPU we define the Wiener amalgam space $W(L^p, L^q)$ to be all functions (equivalent classes) $f : G \to \mathbb{C}$ such that $f\chi_K \in L^p$ for each compact $K \subset G$, and

$$\left(\sum_i \|f\Psi_i\|_p^q\right)^{\frac{1}{q}} < \infty.$$

Throughout Section 3, we will denote $W^p = W\left(L^p, \ell^1\right)$. Let $1 \le p_1,\, q_1 \le \infty$ and $1 \le p_2,\, q_2 \le \infty$. The space of convolution multipliers from $W(L^{p_1}, L^{q_1})$ to $W(L^{p_2}, L^{q_2})$ is denoted by $M_G(W(L^{p_1}, L^{q_1}),\; W(L^{p_2}, L^{q_2}))$.

3. Inclusions among the spaces of multipliers

Theorem 1. *Let G be a locally compact Abelian group, $1 \leq p < \infty$ and let p' be dual index to p. We denote $W^p = W\left(L^p, \ell^1\right)$. If $\frac{1}{w} \in L^{p'}(G)$, then $M_G\left(L^p_w, L^p_w\right) \subset M_G\left(W^p, W^p\right)$.*

Proof. Let $f \in L^p_w(G)$. Since $\frac{1}{w} \in L^{p'}(G)$ and $fw \in L^p(G)$, then $f = (fw)\frac{1}{w} \in L^1(G)$ and hence $L^p_w(G) \subseteq L^1(G)$. By the inclusion $L^p_w(G) \subset L^1(G)$ we obtain

$$L^p_w(G) = W\left(L^p, L^p_w\right)(G) \subseteq W\left(L^p, L^1\right)(G) = W\left(L^p, \ell^1\right)(G) = W^p. \quad (1)$$

Take any $g \in L^p_w(G)$. By the inclusion $L^p_w(G) \subseteq L^1(G)$, there exists $C_1 > 0$ such that

$$\|g\|_1 \leq C_1 \|g\|_{p,w}. \quad (2)$$

Then from (1) and (2),

$$\|g\|_{W(L^p,\ell^1)} = \|g\|_{W^p} = \|F_g\|_1 \leq C_1 \|F_g\|_{p,w} = C_1 \|g\|_{W\left(L^p,L^p_w\right)} = C_1 \|g\|_{p,w}.$$

Since $C_c(G) \subset L^p_w(G)$, using (1) we obtain $C_c(G) \subset W\left(L^p, \ell^1\right)(G) = W^p$. Let $T \in M_G\left(L^p_w, L^p_w\right)$ and $f \in C_c(G)$. Since translation is isometry on W^p, and the sum is finite, then we have

$$\|Tf\|_{W^p} = \left\|T\Big(\sum_n f\Psi_n\Big)\right\|_{W^p} = \left\|\sum_n T\left(f\Psi_n\right)\right\|_{W^p} \quad (3)$$

$$\begin{aligned}
&= \left\|\sum_n T\left(T_{x_n}T_{-x_n}\left(f\Psi_n\right)\right)\right\|_{W^p} = \left\|\sum_n T_{x_n}T\left(T_{-x_n}\left(f\Psi_n\right)\right)\right\|_{W^p} \\
&\leq \sum_n \left\|T_{x_n}T\left(T_{-x_n}\left(f\Psi_n\right)\right)\right\|_{W^p} = \sum_n \left\|T\left(T_{-x_n}\left(f\Psi_n\right)\right)\right\|_{W^p} \\
&\leq \sum_n C_1 \left\|T\left(T_{-x_n}\left(f\Psi_n\right)\right)\right\|_{p,w} \leq C_1 \sum_n \|T\|_{L^p_w \to L^p_w} \left\|T_{-x_n}\left(f\Psi_n\right)\right\|_{p,w} \\
&= C_1 \|T\|_{L^p_w \to L^p_w} \sum_n \left\|T_{-x_n}\left(f\Psi_n\right)\right\|_{p,w},
\end{aligned}$$

where $(\Psi_n)_{i\in I}$ is the uniform partition of unity and $\|T\|_{L^p_w \to L^p_w}$ is the operator norm. By the definition of Wiener amalgam space there exists a compact set Q_0 such that $supp\Psi_n \subset x_n + Q_0$. This implies $suppT_{-x_n}\left(f\Psi_n\right) \subset Q_0$. Thus

$$\left\|T_{-x_n}\left(f\Psi_n\right)\right\|_{p,w} \leq \max_{x\in Q_0} w(x) \left\|T_{-x_n}\left(f\Psi_n\right)\right\|_p. \quad (4)$$

If we use the inequality (4) in (3)

$$\begin{aligned}\|Tf\|_{W^p} &\leq C_1 \|T\|_{L^p_w \to L^p_w} \sum_n \|T_{-x_n}(f\Psi_n)\|_{p,w} \\ &\leq C_1 \|T\|_{L^p_w \to L^p_w} \max_{x\in Q_0} w(x) \sum_n \|T_{-x_n}(f\Psi_n)\|_p \\ &= C_2 \sum_n \|f\Psi_n\|_p = C_2 \|f\|_{W^p},\end{aligned}$$

where $C_2 = C_1 \|T\|_{L^p_w \to L^p_w} \max_{x\in Q_0} w(x)$. Since $C_c(G)$ is dense in $L^p(G)$, then $C_c(G)$ is dense in $W^p = W(L^p, \ell^1)$ by Lemma in 5.5.4 in [6]. Then $M_G(L^p_w, L^p_w) \subseteq M_G(W^p, W^p)$.

Now we show that the inclusion in the statement is proper. Take the Dirac delta function δ_x at any $x \in G$ and any function $f \in L^p_w(G)$. Since $L^p_w(G) \subset L^1(G)$, the convolution $\delta_x * f$ is defined and $\delta_x * f = T_x f$. We know by Lemma 2.2 in [13] that the function $x \to \|T_x f\|_{p,w}$ is equivalent to the weight function w, i.e there exists a constant $C > 0$ such that

$$C^{-1} w(x) \leq \|T_x f\|_{p,w} \leq C w(x).$$

Hence

$$\|\delta_x | M_G(L^p_w, L^p_w)\| = \sup_{\|f\|_{p,w} \leq 1} \frac{\|T_x f\|_{p,w}}{\|f\|_{p,w}} \geq \sup_{\|f\|_{p,w} \leq 1} \frac{w(x)}{C\|f\|_{p,w}} \to \infty,$$

as $x \to \infty$. Then δ_x is not uniformly bounded. Thus $\delta_x \notin M_G(L^p_w, L^p_w)$. On the other hand

$$\|\delta_x | M_G(W^p, W^p)\| = \sup_{\|f\|_{W^p} \leq 1} \frac{\|\delta_x * f\|_{W^p}}{\|f\|_{W^p}} = \sup_{\|f\|_{W^p} \leq 1} \frac{\|T_x f\|_{W^p}}{\|f\|_{W^p}}.$$

From the equality

$$\|T_x f\|_{W^p} = \|F_{T_x f}\|_1 = \|T_x F_f\|_1 = \|F_f\|_1 = \|f\|_{W^p},$$

we obtain

$$\|\delta_x | M_G(W^p, W^p)\| = \sup_{\|f\|_W \leq 1} \frac{\|T_x f\|_W}{\|f\|_W} = \sup_{\|f\|_W \leq 1} \frac{\|f\|_W}{\|f\|_W} = 1.$$

Hence δ_x is uniformly bounded in $M_G(W^p, W^p)$ and thus $\delta_x \in M_G(W^p, W^p)$. That means the inclusion $M_G(L^p_w, L^p_w) \subset M_G(W^p, W^p)$ is proper. ◀

Example 1. *Let $G = \mathbb{R}^n$, $p^{'}$ be dual to p and $s > \frac{n}{p^{'}}$. Define the weight function $w(x) = \left(1 + |x|^2\right)^s$. Then $\frac{1}{w} \in L^{p^{'}}(\mathbb{R}^n)$.*

Theorem 2. *Let G be a locally compact Abelian group, $1 \leq p < \infty$ and $p^{'}$ be dual index to p. Assume that $\frac{1}{w} \in L^{p^{'}}(G)$. Then*

$$M_G\left(L_w^p(G)\right) \subseteq M_G\left(W\left(L^p, L_v^r\right)\right)$$

for $1 \leq r \leq p$, and $0 < \theta < 1$, where

$$\frac{1}{r} = 1 - \frac{\theta}{p^{'}} \text{ and } v = w^{\theta}$$

Proof. For the proof we will use the interpolation Theorem 2.2 and the Corollary 2.3 for Wiener amalgam spaces in [10]. We have for $0 < \theta < 1$,

$$\left[W\left(L^p, L_w^p\right), W\left(L^p, L^1\right)\right]_{[\theta]} = W\left(L^p, L_v^r\right), \tag{5}$$

where

$$v = v_1^{\theta} v_2^{1-\theta} = w^{\theta} \text{ and } \frac{\theta}{p} + \frac{1-\theta}{1} = \frac{1}{r}.$$

This implies

$$\frac{1}{r} = \frac{\theta}{p} + \frac{1-\theta}{1} = 1 - \theta\left(1 - \frac{1}{p}\right) = 1 - \frac{\theta}{p^{'}}.$$

Let $T \in M_G\left(L_w^p(G)\right)$. Then by Theorem 1, $T \in M_G\left(W^p, W^p\right)$. Since $T \in M_G\left(L_w^p(G)\right) = M_G\left(W\left(L^p, L_w^p\right)\right)$ and $T \in M_G\left(W^p, W^p\right) = M_G\left(W\left(L^p, L_w^1\right),\right.$ $\left.W\left(L^p, L_w^1\right)\right)$, the functions

$$\begin{aligned} T &: W\left(L^p, L_w^p\right) \to W\left(L^p, L_w^p\right) \\ T &: W\left(L^p, L_w^1\right) \to W\left(L^p, L_w^1\right) \end{aligned}$$

are bounded. Applying complex interpolation method [3], [24] and using (5), we find that the function

$$T : W\left(L^p, L_v^r\right) \to W\left(L^p, L_v^r\right)$$

is bounded for $0 < \theta < 1$, where

$$\frac{1}{r} = 1 - \frac{\theta}{p^{'}} \text{ and } v = w^{\theta}.$$

Then $T \in M_G\left(W\left(L^p, L_v^r\right), W\left(L^p, L_v^r\right)\right) = M_G\left(W\left(L^p, L_v^r\right)\right)$. ◄

Lemma 1. *Let* $\left(B_1, \|.\|_{B_1}\right), \left(B_2, \|.\|_{B_2}\right)$ *be two normed spaces and let* T *be a bounded linear operator from* $\left(B_1, \|.\|_{B_1}\right)$ *to* $\left(B_2, \|.\|_{B_2}\right)$*. Assume that a normed space* $\left(B_3, \|.\|_{B_3}\right)$ *is continuously embedded into* B_1*, and* B_2 *is continuously embedded into a normed space* $\left(B_4, \|.\|_{B_4}\right)$*. Then* T *defines a bounded linear operator from* B_3 *to* B_4*.*

Proof. Since T is bounded, there exists $C_1 > 0$ such that

$$\|Tx\|_{B_2} \leq C_1 \|x\|_{B_1}, \tag{6}$$

for all $x \in B_1$. Also, since $B_3 \hookrightarrow B_1$ and $B_2 \hookrightarrow B_4$, there exist $C_2 > 0$ and $C_3 > 0$ such that

$$\|u\|_{B_1} \leq C_2 \|u\|_{B_3} \tag{7}$$

and

$$\|v\|_{B_4} \leq C_3 \|v\|_{B_2} \tag{8}$$

for all $u \in B_3$ and $v \in B_4$. By using $(8), (6)$ and (7), we have

$$\|Tx\|_{B_4} \leq C_3 \|Tx\|_{B_2} \leq C_3 C_1 \|x\|_{B_1} \leq C_1 C_2 C_3 \|x\|_{B_3}$$

for all $x \in B_3$. Then T is bounded. ◄

Proposition 1. *Let* w_1, w_2, v_1, v_2 *be weight functions,* $1 \leq p_1, q_1, r_1, s_1 \leq \infty$ *and let* $1 \leq p_2, q_2, r_2, s_2 \leq \infty$*. Assume that* $p_2 \geq p_1$*,* $q_1 \geq q_2$*,* $r_1 \geq r_2$*,* $s_1 \leq s_2$*. If* $w_1 \geq v_1$ *and* $w_2 \leq v_2$*, then*

$$M_G\left(W\left(L^{p_1}, L^{q_1}_{w_1}\right), W\left(L^{r_1}, L^{s_1}_{w_2}\right)\right) \subset M_G\left(W\left(L^{p_2}, L^{q_2}_{v_1}\right), W\left(L^{r_2}, L^{s_2}_{v_2}\right)\right).$$

Proof. By the assumption

$$W\left(L^{p_2}, L^{q_2}_{v_1}\right) \hookrightarrow W\left(L^{p_1}, L^{q_1}_{w_1}\right)$$

and

$$W\left(L^{r_1}, L^{s_1}_{w_2}\right) \hookrightarrow W\left(L^{r_2}, L^{s_2}_{v_2}\right).$$

Then by Lemma 1, the proof is completed. ◄

Lemma 2. *If* $1 \leq p, q < \infty$ *and* $f \in W\left(L^p, L^q\right)$*, then*

$$\lim_{h \to \infty} \|f + T_h f\|_{W(L^p, L^q)} = 2^{\frac{1}{q}} \|f\|_{W(L^p, L^q)}.$$

Proof. Suppose $g \in C_c(G)$ with compact support K. Since the definition of $W(L^p, L^q)$ is independent of choice of the compact set Q, we can choose $Q \subset K$. If $h \notin K - K$, then

$$\sup p\chi_{Q+x} \cap \sup p\chi_{Q+x-h} = \phi,$$

thus

$$\sup pg\chi_{Q+x} \cap \sup pg\chi_{Q+x-h} = \phi,$$

for all $x \in G$. Then we have

$$\begin{aligned} \|(g + T_h g)\,\chi_{Q+x}\|_p &= \|g\chi_{Q+x} + (T_h g)\,\chi_{Q+x}\|_p \\ &= \|g\chi_{Q+x}\|_p + \|g\chi_{Q+x-h}\|_p = F_g(x) + T_h F_g(x). \end{aligned} \tag{9}$$

Since F_g and $T_h F_g$ belong to $L^q(G)$, by Lemma 3.5.1 in [23] we have

$$\lim_{h\to\infty} \|F_g + T_h F_g\|_q = 2^{\frac{1}{q}} \|F_g\|_q = 2^{\frac{1}{q}} \|f\|_{W(L^p,L^q)}. \tag{10}$$

Thus by (9) and (10) ,we obtain

$$\|g + T_h g\|_{W(L^p,L^q)} = \left\| \|(g + T_h g)\,\chi_{Q+x}\|_p \right\|_q = 2^{\frac{1}{q}} \|f\|_{W(L^p,L^q)}. \tag{11}$$

It is known that $C_c(G)$ is dense in $W(L^p, L^q)$, [6] . Then for any $f \in W(L^p, L^q)$, and any $\varepsilon > 0$, there exists $g \in C_c(G)$ such that

$$\|f - g\|_{W(L^p,L^q)} < \frac{\varepsilon}{2\left(2 + 2^{\frac{1}{q}}\right)}. \tag{13}$$

Take any $h \notin K - K$ such that

$$\left| \|T_h g - g\|_{W(L^p,L^q)} - 2^{\frac{1}{q}} \|g\|_{W(L^p,L^q)} \right| \le \frac{\varepsilon}{2}.$$

Then it is easily shown that for all $h \notin K - K$

$$\left| \|f - T_h f\|_{W(L^p,L^q)} - 2^{\frac{1}{q}} \|f\|_{W(L^p,L^q)} \right| = \tag{14}$$

$$\begin{aligned} &= \Big| \; \|f - g + g - T_h g + T_h g - T_h f\|_{W(L^p,L^q)} + \\ &\quad + 2^{\frac{1}{q}} \|g\|_{W(L^p,L^q)} - 2^{\frac{1}{q}} \|g\|_{W(L^p,L^q)} - 2^{\frac{1}{q}} \|f\|_{W(L^p,L^q)} \; \Big| \\ &\le \Big| \; \|f - g\|_{W(L^p,L^q)} + \|T_h f - T_h g\|_{W(L^p,L^q)} + \|T_h g - g\|_{W(L^p,L^q)} + \\ &\quad + 2^{\frac{1}{q}} \|g\|_{W(L^p,L^q)} - 2^{\frac{1}{q}} \|g\|_{W(L^p,L^q)} - 2^{\frac{1}{q}} \|f\|_{W(L^p,L^q)} \; \Big| . \end{aligned}$$

Since the Wiener amalgam space is strongly translation invariant (i.e. $\|T_h g\|_{W(L^p,L^q)} = \|g\|_{W(L^p,L^q)}$), from (13) and (14) we have

$$
\begin{aligned}
& \left|\|f-T_hf\|_{W(L^p,L^q)}-2^{\frac{1}{q}}\|f\|_{W(L^p,L^q)}\right| \\
\leq\; & 2\|f-g\|_{W(L^p,L^q)}+\left|\|T_hg-g\|_{W(L^p,L^q)}-2^{\frac{1}{q}}\|g\|_{W(L^p,L^q)}\right|+ \\
& +\left|2^{\frac{1}{q}}\|g\|_{W(L^p,L^q)}-2^{\frac{1}{q}}\|f\|_{W(L^p,L^q)}\right|\leq 2\|f-g\|_{W(L^p,L^q)} \\
+\; & 2^{\frac{1}{q}}\|f-g\|_{W(L^p,L^q)}+\left|\|T_hg-g\|_{W(L^p,L^q)}-2^{\frac{1}{q}}\|g\|_{W(L^p,L^q)}\right| \\
=\; & \left(2+2^{\frac{1}{q}}\right)\|f-g\|_{W(L^p,L^q)}+\left|\|T_hg-g\|_{W(L^p,L^q)}-2^{\frac{1}{q}}\|g\|_{W(L^p,L^q)}\right| \\
\leq\; & \frac{\varepsilon}{2\left(2+2^{\frac{1}{q}}\right)}\left(2+2^{\frac{1}{q}}\right)+\frac{\varepsilon}{2}=\frac{\varepsilon}{2}+\frac{\varepsilon}{2}=\varepsilon.
\end{aligned}
$$

◄

Proposition 2. *If $A \in M_G\left(W\left(L^p, L^q\right), W\left(L^r, L^s\right)\right)$ and $q > s$, then $A = 0$.*

Proof. Since A is bounded, there exists a smallest constant $C > 0$ such that

$$\|Af\|_{W(L^r,L^s)} \leq C\|f\|_{W(L^p,L^q)} \tag{15}$$

for all $f \in W\left(L^p, L^q\right)$. Then from (15),

$$
\begin{aligned}
\|Af+T_h(Af)\|_{W(L^r,L^s)} &= \|Af+A(T_hf)\|_{W(L^r,L^s)} \\
&= \|A(f+T_hf)\|_{W(L^r,L^s)} \leq C\|f+T_hf\|_{W(L^p,L^q)}.
\end{aligned} \tag{16}
$$

By Lemma 2, we have

$$\lim_{h\to\infty}\|Af+T_h(Af)\|_{W(L^r,L^s)} = 2^{\frac{1}{s}}\|Af\|_{W(L^r,L^s)}, \tag{17}$$

and

$$\lim_{h\to\infty}\|f+T_hf\|_{W(L^p,L^q)} = 2^{\frac{1}{q}}\|f\|_{W(L^p,L^q)}. \tag{18}$$

Then from (16), (17), and (18), we have

$$2^{\frac{1}{s}}\|Af\|_{W(L^r,L^s)} \leq C2^{\frac{1}{q}}\|f\|_{W(L^p,L^q)},$$

and hence

$$\|Af\|_{W(L^r,L^s)} \leq 2^{\frac{1}{q}-\frac{1}{s}}C\|f\|_{W(L^p,L^q)}. \tag{19}$$

Since $q > s$, we have $\frac{1}{q} - \frac{1}{s} < 0$, and so $2^{\frac{1}{q}-\frac{1}{s}}C < C$. But this contradicts the assumption that C is a smallest constant satisfying (19). ◄

Proposition 3. *If* $\frac{1}{w} \in L^{s'}$ *and* $A \in M_G\left(W\left(L^p, L^q\right), W\left(L^r, L_w^s\right)\right)$, *then* $A = 0$.

Proof. The assumption $\frac{1}{w} \in L^{s'}$ implies that $L_w^s \subset L^1$, and thus $W\left(L^r, L_w^s\right) \hookrightarrow W\left(L^r, L^1\right)$. Then the inclusion

$$M_G\left(W\left(L^p, L^q\right), W\left(L^r, L_w^s\right)\right) \subset M_G\left(W\left(L^p, L^q\right), W\left(L^r, L^1\right)\right) \tag{20}$$

is obtained by Lemma 1. Hence by the inclusion (20), we have $A \in M_G\left(W\left(L^p, L^q\right), W\left(L^r, L^1\right)\right)$. Since $q > 1$, by Proposition 2, we obtain $A = 0$. ◄

4. Noninclusions among the spaces of multipliers

In this section we will discuss the noninclusions among the spaces of multipliers.

We need the following Lemma (see Lemma 17 in [5]).

Lemma 3. *(M.G. Cowling and J.J.F. Fournier). Suppose G is a nondiscrete locally compact group. There exists a sequence of relatively compact neighbourhoods (U_n) of the identity in G such that*

$$m\left(U_n + U_n\right) \leq Cm\left(U_n\right), \ n = 1, 2, ...; \ m\left(U_n\right) \to 0 \ \text{as } n \to \infty,$$

where C is a constant independent of n, and $m\left(U_n\right)$ is the Haar measure of the set U_n.

Theorem 3. *Let G be a nondiscrete locally compact Abelian group. Suppose $1 \leq p, q, r, s, p_1, q_1, r_1, s_1 \leq \infty$, $1 \leq r_1 \leq q \leq s_1 \leq \infty$, and $1 \leq r_1 \leq r \leq s_1 \leq \infty$. If*

$$0 \leq \frac{1}{p} - \frac{1}{q} < \frac{1}{r} - \frac{1}{s},$$

then $M_G\left(L^p, W\left(L^{r_1}, L^{s_1}\right)\right)$ is not contained in $M_G\left(W\left(L^{r_1}, L^{s_1}\right), L^s\right)$.

Proof. Since G is a nondiscrete locally compact Abelian group, by Lemma 3, there exists a sequence of relatively compact neighbourhoods $\left(U_n\right)_{n\in\mathbb{N}}$ of the identity in G such that

$$\mu\left(U_n + U_n\right) \leq C\mu\left(U_n\right), \ n \in \mathbb{N}; \ \mu\left(U_n\right) \to 0, \ \text{as } n \to \infty,$$

where C is a constant independent of n. We estimate the $M_G\left(L^p, W\left(L^{r_1}, L^{s_1}\right)\right)$ and $M_G\left(L^p, L^q\right)$ norms of characteristic function χ_{U_n} of the set U_n, where U_n is the term of the sequence $(U_n)_{\in\mathbb{N}}$. Since $r_1 \leq q \leq s_1$, we have $L^q \subset W(L^{r_1}, L^{s_1})$ and there exists $C_1 > 0$ such that

$$\|g\|_{W(L^{r_1},L^{s_1})} \leq C_1 \|g\|_q$$

for all $g \in L^q$. Then

$$\|\chi_{U_n}|M_G(L^p, W(L^{r_1}, L^{s_1}))\| = \sup_{f\in L^p} \frac{\|\chi_{U_n}(f)\|_{W(L^{r_1},L^{s_1})}}{\|f\|_p} \tag{21}$$

$$\leq \sup_{f\in L^p} \frac{C_1 \|\chi_{U_n}(f)\|_q}{\|f\|_p} = C_1 \|\chi_{U_n}|M_G(L^p, L^q)\|.$$

Note that

$$\|\chi_{U_n}|M_G(L^p, L^q)\| \leq \mu(U_n)^{1-\frac{1}{p}+\frac{1}{q}}. \tag{22}$$

Indeed, if we take a number t such that

$$1 - \frac{1}{t} = \frac{1}{p} - \frac{1}{q},$$

then L^t is embedded continuously in $M_G(L^p, L^q)$ and

$$\|\chi_{U_n} * f\|_q \leq \|\chi_{U_n}\|_t \|f\|_p$$

for all $f \in L^p$. Then

$$\|\chi_{U_n}|M_G(L^p, L^q)\| \leq \|\chi_{U_n}\|_t = \mu(U_n)^{1-\frac{1}{p}+\frac{1}{q}}.$$

Combining (21) and (22), we obtain

$$\|\chi_{U_n}|M_G(L^p, W(L^{r_1}, L^{s_1}))\| \leq C_1 \|\chi_{U_n}|M_G(L^p, L^q)\| \leq C_1\mu(U_n)^{1-\frac{1}{p}+\frac{1}{q}}. \tag{23}$$

On the other hand, from the inequality $r_1 \leq r \leq s_1$ we have $L^r \subset W(L^{r_1}, L^{s_1})$ and there exists $C_2 > 0$ such that

$$\|g\|_{W(L^{r_1},L^{s_1})} \leq C_2 \|g\|_r \tag{24}$$

for all $g \in L^r$. Then by (23),

$$\|\chi_{U_n}|M_G(W(L^{r_1}, L^{s_1}), L^s)\| = \sup_{\substack{f\in W(L^{r_1},L^{s_1}) \\ f\neq 0}} \frac{\|\chi_{U_n}(f)\|_s}{\|f\|_{W(L^{r_1},L^{s_1})}} \geq$$

$$\geq \sup_{\substack{f\in L^r \\ f\neq 0}} \frac{\|\chi_{U_n}(f)\|_s}{C_2\|f\|_r} = \frac{1}{C_2}\|\chi_{U_n}|M_G(L^r,L^s)\|. \tag{25}$$

Again as in (22), let t be the number such that

$$1-\frac{1}{t}=\frac{1}{p}-\frac{1}{q}.$$

It is easy to show that

$$\mu(U_n)\chi_{-U_n} \leq \chi_{U_n} * \chi_{-U_n-U_n}.$$

Then

$$\begin{aligned}\mu(U_n)\|\chi_{-U_n}\|_s &\leq \|\chi_{U_n} * \chi_{-U_n-U_n}\|_s \\ &\leq \|\chi_{U_n}|M_G(L^r,L^s)\|\,\|\chi_{-U_n-U_n}\|_r,\end{aligned}$$

this implies

$$\mu(U_n)(\mu(U_n))^{\frac{1}{s}} \leq \|\chi_{U_n}|M_G(L^r,L^s)\|(\mu(-U_n-U_n))^{\frac{1}{r}}. \tag{26}$$

By Lemma 3, $\mu(-U_n-U_n) \leq C\mu(U_n)$ for some constant $C>0$. Thus from (22)

$$\mu(U_n)(\mu(U_n))^{\frac{1}{s}} \leq \|\chi_{U_n}|M_G(L^r,L^s)\|(C\mu(U_n))^{\frac{1}{r}},$$

and so we have

$$\|\chi_{U_n}|M_G(L^r,L^s)\| \geq C^{-\frac{1}{r}}\mu(U_n)^{1-\frac{1}{r}+\frac{1}{s}}. \tag{27}$$

Combining (25) and (27), we have

$$\|\chi_{U_n}|M_G(W(L^{r_1},L^{s_1}),L^s)\| \geq \frac{1}{C_2}\|\chi_{U_n}|M_G(L^r,L^s)\| \geq \frac{1}{C_2C^{\frac{1}{r}}}\mu(U_n)^{1-\frac{1}{r}+\frac{1}{s}}. \tag{28}$$

Finally, by using the estimates (23) and (28), we obtain

$$\begin{aligned}\frac{\|\chi_{U_n}|M_G(W(L^{r_1},L^{s_1}),L^s)\|}{\|\chi_{U_n}|M_G(L^p,W(L^{r_1},L^{s_1}))\|} &\geq \frac{\frac{1}{C_2C^{\frac{1}{r}}}\mu(U_n)^{1-\frac{1}{r}+\frac{1}{s}}}{C_1\mu(U_n)^{1-\frac{1}{p}+\frac{1}{q}}} \\ &= \frac{1}{C_1C_2C^{\frac{1}{r}}\mu(U_n)^{\left[\left(\frac{1}{r}-\frac{1}{s}\right)-\left(\frac{1}{p}-\frac{1}{q}\right)\right]}}.\end{aligned} \tag{29}$$

Since $0 \leq \frac{1}{p} - \frac{1}{q} < \frac{1}{r} - \frac{1}{s}$, the right- hand side of (29) tends to ∞ as $n \to \infty$. That means we haven't any constant $C_0 > 0$ such that

$$\frac{\|\chi_{U_n}|M_G\left(W\left(L^{r_1}, L^{s_1}\right), L^s\right)\|}{\|\chi_{U_n}|M_G\left(L^p, W\left(L^{r_1}, L^{s_1}\right)\right)\|} \leq C_0.$$

for all $(U_n)_{\in \mathbb{N}}$. This implies that $M_G\left(L^p, W\left(L^{r_1}, L^{s_1}\right)\right)$ is not contained in $M_G\left(W\left(L^{r_1}, L^{s_1}\right), L^s\right)$. ◄

Corollary 1. *Let G be a nondiscrete locally compact Abelian group and let $1 \leq p$, q, r, s, p_1, q_1, r_1, $s_1 \leq \infty$. If $r_1 \leqq$, $r \leq s_1$, $s_2 \leq s \leq r_2$ and*

$$0 \leq \frac{1}{p} - \frac{1}{q} < \frac{1}{r} - \frac{1}{s},$$

then $M_G\left(L^p, W\left(L^{r_1}, L^{s_1}\right)\right)$ is not contained in $M_G\left(W\left(L^{r_1}, L^{s_1}\right), W\left(L^{r_2}, L^{s_2}\right)\right)$.

Proof. *Assume that*

$$M_G\left(L^p, W\left(L^{r_1}, L^{s_1}\right)\right) \subset M_G\left(W\left(L^{r_1}, L^{s_1}\right), W\left(L^{r_2}, L^{s_2}\right)\right). \tag{30}$$

Since $s_2 \leq s \leq r_2$, we have $W\left(L^{r_2}, L^{s_2}\right) \subset L^s$. Thus there exists $C_1 > 0$ such that

$$\|f\|_s \leq C_1 \|f\|_{W(L^{r_2}, L^{s_2})}$$

for all $f \in W\left(L^{r_2}, L^{s_2}\right)$. Let $A \in M_G\left(W\left(L^{r_1}, L^{s_1}\right), W\left(L^{r_2}, L^{s_2}\right)\right)$. Then by (30),

$$\|Af\|_s \leq C_1 \|Af\|_{W(L^{r_2}, L^{s_2})} \leq C_1 C_2 \|f\|_{W(L^{r_1}, L^{s_1})}$$

for some $C_2 > 0$. This implies $A \in M_G\left(W\left(L^{r_1}, L^{s_1}\right), L^s\right)$. Hence

$$M_G\left(W\left(L^{r_1}, L^{s_1}\right), W\left(L^{r_2}, L^{s_2}\right)\right) \subset M_G\left(W\left(L^{r_1}, L^{s_1}\right), L^s\right). \tag{31}$$

Combining (30) and (31), we have

$$\begin{aligned} M_G\left(L^p, W\left(L^{r_1}, L^{s_1}\right)\right) &\subset M_G\left(W\left(L^{r_1}, L^{s_1}\right), W\left(L^{r_2}, L^{s_2}\right)\right) \\ &\subset M_G\left(W\left(L^{r_1}, L^{s_1}\right), L^s\right). \end{aligned}$$

But this inclusion is a conradiction with the Theorem 3. Thus the inclusion (26) is not true. ◄

Theorem 4. *(M.G. Cowling and J.J.F. Fournier, [5], Theorem 7). Let G be a noncompact, unimodular, locally compact group. Let $1 \leq p, q, r, s \leq \infty$. Suppose that $p \leq q$ and $\min\left(s, r^{'}\right) < \min\left(q, p^{'}\right)$. Then $M_G\left(L^p, L^q\right)$ is not included in $M_G\left(L^r, L^s\right)$.*

Theorem 5. *Let* $s_1 \leq p \leq r_1$, $r_2 \leq q \leq s_2$ *and let* $r_3 \leq r \leq s_3$, $s_4 \leq s \leq r_4$. *Suppose that* $p \leq q$ *and* $\min\left(s, r'\right) < \min\left(q, p'\right)$. *Then* $M_G\left(W\left(L^{r_1}, L^{s_1}\right), W\left(L^{r_2}, L^{s_2}\right)\right)$ *is not included in* $M_G\left(W\left(L^{r_3}, L^{s_3}\right), W\left(L^{r_4}, L^{s_4}\right)\right)$.

Proof. By Theorem 4, $M_G\left(L^p, L^q\right)$ is not included in $M_G\left(L^r, L^s\right)$. Then there exists at least one element $T \in M_G\left(L^p, L^q\right)$ such that $T \notin W\left(L^r, L^s\right)$. By the assumptions $W\left(L^{r_1}, L^{s_1}\right) \hookrightarrow L^p$, $L^q \hookrightarrow W\left(L^{r_2}, L^{s_2}\right)$ and also $L^r \hookrightarrow W\left(L^{r_3}, L^{s_3}\right)$ and $W\left(L^{r_4}, L^{s_4}\right) \hookrightarrow L^s$. Then by Lemma 1, we have the inclusions

$$M_G\left(L^p, L^q\right) \subset M_G\left(W\left(L^{r_1}, L^{s_1}\right), W\left(L^{r_2}, L^{s_2}\right)\right),$$

$$M_G\left(W\left(L^{r_3}, L^{s_3}\right), W\left(L^{r_4}, L^{s4}\right)\right) \subset M_G\left(L^r, L^s\right).$$

Since $T \in M_G\left(W\left(L^{r_1}, L^{s_1}\right), W\left(L^{r_2}, L^{s_2}\right)\right)$ but $T \notin M_G\left(W\left(L^{r_3}, L^{s_3}\right), W\left(L^{r_4}, L^{s_4}\right)\right)$, the space $M_G\left(W\left(L^{r_1}, L^{s_1}\right), W\left(L^{r_2}, L^{s_2}\right)\right)$ is not included in $M_G\left(W\left(L^{r_3}, L^{s_3}\right), W\left(L^{r_4}, L^{s_4}\right)\right)$. ◄

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