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The Existence of Solutions to Boundary Value Problems for Differential Equations of Variable Order

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Abstract. In this paper, we discuss the existence of solutions to a boundary value problem for differential equations of variable order. Our results are based on the Schauder fixed point theorem. Some examples are given to illustrate the effectiveness of our results.

Key Words and Phrases: derivatives and integrals of variable order, differential equations of variable order, piecewise constant functions, existence.

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1. Introduction

In this paper, we consider the existence of solution to boundary value problem for differential equation of variable order

$$\begin{cases} D_{0+}^{q(t)}x(t) = f(t, x(t)), \quad 0 < t < T, \\ x(0) = 0, \quad x(T) = 0, \end{cases} \tag{1}$$

where $0 < T < +\infty$, $D_{0+}^{q(t)}$ denotes derivative of variable order defined by

$$D_{0+}^{q(t)}x(t) = \frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q(t)}}{\Gamma(2-q(t))}x(s)ds, \quad t > 0, \tag{2}$$

and

$$I_{0+}^{2-q(t)}x(t) = \int_0^t \frac{(t-s)^{1-q(t)}}{\Gamma(2-q(t))}x(s)ds, \quad t > 0 \tag{3}$$

denotes integral of variable order $2 - q(t)$, $1 < q(t) \leq 2$, $0 \leq t \leq T$.

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The subject of fractional calculus has gained considerable popularity and importance due to its frequent appearance in different research areas and engineering, such as physics, chemistry, control of dynamical systems, etc. Recently, many people paid attention to the existence and uniqueness of solutions to boundary value problem for fractional differential equations, such as [1-4].

The operators of variable order, which fall into a more complex operator category, are the derivatives and integrals whose order is the function of some variables. The variable order fractional derivative is an extension of constant order fractional derivative. In recent years, the operator and differential equations of variable order have been applied in engineering more and more frequently (for the examples and details, see [5-20], [22], [24-27]).

Although the existing literature on solutions of FBVPs is quite wide, few papers deal with the existence of solutions to BVPs with variable order. According to (1), (2) and (3), it is clear that when $q(t)$ is a constant function, i.e. $q(t) \equiv q$ (q is a finite positive constant), then $I_{0+}^{q(t)}, D_{0+}^{q(t)}$ are the usual Riemann-Liouville fractional integral and derivative [21].

The following properties of fractional calculus operators D_{0+}^{q}, I_{0+}^{q} play an important part in discussing the existence of solutions of fractional differential equations.

Proposition 1. [21] The equality $I_{0+}^{\gamma} I_{0+}^{\delta} f(t) = I_{0+}^{\gamma+\delta} f(t)$, $\gamma > 0, \delta > 0$ holds for $f \in L(0, b), 0 < b < +\infty$.

Proposition 2. [21] The equality $D_{0+}^{\gamma} I_{0+}^{\gamma} f(t) = f(t)$, $\gamma > 0$ holds for $f \in L(0, b), 0 < b < +\infty$.

Proposition 3. [21] Let $1 < \alpha \leq 2$. Then the differential equation

$$D_{0+}^{\alpha} u = 0$$

has a unique solution

$$u(t) = c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, c_1, c_2 \in R.$$

Proposition 4. [21] Let $1 < \alpha \leq 2$, $u(t) \in L(0, b)$, $D_{0+}^{\alpha} u \in L(0, b)$. Then the following equality holds

$$I_{0+}^{\alpha} D_{0+}^{\alpha} u(t) = u(t) + c_1 t^{\alpha-1} + c_2 t^{\alpha-2}, c_1, c_2 \in R.$$

A key point is whether the above properties of fractional calculus operators remain true for the operators of variable order.

Let's consider Proposition 1 for example. To begin with the simplest case, we let $f(t) \equiv 1, t \in [0,T]$ and calculate $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)$.
According to (3), we have

$$\begin{aligned} I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) &= \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds \\ &= \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds \\ &= \int_0^t \frac{(t-s)^{p(t)-1} s^{q(s)}}{\Gamma(p(t))\Gamma(1+q(s))} ds \\ &= \frac{t^{p(t)}}{\Gamma(p(t))} \int_0^1 \frac{t^{q(tr)}}{\Gamma(1+q(tr))} (1-r)^{p(t)-1} r^{q(tr)} dr. \end{aligned}$$

And secondly, we have

$$I_{0+}^{p(t)+q(t)} f(t) = \int_0^t \frac{(t-s)^{p(t)+q(t)-1}}{\Gamma(p(t)+q(t))} f(s) ds = \frac{t^{p(t)+q(t)}}{\Gamma(1+p(t)+q(t))}.$$

Thus, for $f(t) \equiv 1$, we can obtain

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) = I_{0+}^{p(t)+q(t)} f(t),$$

if

$$\int_0^1 \frac{t^{q(tr)}}{\Gamma(1+q(tr))} (1-r)^{p(t)-1} r^{q(tr)} dr = \frac{\beta(p(t), q(t)+1)}{\Gamma(q(t)+1)} t^{q(t)}.$$

However, we can't assert that the above equality is true.
What we can get is

$$\frac{1}{\Gamma(1+q(t))} \int_0^1 t^{q(t)} r^{q(t)} (1-r)^{p(t)-1} dr = \frac{\beta(p(t), q(t)+1)}{\Gamma(q(t)+1)} t^{q(t)}.$$

Therefore, for general functions $p(t), q(t)$ and $f(t)$, we have

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) \neq I_{0+}^{p(t)+q(t)} f(t). \tag{4}$$

In particular,for general functions $0 < p(t) < 1$ and $f(t)$, we have

$$I_{0+}^{p(t)} I_{0+}^{1-p(t)} f(t) \neq I_{0+}^{p(t)+1-p(t)} f(t) = I_{0+}^{1} f(t).$$

The following example illustrates that inequality (4) is valid.

Example 1. *Let $p(t) = t,\ f(t) = 1,\quad 0 \le t \le 6$ and the function $q(t)$ be defined by*

$$q(t) = \begin{cases} \frac{t}{2}, & 0 \le t \le 2, \\ 1, & 2 < t \le 3, \\ \frac{t}{3}, & 3 < t \le 6. \end{cases}$$

Now, we calculate $I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3}$ and $I_{0+}^{p(t)+q(t)} f(t)|_{t=3}$ defined by (3).

$$\begin{aligned}
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t) &= \int_0^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} f(\tau) d\tau ds \\
&= \int_0^2 \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds + \\
&\qquad \int_2^t \frac{(t-s)^{p(t)-1}}{\Gamma(p(t))} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds \\
&= \int_0^2 \frac{(t-s)^{t-1}}{\Gamma(t)} \int_0^s \frac{(s-\tau)^{\frac{s}{2}-1}}{\Gamma(\frac{s}{2})} d\tau ds + \\
&\qquad \int_2^t \frac{(t-s)^{t-1}}{\Gamma(t)} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds \\
&= \int_0^2 \frac{(t-s)^{t-1} s^{\frac{s}{2}}}{\Gamma(t)\Gamma(1+\frac{s}{2})} ds + \int_2^t \frac{(t-s)^{t-1}}{\Gamma(t)} \int_0^s \frac{(s-\tau)^{q(s)-1}}{\Gamma(q(s))} d\tau ds.
\end{aligned}$$

So, we have

$$\begin{aligned}
I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} &= \int_0^2 \frac{(3-s)^{3-1} s^{\frac{s}{2}}}{\Gamma(3)\Gamma(1+\frac{s}{2})} ds + \int_2^3 \frac{(3-s)^{3-1}}{\Gamma(3)} \int_0^s \frac{(s-\tau)^{1-1}}{\Gamma(1)} d\tau ds \\
&= \int_0^2 \frac{(3-s)^2 s^{\frac{s}{2}}}{\Gamma(3)\Gamma(1+\frac{s}{2})} ds + \int_2^3 \frac{(3-s)^2 s}{\Gamma(3)} ds \\
&= \int_0^2 \frac{(3-s)^2 s^{\frac{s}{2}}}{\Gamma(3)\Gamma(1+\frac{s}{2})} ds + \frac{3}{8} \\
&\approx 4.660 + 0.375 \\
&= 5.035.
\end{aligned}$$

On the other hand, we get

$$\begin{aligned}
I_{0+}^{p(t)+q(t)} f(t)|_{t=3} &= \int_0^3 \frac{(3-s)^{p(3)+q(3)-1}}{\Gamma(p(3)+q(3))} f(s) ds \\
&= \int_0^3 \frac{(3-s)^{3+1-1}}{\Gamma(3+1)} ds
\end{aligned}$$

$$= \frac{27}{8} = 3.375.$$

Obviously,

$$I_{0+}^{p(t)} I_{0+}^{q(t)} f(t)|_{t=3} \neq I_{0+}^{p(t)+q(t)} f(t)|_{t=3}.$$

Now, we can conclude that Propositions 1-4 do not hold for $D_{0+}^{q(t)}$ and $I_{0+}^{q(t)}$.

So, one can not transform a differential equation of variable order into an equivalent interval equation without the Propositions $1, 2, 3$ and 4. It is a difficulty for us in dealing with the boundary value problems for differential equations of variable order. Since the equations described by the variable order derivatives are highly complex, difficult to handle analytically, it is necessary and significant to investigate their solutions.

In [22], the authors study the Cauchy problem for variable order differential equations with a piecewise constant order function. In this paper, we study the boundary value problem (1) for variable order differential equations with a piecewise constant order function $q(t)$.

The paper is organized as follows. In Section 2, we provide some necessary definitions associated with the problem (1). In Section 3, we establish the existence of solutions for (1) by using the Schauder fixed point theorem. In Section 4, some examples are presented to illustrate the main results.

2. Preliminaries

For the convenience of the reader, we present here some necessary definitions that will be used to prove our main theorems.

Definition 1. *A generalized interval is a subset I of R which is either an interval (i.e. a set of the form $[a,b], (a,b), [a,b)$ or $(a,b]$); a point $\{a\}$; or the empty set $\emptyset$.*

Definition 2. *Let I be a generalized interval. A partition of I is a finite set P of generalized intervals contained in I, such that every x in I lies in exactly one of the generalized intervals J in P.*

Example 2. *The set $P = \big\{\{1\}, (1,6), [6,7), \{7\}, (7,8]\big\}$ of generalized intervals is a partition of $[1,8]$.*

Definition 3. *[22, 23] Let I be a generalized interval, $f : I \to R$ be a function, and P be a partition of I. f is said to be piecewise constant with respect to P if for every $J \in P$, f is constant on J.*

Example 3. *The function $f : [1, 6] \to R$ defined by*

$$f(x) = \begin{cases} 3, & 1 \le x < 3, \\ 4, & x = 3, \\ 5, & 3 < x < 6, \\ 2, & x = 6, \end{cases}$$

is piecewise constant with respect to the partition $\{[1, 3), \{3\}, (3, 6), \{6\}\}$ *of* $[1, 6]$.

3. Main Results

We need the following assumptions:

(H_1) Let $P = \{[0, T_1], (T_1, T_2], (T_2, T_3], \cdots, (T_{N-1}, T]\}$ be a partition of the interval $[0, T]$, and let $q(t) : [0, T] \to (1, 2]$ be a piecewise constant function with respect to P, i.e.

$$q(t) = \sum_{k=1}^{N} q_k I_k(t) = \begin{cases} q_1, & 0 \le t \le T_1, \\ q_2, & T_1 < t \le T_2, \\ \cdots, & \cdots, \\ q_N, & T_{N-1} < t \le T_N = T, \end{cases} \tag{5}$$

where $1 < q_k \le 2$ $(k = 1, 2, \cdots, N)$ are constants, and I_k is the indicator of the interval $[T_{k-1}, T_k]$, $k = 1, 2, \cdots, N$ (here $T_0 = 0, T_N = T$), that is, $I_k(t) = 1$ for $t \in [T_{k-1}, T_k]$ and $I_k(t) = 0$ for elsewhere.

(H_2) Let $t^r f : [0, T] \times R \to R$ be a continuous function $(0 \le r < 1)$.

(H_3) There exist constants $c_1 > 0, c_2 > 0$, $0 < \gamma < 1$ such that

$$t^r |f(t, x)| \le c_1 + c_2 |x|^\gamma, 0 \le t \le T, x \in R.$$

(H_4) There exist constants $d_1 > 0, d_2 > 0$ satisfying

$$d_2 < \frac{\Gamma_q}{4\Gamma(1 - r)T^*},$$

such that

$$t^r |f(t, x)| \le d_1 + d_2 |x|, \quad 0 \le t \le T, x \in R,$$

where

$$T^* = \max\{T^{1-r}, T^{2-r}\}, \Gamma_q = \min\{\Gamma(1 - r + q_1), \Gamma(1 - r + q_2), \cdots, \Gamma(1 - r + q_N)\}.$$

(H_5) There exist constants $e_1 \geq 0, e_2 > 0,\ \mu > 1$ satisfying

$$\frac{4e_1\Gamma(1-r)T^*}{\Gamma_q} < \left(\frac{\Gamma_q}{4e_2\Gamma(1-r)T^*}\right)^{\frac{1}{\mu-1}},$$

such that

$$t^r|f(t,x)| \leq e_1 + e_2|x|^\mu, \quad 0 \leq t \leq T, x \in R.$$

In order to obtain our main results, we first carry out essential analysis of (1).
According to (H_1), we have

$$\int_0^t \frac{(t-s)^{1-q(t)}}{\Gamma(2-q(t))}x(s)ds = \sum_{k=1}^N I_k(t)\int_0^t \frac{(t-s)^{1-q_k}}{\Gamma(2-q_k)}x(s)ds.$$

Then, the equation (1) can be written as

$$\frac{d^2}{dt^2}\sum_{k=1}^N I_k(t)\int_0^t \frac{(t-s)^{1-q_k}}{\Gamma(2-q_k)}x(s)ds = f(t,x(t)), \quad 0 < t < T. \tag{6}$$

Moreover, equation (6) in the interval $[0, T_1]$ can be written as

$$D_{0+}^{q_1}x(t) = f(t,x(t)), \ \ 0 < t \leq T_1. \tag{7}$$

Equation (6) in the interval $(T_1, T_2]$ can be written as

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds = f(t,x), \ \ T_1 < t \leq T_2, \tag{8}$$

and equation (6) in the interval $(T_2, T_3]$ can be written as

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_3}}{\Gamma(2-q_3)}x(s)ds = f(t,x), \quad T_2 < t \leq T_3. \tag{9}$$

In the same way, equation (6) in the interval $(T_i, T_{i+1}],\ i = 3, 4, \cdots, N-2$ can be written as

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_{i+1}}}{\Gamma(2-q_{i+1})}x(s)ds = f(t,x), \ \ T_i < t \leq T_{i+1}. \tag{10}$$

As for the last interval $(T_{N-1}, T]$, similar to above argument, equation (6) can be written as

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_N}}{\Gamma(2-q_N)}x(s)ds = f(t,x), \ \ T_{N-1} < t < T. \tag{11}$$

Now, we present definition of solution to problem (1), which is fundamental in our work.

Definition 4. *We say the boundary value problem* (1) *has a solution, if there exist functions* $u_i(t), i = 1, 2, \cdots, N$ *such that* $u_1 \in C[0, T_1]$ *satisfying equation* (7) *and* $u_1(0) = u_1(T_1) = 0$*;* $u_2 \in C[0, T_2]$ *satisfying equation* (8) *and* $u_2(0) = u_2(T_2) = 0$*;* $u_3 \in C[0, T_3]$ *satisfying equation* (9) *and* $u_3(0) = u_3(T_3) = 0$*;* $u_i \in C[0, T_i]$ *satisfying equation* (10) *and* $u_i(0) = u_i(T_i) = 0 (i = 4, 5, \cdots, N-1)$*;* $u_N \in C[0, T]$ *satisfying equation* (11) *and* $u_N(0) = u_N(T) = 0$.

Theorem 1. *Assume that conditions* $(H_1) - (H_3)$ *hold. Then problem* (1) *has one solution.*

Proof. According to above analysis, equation of problem (1) can be written as equation (6). Equation (6) in the interval $[0, T_1]$ can be written as

$$D_{0+}^{q_1} x(t) = f(t, x(t)), \quad 0 < t \leq T_1.$$

Applying the operator $I_{0+}^{q_1}$ to both sides of the above equation, by Propositions $1-4$, we have

$$x(t) = c_1 t^{q_1-1} + c_2 t^{q_1-2} + \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} f(s, x(s)) ds, \quad 0 \leq t \leq T_1.$$

By $x(0) = 0$ and the assumption on function f, we get $c_2 = 0$. Let $x(t)$ satisfy $x(T_1) = 0$. Then we get $c_1 = -I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1}$. We have

$$x(t) = -I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1} t^{q_1-1} + I_{0+}^{q_1} f(t, x(t)), \quad 0 \leq t \leq T_1.$$

Define the operator $T : C[0, T_1] \to C[0, T_1]$ by

$$Tx(t) = -I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1} t^{q_1-1} + I_{0+}^{q_1} f(t, x(t)), \quad 0 \leq t \leq T_1. \tag{12}$$

It follows from the properties of fractional integrals and assumptions on function f that the operator $T : C[0, T_1] \to C[0, T_1]$ defined in (12) is well defined. By the standard arguments, we can verify that $T : C[0, T_1] \to C[0, T_1]$ is a completely continuous operator.

Let $\Omega_1 = \{x \in C[0, T_1] : \|x\| \leq R_1\}$ be a bounded closed convex subset of $C[0, T_1]$, where

$$R_1 = \max\left\{\frac{4c_1 T_1^{q_1-r}\Gamma(1-r)}{\Gamma(1+q_1-r)}, \left(\frac{4c_2 T_1^{q_1-r}\Gamma(1-r)}{\Gamma(1+q_1-r)}\right)^{\frac{1}{1-\gamma}}\right\}.$$

For $x \in \Omega_1$ and by (H_3), we have

$$|Tx(t)| \leq \quad \frac{T_1^{1-q_1} t^{q_1-1}}{\Gamma(q_1)} \int_0^{T_1} (T_1 - s)^{q_1-1} |f(s, x(s))| ds$$

$$
\begin{aligned}
&+\frac{1}{\Gamma(q_1)}\int_0^t (t-s)^{q_1-1}|f(s,x(s))|ds \\
\leq\ & \frac{2}{\Gamma(q_1)}\int_0^{T_1} (T_1-s)^{q_1-1}|f(s,x(s))|ds \\
\leq\ & \frac{2}{\Gamma(q_1)}\int_0^{T_1} (T_1-s)^{q_1-1}s^{-r}(c_1+c_2|x(s)|^\gamma)ds \\
\leq\ & \frac{2\Gamma(1-r)T_1^{q_1-r}}{\Gamma(1+q_1-r)}(c_1+c_2\|x\|^\gamma) \\
\leq\ & \frac{2\Gamma(1-r)T_1^{q_1-r}}{\Gamma(1+q_1-r)}(c_1+c_2R_1R_1^{\gamma-1}) \\
\leq\ & \frac{R_1}{2}+\frac{R_1}{2}=R_1,
\end{aligned}
$$

which means that $T(\Omega_1)\subseteq\Omega_1$. Then the Schauder fixed point theorem assures that the operator T has one fixed point $x_1(t)\in\Omega_1$, which is a solution of equation (7).

Also, we have obtained that the equation (6) in the interval $(T_1, T_2]$ can be written as (8). To consider the existence of solution to (8), we may discuss the following equation defined on interval $[0, T_2]$:

$$
\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds = D_{0+}^{q_2}x(t) = f(t,x), \quad 0<t\leq T_2. \tag{13}
$$

It is clear that if function $x(t)\in C[0,T_2]$ satisfies equation (13), then $x(t)$ must satisfy equation (8). In fact, if $x\in C[0,T_2]$ with $x(0)=x(T_2)=0$ is a solution of equation (13), then

$$
\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds = f(t,x), \quad 0<t\leq T_2.
$$

As a result, we have $x(t)\in C[0,T_2]$ with $x(0)=x(T_2)=0$ satisfying the equation

$$
\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds = f(t,x), \quad T_1\leq t\leq T_2,
$$

which means the function $x(t)\in C[0,T_2]$ with $x(0)=x(T_2)=0$ is a solution of equation (8).

Based on this fact, we will consider the existence of solution to equation (13) with the conditions $x(0) = x(T_2) = 0$. Applying operator $I^{q_2}_{0+}$ to both sides of (13) and by Propositions $1-4$, we have

$$x(t) = c_1 t^{q_2-1} + c_2 t^{q_2-2} + \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} f(s, x(s))ds, \quad 0 \le t \le T_2. \quad (14)$$

By $x(0) = 0 = x(T_2)$, we have $c_2 = 0$ and $c_1 = -I^{q_2}_{0+} f(T_2, x) T_2^{1-q_2}$.

Define the operator $T : C[0, T_2] \to C[0, T_2]$ by

$$Tx(t) = -I^{q_2}_{0+} f(T_2, x)) T_2^{1-q_2} t^{q_2-1} + \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} f(s, x(s))ds.$$

It follows from the continuity of functions $t^r f(t, x(t))$ that the operator $T : C[0, T_2] \to C[0, T_2]$ is well defined. We note that $T : C[0, T_2] \to C[0, T_2]$ is a completely continuous operator.

Let $\Omega_2 = \{x \in C[0, T_2] : \|x\| \le R_2\}$ be a bounded closed convex subset of $C[0, T_2]$, where

$$R_2 = \max\left\{\frac{4c_1\Gamma(1-r)T_2^{q_2-r}}{\Gamma(1+q_2-r)}, \left(\frac{4c_2\Gamma(1-r)T_2^{q_2-r}}{\Gamma(1+q_2-r)}\right)^{\frac{1}{1-\gamma}}\right\}.$$

For $x \in \Omega_2$, by (H_3), we get

$$\begin{aligned}
|Tx(t)| \le \quad & \frac{T_2^{1-q_2} t^{q_2-1}}{\Gamma(q_2)} \int_0^{T_2} (T_2-s)^{q_2-1} |f(s, x(s))| ds \\
& + \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} |f(s, x(s))| ds \\
\le \quad & \frac{2}{\Gamma(q_2)} \int_0^{T_2} (T_2-s)^{q_2-1} |f(s, x(s))| ds \\
\le \quad & \frac{2}{\Gamma(q_2)} \int_0^{T_2} (T_2-s)^{q_2-1} s^{-r} (c_1 + c_2 |x(s)|^{\gamma}) ds \\
\le \quad & \frac{2\Gamma(1-r)T_2^{q_2-r}}{\Gamma(1+q_2-r)} (c_1 + c_2 R_2 R_2^{\gamma-1}) \\
\le \quad & \frac{R_2}{2} + \frac{R_2}{2} = R_2,
\end{aligned}$$

which means that $T(\Omega_2) \subseteq \Omega_2$. Then the Schauder fixed point theorem assures that the operator T has one fixed point $x_2(t) \in \Omega_2$, that is

$$x_2(t) = -I_{0+}^{q_2} f(T_2, x_2) T_2^{1-q_2} t^{q_2-1} + \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} f(s, x_2(s)) ds, 0 \le t \le T_2. \tag{15}$$

Applying operator $D_{0+}^{q_2}$ to both sides of (15), by Proposition 2, we can obtain that

$$D_{0+}^{q_2} x_2(t) = f(t, x_2), \quad 0 < t \le T_2,$$

that is, $x_2(t)$ satisfies the following equation

$$\frac{d^2}{dt^2} \frac{1}{\Gamma(2-q_2)} \int_0^t (t-s)^{1-q_2} x_2(s) ds = f(t, x_2), \quad 0 < t \le T_2.$$

From the previous arguments, we know that $x_2(t) \in \Omega_2$ satisfies equation (8).

Again, we know that the equation (6) in the interval $(T_2, T_3]$ can be written as (9). In order to consider the existence of solution to equation (9), we may investigate the following equation defined on interval $[0, T_3]$:

$$\frac{d^2}{dt^2} \int_0^t \frac{(t-s)^{1-q_3}}{\Gamma(2-q_3)} x(s) ds = D_{0+}^{q_3} x(t) = f(t, x), \quad 0 < t \le T_3. \tag{16}$$

From the previous arguments, we note that if function $x(t) \in C[0, T_3]$ satisfies equation (16), then $x(t)$ must satisfy equation (9). Now, we will consider the existence of solution to equation (16)with the boundary condition $x(0) = 0 = x(T_3)$.

Applying the operator $I_{0+}^{q_3}$ to both sides of (16), by Propositions $1-4$, we get

$$x(t) = c_1 t^{q_3-1} + c_2 t^{q_3-2} + \frac{1}{\Gamma(q_3)} \int_0^t (t-s)^{q_3-1} f(s, x(s)) ds, \quad 0 \le t \le T_3.$$

By $x(0) = 0 = x(T_3)$, we get $c_2 = 0$ and $c_1 = -I_{0+}^{q_3} f(T_3, x) T_3^{1-q_3}$.

Define the operator $T : C[0, T_3] \to C[0, T_3]$ by

$$Tx(t) = -I_{0+}^{q_3} f(T_3, x) T_3^{1-q_3} t^{q_3-1} + \frac{1}{\Gamma(q_3)} \int_0^t (t-s)^{q_3-1} f(s, x(s)) ds, \quad 0 \le t \le T_3.$$

It follows from the continuity of functions $t^r f(t, x(t))$ that the operator $T : [0, T_3] \to C[0, T_3]$ is well defined. Furthermore, we can obtain that $T : C[0, T_3] \to C[0, T_3]$ is completely continuous.

Let $\Omega_3 = \{x \in C[0,T_3] : \|x\| \leq R_3\}$ be a bounded closed convex subset of $C[0,T_3]$, where

$$R_3 = \max\left\{\frac{4c_1\Gamma(1-r)T_3^{q_3-r}}{\Gamma(1+q_3-r)}, \left(\frac{4c_2\Gamma(1-r)T_3^{q_3-r}}{\Gamma(1+q_3-r)}\right)^{\frac{1}{1-\gamma}}\right\}.$$

For $x \in \Omega_3$, by (H_3), we have

$$\begin{aligned}|Tx(t)| \leq & \frac{2}{\Gamma(q_3)}\int_0^{T_3}(T_3-s)^{q_3-1}|f(s,x(s))|ds\\ \leq & \frac{2}{\Gamma(q_3)}\int_0^{T_3}(T_3-s)^{q_3-1}s^{-r}(c_1+c_2|x(s)|^{\gamma})ds\\ \leq & \frac{2\Gamma(1-r)T_3^{q_3-r}}{\Gamma(1+q_3-r)}(c_1+c_2R_3R_3^{\gamma-1})\\ \leq & \frac{R_3}{2}+\frac{R_3}{2}=R_3,\end{aligned}$$

which means that $T(\Omega_3) \subseteq \Omega_3$. Then the Schauder fixed point theorem assures that the operator T has one fixed point $x_3(t) \in \Omega_3$, that is,

$$x_3(t) = -I_{0+}^{q_3}f(T_3,x_3)T_3^{1-q_3}t^{q_3-1} + \frac{1}{\Gamma(q_3)}\int_0^t(t-s)^{q_3-1}f(s,x_3(s))ds. \quad 0 \leq t \leq T_3, \tag{17}$$

Applying the operator $D_{0+}^{q_3}$ to both sides of (17), by Proposition 2, we can obtain that

$$D_{0+}^{q_3}x_3(t) = f(t,x_3), \quad 0 < t \leq T_3,$$

that is, $x_3(t)$ satisfies the following equation

$$\frac{d^2}{dt^2}\frac{1}{\Gamma(2-q_3)}\int_0^t(t-s)^{1-q_3}x_3(s)ds = f(t,x_3(t)), \quad 0 < t \leq T_3.$$

From the previous arguments, we know that $x_3(t) \in \Omega_3$ satisfies equation (9).

In a similar way, we can obtain that the equation (10) defined on $[T_{i-1},T_i]$ has solution $x_i(t) \in \Omega_i$ with $x_i(0) = x_i(T_i) = 0$, $i = 4,5,\cdots,N-1$, where

$$R_i = \max\left\{\frac{4c_1\Gamma(1-r)T_i^{q_i-r}}{\Gamma(1+q_i-r)}, \left(\frac{4c_2\Gamma(1-r)T_i^{q_i-r}}{\Gamma(1+q_i-r)}\right)^{\frac{1}{1-\gamma}}\right\}.$$

Similar to the above argument, in order to consider the existence of solution to equation (11), we may consider the following equation defined on interval $[0,T]$:

$$\frac{d^2}{dt^2}\int_0^t\frac{(t-s)^{1-q_N}}{\Gamma(2-q_N)}x(s)ds = D_{0+}^{q_N}x(t) = f(t,x), \quad 0 < t < T. \tag{18}$$

From the previous arguments, we know, if function $x(t) \in C[0,T]$ satisfies equation (18), then $x(t)$ must satisfy equation (11). Now, we consider the existence of solution to equation (18) with boundary conditions $x(0)=0, x(T)=0$. Applying the operator $I_{0+}^{q_N}$ to both sides of (18), by Propositions $1-4$, we have

$$x(t) = c_1 t^{q_N-1} + c_2 t^{q_N-2} + \frac{1}{\Gamma(q_N)} \int_0^t (t-s)^{q_N-1} f(s,x(s))ds, \quad 0 \le t \le T.$$

By conditions $x(0)=0, x(T)=0$, we obtain $c_2=0$ and $c_1 = -I_{0+}^{q_N} f(T,x) T^{1-q_N}$.

Define the operator $T : C[0,T] \to C[0,T]$ by

$$Tx(t) = -I_{0+}^{q_N} f(T,x) T^{1-q_N} t^{q_N-1} + \frac{1}{\Gamma(q_N)} \int_0^t (t-s)^{q_N-1} f(s,x(s))ds.$$

It follows from the continuity of functions $t^r f(t,x(t))$ that the operator $T : [0,T] \to C[0,T]$ is well defined. By the standard arguments, we can obtain that $T : C[0,T] \to C[0,T]$ is completely continuous.

Let $\Omega_N = \{x \in C[0,T] : \|x\| \le R_N\}$ be a bounded closed convex subset of $C[0,T]$, where

$$R_N = \max\left\{ \frac{4c_1 \Gamma(1-r) T^{q_N-r}}{\Gamma(1+q_N-r)}, \left(\frac{4c_2 \Gamma(1-r) T^{q_N-r}}{\Gamma(1+q_N-r)} \right)^{\frac{1}{1-\gamma}} \right\}.$$

For $x(t) \in \Omega_N$, by (H_3), we have

$$\begin{aligned} |Tx(t)| \le & \frac{2\Gamma(1-r) T^{q_N-r}}{\Gamma(1+q_N-r)} (c_1 + c_2 R_N R_N^{\gamma-1}) \\ \le & \frac{R_N}{2} + \frac{R_N}{2} = R_N, \end{aligned}$$

which means that $T(\Omega_N) \subseteq \Omega_N$. Then the Schauder fixed point theorem assures that the operator T has one fixed point $x_N(t) \in \Omega_N$, that is,

$$x_N(t) = -I_{0+}^{q_N} f(T,x_N) T^{1-q_N} t^{q_N-1} + \frac{1}{\Gamma(q_N)} \int_0^t (t-s)^{q_N-1} f(s,x_N(s))ds, 0 \le t \le T.$$

Applying operator $D_{0+}^{q_N}$ to both sides of the above equation, by Proposition 2, we can obtain that

$$D_{0+}^{q_N} x_N(t) = f(t, x_N(t)), \quad 0 < t \le T,$$

that is, $x_N(t)$ satisfies the following equation

$$\frac{d^2}{dt^2} \frac{1}{\Gamma(2-q_N)} \int_0^t (t-s)^{1-q_N} x_N(s) ds = f(t, x_N(t)), \quad 0 < t \le T.$$

From the previous arguments, we know that $x_N(t) \in \Omega_N$ satisfies equation (11).

As a result, we know that the problem (1) has a solution. Thus we complete the proof. ◄

Using similar arguments, we can obtain the following results.

Theorem 2. *Assume the conditions* $(H_1), (H_2)$ *and* (H_4) *hold. Then problem* (1) *has one solution.*

Proof. The proof is similar to that of Theorem 1. By (H_1), we obtain that the equation of problem (1) can be written as (6). And, (6) in the interval $[0, T_1]$ can be written as (7).
Applying the operator $I_{0+}^{q_1}$ to both sides of (8), by Propositions $1-4$, we have that

$$x(t) = c_1 t^{q_1-1} + c_2 t^{q_1-2} + \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1-1} f(s, x(s)) ds, \quad 0 \le t \le T_1.$$

We will consider the existence of solution $x(t)$ defined on $[0, T_1]$. By the boundary condition $x(0) = 0$ and the assumption on function f, we get $c_2 = 0$. Setting $x(T_1) = 0$, we have $c_1 = -I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1}$. Then, we have

$$x(t) = -I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1} t^{q_1-1} + I_{0+}^{q_1} f(t, x), \quad 0 \le t \le T_1.$$

Define the operator $T : C[0, T_1] \to C[0, T_1]$ by

$$Tx(t) = -I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1} t^{q_1-1} + I_{0+}^{q_1} f(t, x), \quad 0 \le t \le T_1.$$

It follows from the properties of fractional integrals and assumptions on function f that the operator $T : C[0, T_1] \to C[0, T_1]$ is well defined. By the standard arguments, we can verify that $T : C[0, T_1] \to C[0, T_1]$ is a completely continuous operator.

Let $\Omega_1 = \{x \in C[0, T_1] : \|x\| \le R_1\}$ be a bounded closed convex subset of $C[0, T_1]$, where

$$R_1 > \frac{4 d_1 \Gamma(1-r) T^*}{\Gamma_q}.$$

For $x \in \Omega_1$, by (H_4), we have

$$\begin{aligned} |Tx(t)| &\le \frac{2\Gamma(1-r) T_1^{q_1-r}}{\Gamma(1+q_1-r)} (d_1 + d_2 R_1) \\ &\le \frac{2\Gamma(1-r) T^{q_1-r}}{\Gamma_q} (d_1 + d_2 R_1) \end{aligned}$$

$$\begin{aligned} &\le \frac{2\Gamma(1-r)T^*}{\Gamma_q}(d_1+d_2R_1) \\ &\le \frac{R_1}{2}+\frac{R_1}{2}=R_1, \end{aligned}$$

which means that $T(\Omega_1) \subseteq \Omega_1$. Then the Schauder fixed point theorem assures that the operator T has one fixed point $x_1(t) \in \Omega_1$, which is one solution of equation (7).

Equation (6) in the interval $(T_1, T_2]$ can be written as (8). In order to discuss the existence of solution to equation (8), we consider the following equation defined on interval $[0, T_2]$

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds = D_{0+}^{q_2}x(t) = f(t,x(t)), \quad 0 < t \le T_2. \tag{19}$$

We see that if function $x(t) \in C[0, T_2]$ satisfies equation (19), then $x(t)$ must satisfy equation (8). In fact, if $x \in C[0, T_2]$ with $x(0) = x(T_2) = 0$ is a solution of equation (19), then

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds = f(t,x), \quad 0 < t \le T_2.$$

As a result, we obtain that $x(t) \in C[0, T_2]$ with $x(0) = x(T_2) = 0$ satisfies

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)}x(s)ds = f(t,x), \quad T_1 \le t \le T_2,$$

which means that $x(t) \in C[0, T_2]$ with $x(0) = x(T_2) = 0$ is a solution of equation (8).

Next, we will consider the existence of solution to equation (3.15) with conditions $x(0) = 0 = x(T_2)$.
Applying operator $I_{0+}^{q_2}$ to both sides of (19), by Propositions $1-4$, we have

$$x(t) = c_1t^{q_2-1} + c_2t^{q_2-2} + \frac{1}{\Gamma(q_2)}\int_0^t (t-s)^{q_2-1}f(s,x(s))ds, \quad 0 \le t \le T_2.$$

By conditions $x(0) = 0 = x(T_2) = 0$, we have $c_2 = 0$ and $c_1 = -I_{0+}^{q_2}f(T_2,x)T_2^{1-q_2}$.

Define the operator $T : C[0, T_2] \to C[0, T_2]$ by

$$\begin{aligned} Tx(t) = &-I_{0+}^{q_2}f(T_2,x))T_2^{1-q_2}t^{q_2-1} + \\ &\frac{1}{\Gamma(q_2)}\int_0^t (t-s)^{q_2-1}f(s,x(s))ds, \quad 0 \le t \le T_2. \end{aligned}$$

It follows from the continuity of functions $t^r f(t, x(t))$ that the operator $T : C[0, T_2] \to C[0, T_2]$ is well defined. Moreover, we can obtain that $T : C[0, T_2] \to C[0, T_2]$ is a completely continuous operator.

Let $\Omega_2 = \{x \in C[0, T_2] : \|x\| \le R_2\}$ be a bounded closed convex subset of $C[0, T_2]$, where

$$R_2 > \frac{4d_1\Gamma(1-r)T^*}{\Gamma_q}.$$

For $x \in \Omega_2$, by (H_4), we have

$$\begin{aligned}|Tx(t)| \le & \frac{2\Gamma(1-r)T_2^{q_2-r}}{\Gamma(1+q_2-r)}(d_1 + d_2R_2)\\ \le & \frac{2\Gamma(1-r)T^*}{\Gamma_q}(d_1 + d_2R_2)\\ \le & \frac{R_2}{2} + \frac{R_2}{2} = R_2,\end{aligned}$$

which means that $T(\Omega_2) \subseteq \Omega_2$. Then the Schauder fixed point theorem assures that the operator T has one fixed point $x_2(t) \in \Omega_2$. By the same arguments as in the proof of Theorem 1, we obtain that $x_2(t) \in \Omega_2$ satisfies equation (8).

In a similar way, we obtain that equation (10) defined on $[T_{i-1}, T_i]$ has a solution $x_i(t) \in C[0, T_i]$ with $x_i(0) = x_i(T_i) = 0$, $i = 3, 4, \cdots, N-1$.

Equation (6) in the interval $(T_{N-1}, T]$ can be written as (11). In order to consider the existence of solution to equation (11), we may consider the following equation defined on interval $[0, T]$:

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_N}}{\Gamma(2-q_N)}x(s)ds = D_{0+}^{q_N}x(t) = f(t, x(t)), \quad 0 < t < T. \tag{20}$$

From the previous arguments, we know, if function $x(t) \in C[0, T]$ satisfies equation (20), then $x(t)$ must satisfy equation (11). Based on this fact, we will consider existence of solution to equation (20) with boundary conditions $x(0) = 0, x(T) = 0$.

Applying the operator $I_{0+}^{q_N}$ to both sides of (3.16), by Propositions $1-4$, we have

$$x(t) = c_1 t^{q_N-1} + c_2 t^{q_N-2} + \frac{1}{\Gamma(q_N)}\int_0^t (t-s)^{q_N-1} f(s, x(s))ds, \quad 0 \le t \le T.$$

By $x(0) = 0, x(T) = 0$, we have $c_2 = 0$ and $c_1 = -I_{0+}^{q_N} f(T, x)T^{1-q_N}$.

Define the operator $T : C[0, T] \to C[0, T]$ by

$$Tx(t) = -I_{0+}^{q_N} f(T, x)T^{1-q_N}t^{q_N-1} + \frac{1}{\Gamma(q_N)}\int_0^t (t-s)^{q_N-1} f(s, x(s))ds, \quad 0 \le t \le T.$$

It follows from the continuity of functions $t^r f(t, x(t))$ that the operator $T : [0, T] \to C[0, T]$ is well defined. By the standard arguments, we can obtain that $T : C[0, T] \to C[0, T]$ is completely continuous.

Let $\Omega_N = \{x \in C[0, T] : \|x\| \leq R_N\}$ be a bounded closed convex subset of $C[0, T]$, where

$$R_N > \frac{4d_1\Gamma(1-r)T^*}{\Gamma_q}.$$

For $x \in \Omega_N$, by (H_4), we have

$$\begin{aligned} |Tx(t)| \leq & \frac{2\Gamma(1-r)T^{q_N - r}}{\Gamma(1+q_N-r)}(d_1 + d_2R_N) \\ \leq & \frac{2\Gamma(1-r)T^*}{\Gamma_q}(d_1 + d_2R_N) \\ \leq & \frac{R_N}{2} + \frac{R_N}{2} = R_N, \end{aligned}$$

which means that $T(\Omega_N) \subseteq \Omega_N$. Then the Schauder fixed point theorem assures that the operator T has one fixed point $x_N(t) \in \Omega_N$. By the same arguments as in the proof of Theorem 1, we obtain that $x_N(t) \in \Omega_N$ satisfies equation (11).

As a result, we conclude that the problem (1) has a solution. Thus we complete this proof. ◄

Theorem 3. *Assume that conditions $(H_1), (H_2)$ and (H_5) hold. Then problem (1) has one solution.*

Proof. This proof is also similar to that of Theorem 1. Equation of (1) can be written as (6). And, equation (6) in the interval $[0, T_1]$ can be written as (6). Applying operator $I_{0+}^{q_1}$ to both sides of (7), by Propositions $1 - 4$, we have

$$x(t) = c_1 t^{q_1 - 1} + c_2 t^{q_1 - 2} + \frac{1}{\Gamma(q_1)} \int_0^t (t-s)^{q_1 - 1} f(s, x(s)) ds, \quad 0 \leq t \leq T_1.$$

We will consider the existence of solution $x(t)$ defined on $[0, T_1]$. By the boundary condition $x(0) = 0$ and assumption on function f, we get $c_2 = 0$. Let $x(T_1) = 0$. Then, $c_1 = -I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1}$.
As a result, we have

$$x(t) = -I_{0+}^{q_1} f(T_1, x) T_1^{1-q_1} t^{q_1 - 1} + I_{0+}^{q_1} f(t, x), \quad 0 \leq t \leq T_1.$$

Define the operator $T : C[0,T_1] \to C[0,T_1]$ by

$$Tx(t) = -I_{0+}^{q_1} f(T_1,x) T_1^{1-q_1} t^{q_1-1} + I_{0+}^{q_1} f(t,x), \quad 0 \le t \le T_1.$$

It follows from the properties of fractional integrals and assumptions on function f that the operator $T : C[0,T_1] \to C[0,T_1]$ is well defined. By the standard arguments, we can verify that $T : C[0,T_1] \to C[0,T_1]$ is a completely continuous operator.

Let $\Omega_1 = \{x \in C[0,T_1] : \|x\| \le R_1\}$ be a bounded closed convex subset of $C[0,T_1]$, where

$$\frac{4e_1\Gamma(1-r)T^*}{\Gamma_q} < R_1 < \left(\frac{\Gamma_q}{4e_2\Gamma(1-r)T^*}\right)^{\frac{1}{\mu-1}}.$$

For $x \in \Omega_1$, by (H_5) , we have

$$\begin{aligned} |Tx(t)| \le & \frac{2\Gamma(1-r)T_1^{q_1-r}}{\Gamma(1+q_1-r)}(e_1 + e_2 R_1 R_1^{\mu-1}) \\ \le & \frac{2\Gamma(1-r)T^*}{\Gamma_q}(e_1 + e_2 R_1 R_1^{\mu-1}) \\ \le & \frac{R_1}{2} + \frac{R_1}{2} = R_1, \end{aligned}$$

which means that $T(\Omega_1) \subseteq \Omega_1$. Then the Schauder fixed point theorem assures that the operator T has one fixed point $x_1 \in \Omega_1$, which is one solution of equation (7).

Equation (6) in the interval $(T_1,T_2]$ can be written as (8). In order to consider the existence of solution to equation (8), we consider the following equation defined on interval $[0,T_2]$:

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{1-q_2}}{\Gamma(2-q_2)} x(s)ds = D_{0+}^{q_2} x(t) = f(t,x), \quad 0 < t \le T_2. \tag{21}$$

Using same arguments as in the proof of Theorem 1, we see that if function $x(t) \in C[0,T_2]$ satisfies equation (21), then $x(t)$ must satisfy equation (8). Now, we will consider the existence of solution to equation (21) with conditions $x(0) = 0, x(T_2) = 0$.

Applying operator $I_{0+}^{q_2}$ to both sides of (21), by Propositions $1-4$, we have

$$x(t) = c_1 t^{q_2-1} + c_2 t^{q_2-2} + \frac{1}{\Gamma(q_2)}\int_0^t (t-s)^{q_2-1} f(s,x(s))ds, \quad 0 \le t \le T_2.$$

By conditions $x(0) = 0, x(T_2) = 0$, we have $c_2 = 0$ and $c_1 = -I_{0+}^{q_2} f(T_2,x) T_2^{1-q_2}$.

Define the operator $T : C[0, T_2] \to C[0, T_2]$ by

$$Tx(t) = -I_{0+}^{q_2} f(T_2, x) T_2^{1-q_2} t^{q_2-1} + \frac{1}{\Gamma(q_2)} \int_0^t (t-s)^{q_2-1} f(s, x(s)) ds.$$

It follows from the continuity of functions $t^r f(t, x(t))$ that the operator $T : C[0, T_2] \to C[0, T_2]$ is well defined. Then we can obtain $T : C[0, T_2] \to C[0, T_2]$ is a completely continuous operator.

Let $\Omega_2 = \{x \in C[0, T_2] : \|x\| \le R_2\}$ be a bounded closed convex subset of $C[0, T_2]$, where

$$\frac{4e_1\Gamma(1-r)T^*}{\Gamma_q} < R_2 < \left(\frac{\Gamma_q}{4e_2\Gamma(1-r)T^*}\right)^{\frac{1}{\mu-1}}.$$

For $x(t) \in \Omega_2$, by (H_5) we get

$$\begin{aligned} |Tx(t)| \le & \quad \frac{2\Gamma(1-r)T_2^{q_2-r}}{\Gamma(1+q_2-r)}(e_1 + e_2 R_2 R_2^{\mu-1}) \\ \le & \quad \frac{2\Gamma(1-r)T^*}{\Gamma_q}(e_1 + e_2 R_2 R_2^{\mu-1}) \\ \le & \quad \frac{R_2}{2} + \frac{R_2}{2} = R_2, \end{aligned}$$

which means that $T(\Omega_2) \subseteq \Omega_2$. Then the Schauder fixed point theorem assures that the operator T has one fixed point $x_2(t) \in \Omega_2$. By the same arguments as in the proof of Theorem 1, we obtain that $x_2(t) \in \Omega_2$ satisfies equation (8).

In a similar way, we obtain that thee equation (10) defined on $[T_{i-1}, T_i]$ ($T_N = T$), has solution $x_i(t) \in \Omega_i = \{x \in C[0, T_i]; \|x\| \le R_i\}$, $i = 3, 4, \cdots, N$, where

$$\frac{4e_1\Gamma(1-r)T^*}{\Gamma_q} < R_i < \left(\frac{\Gamma_q}{4e_2\Gamma(1-r)T^*}\right)^{\frac{1}{\mu-1}},$$

Hence, problem (1) has a solution. The proof is completed. ◄

Consider the following problem

$$\begin{cases} D_{0+}^{q(t)} x(t) = b_1(t)x^\gamma + b_2(t)x^\mu + b_3(t)x, \quad 0 < t < T, \\ x(0) = 0, x(T) = 0, \end{cases} \tag{22}$$

where $0 < T < +\infty$, $q(t)$ satisfies (H_1), $0 < \gamma < 1$, $\mu > 1$, $t^r b_i(t) \in C[0, T]$.

Combining Theorems 1, 2 and 3, we have the following result.

Theorem 4. *Let* $0 < \gamma < 1$, $\mu > 1$, $t^r b_i(t) \in C[0,T], i = 1,2,3$ *with* $\max_{0\leq t\leq T} t^r|b_3(t)| < \frac{\Gamma_q}{4\Gamma(1-r)T^*}$, *and assume that the condition* (H_1) *holds. Then problem* (22) *has one solution.*

Proof. It follows from Theorems 1, 2 and 3 that two-point boundary value problems

$$\begin{cases} D_{0+}^{q(t)}x(t) = b_1(t)x^{\gamma}, & 0 < t < T, \\ x(0) = 0, x(T) = 0, \end{cases}, \begin{cases} D_{0+}^{q(t)}x(t) = b_3(t)x, & 0 < t < T, \\ x(0) = 0, x(T) = 0, \end{cases},$$

$$\begin{cases} D_{0+}^{q(t)}x(t) = b_2(t)x^{\mu}, & 0 < t < T, \\ x(0) = 0, x(T) = 0, \end{cases}$$

have one solution $u_1(t)$, $u_2(t)$ and $u_3(t)$, respectively. Based on the linearity of variable order fractional derivative $D_{0+}^{q(t)}$, we obtain that the function $u(t) = u_1(t) + u_2(t) + u_3(t)$ is one solution of the problem (22). ◀

4. Some examples

In this section, we provide several examples to demonstrate the utility of our results.

Example 4. Let us consider the following linear boundary value problem

$$\begin{cases} D_{0^+}^{q(t)}x(t) = t^{-0.4}, \quad 0 < t < 3, \\ u(0) = 0, \ u(3) = 0, \end{cases} \tag{23}$$

where

$$q(t) = \begin{cases} 1.2, & 0 \leq t \leq 1, \\ 1.5, & 1 < t \leq 2, \\ 1.8, & 2 < t \leq 3. \end{cases}$$

We see that $q(t)$ satisfies condition (H_1); $t^{0.5}f(t,x) = t^{0.1} : [0,3] \times R \to R$ is continuous. Moreover, $\max_{0\leq t\leq 3} t^{0.5}f(t,x) = \max_{0\leq t\leq 3} t^{0.1} = 3^{0.1}$, thus we can take suitable constants to verify $f(t,x) = t^{-0.4}$ satisfies conditions $(H_2) - (H_5)$. Then Theorem 1 or Theorem 2 or Theorem 3 assures problem (23) has a solution.

In fact, by the above arguments, we obtain that the equation of (23) can be divided into three expressions as follows:

$$D_{0+}^{1.2}x(t) = t^{-0.4}, \quad 0 < t \leq 1, \tag{24}$$

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{-0.5}}{\Gamma(0.5)}x(s)ds = t^{-0.4}, \quad 1 < t \le 2, \tag{25}$$

$$\frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{-0.8}}{\Gamma(0.2)}x(s)ds = t^{-0.4}, \quad 2 < t \le 3. \tag{26}$$

By [21], we can easily obtain that the problems

$$\begin{cases} D_{0+}^{1.2}x(t) = t^{-0.4}, & 0 < t \le 1, \\ x(0) = 0, x(1) = 0 \end{cases}$$

$$\begin{cases} D_{0+}^{1.5}x(t) = \frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{-0.5}}{\Gamma(0.5)}x(s)ds = t^{-0.4}, & 0 < t < 2, \\ x(0) = 0, x(2) = 0 \end{cases}$$

$$\begin{cases} D_{0+}^{1.8}x(t) = \frac{d^2}{dt^2}\int_0^t \frac{(t-s)^{-0.8}}{\Gamma(0.2)}x(s)ds = t^{-0.4}, & 0 < t < 3, \\ x(0) = 0, x(3) = 0 \end{cases}$$

have, respectively, the following solutions

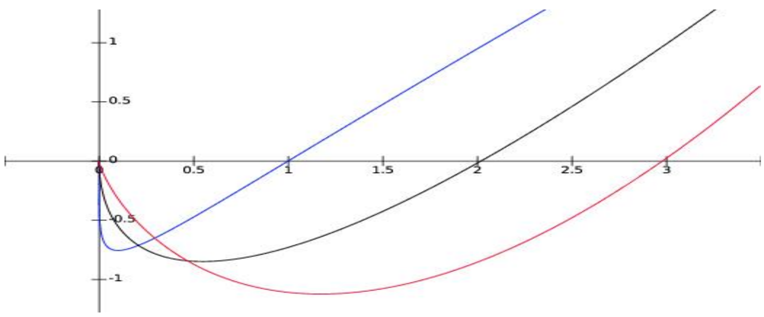
$$x_1(t) = \frac{\Gamma(0.6)}{\Gamma(1.8)}\left(t^{0.8} - t^{0.2}\right) \in C[0,1];$$

$$x_2(t) = \frac{\Gamma(0.6)}{\Gamma(2.1)}t^{1.1} - \frac{\Gamma(0.6)2^{0.6}}{\Gamma(2.1)}t^{0.5} \in C[0,2];$$

$$x_3(t) = \frac{\Gamma(0.6)}{\Gamma(2.4)}t^{1.4} - \frac{\Gamma(0.6)3^{0.6}}{\Gamma(2.4)}t^{0.8} \in C[0,3].$$

By calculation we obtain that $x_1(t)$, $x_2(t)$ and $x_3(t)$ are the solutions of (24), (25) and (26), respectively. As a result, problem (23) has a solution.

To facilitate the intuitionistic descriptions of $x_i(t), i = 1,2,3$, we give their function images. The blue curve is x_1's image; the black curve is x_2's image and the red curve is x_3's image.



Example 5. *Let us consider the following nonlinear boundary value problem*

$$\begin{cases} D_{0^+}^{q(t)} x(t) = t^{-0.3} \frac{|x|^{\frac{1}{3}}}{1+x^2}, & 0 < t < 2, \\ u(0) = 0, \ u(2) = 0, \end{cases} \tag{27}$$

where

$$q(t) = \begin{cases} 1.5, & 0 \le t \le 1, \\ 1.7, & 1 < t \le 2. \end{cases}$$

We see that $q(t)$ *satisfies condition* (H_1)*;* $t^{0.5} f(t,x) = t^{0.2} \frac{|x|^{\frac{1}{3}}}{1+x^2} : [0,2] \times R \to R$ *is continuous. Moreover, we have*

$$\max_{0 \le t \le 2} t^{0.5} |f(t,x)| = \max_{0 \le t \le 2} t^{0.2} \frac{|x|^{\frac{1}{3}}}{1+x^2} \le 2^{0.2} |x|^{\frac{1}{3}}.$$

Let $r = 0.5$*,* $c_1 = 1$*,* $c_2 = 2^{0.2}$ *and* $\gamma = \frac{1}{3}$*. We can verify that* $f(t,x) = t^{0.2} \frac{|x|^{\frac{1}{3}}}{1+x^2}$ *satisfies condition* (H_2)*. The assumption* (H_3) *of Theorem 1 is also satisfied. This suggests that the problem* (27) *has a solution by virtue of Theorem 1.*

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