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Inverse Problem of Spectral Analysis for Diffusion Operator with Nonseparated Boundary Conditions and Spectral Parameter in Boundary Condition

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Abstract. This work deals with an inverse problem for a diffusion operator with non-separated boundary conditions, one of which contains a spectral parameter. Uniqueness theorem is proved, solution algorithm is constructed and sufficient conditions for solvability of inverse problem are obtained. As spectral data, we use the spectra of two boundary value problems and some sequence of signs.

Key Words and Phrases: diffusion operator, nonseparated boundary conditions, eigenvalues, inverse problem.

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1. Introduction. Inverse problem statement

Inverse problems of spectral analysis, which require the recovery of operators from some of their given spectral data, are of great interest as they are very often used in mathematics and various branches of natural science and technical science. Many authors have considered this kind of problems for differential and difference operators. For main results in the theory of inverse problems and their applications we refer the readers to [1-8].

The process of solving inverse spectral problem usually consists of three major steps:

1) finding out which spectral data uniquely determine the operator and proving the corresponding uniqueness theorems;

2) constructing solution method and recovery algorithm for the operator using the chosen spectral data;

3) finding solvability conditions for the inverse problem.

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As main spectral data, one uses, in particular, one, two or more spectra, spectral functions, normalizing numbers, Weyl function, scattering data. The statements of inverse problems of spectral analysis differ depending on the choice of spectral data.

Consider the boundary value problem generated on the interval $[0,\pi]$ by differential diffusion equation

$$y'' + \left[\lambda^2 - 2\lambda p\left(x\right) - q\left(x\right)\right]\, y = 0 \tag{1}$$

and general boundary conditions

$$\begin{array}{l} a_{11}y\left(0\right) + a_{12}y'\left(0\right) + a_{13}y\left(\pi\right) + a_{14}y'\left(\pi\right) = 0, \\ a_{21}y\left(0\right) + a_{22}y'\left(0\right) + a_{23}y\left(\pi\right) + a_{24}y'\left(\pi\right) = 0, \end{array} \tag{2}$$

where λ is a spectral parameter, $p\ (x) \in W_2^1\left[0,\pi\right]$, $q\ (x) \in L_2\left[0,\pi\right]$ are real functions, a_{jm} are arbitrary complex numbers, $m = \overline{1,4}$, and j takes on values 1 and 2 here and throughout this paper. By $W_2^n\left[0,\pi\right]$ we denote a Sobolev space of complex-valued functions on $[0,\pi]$, whose derivatives up to $n-1$ th order are absolutely continuous and the n-th derivative is square summable on $[0,\pi]$.

Inverse spectral problems for Sturm-Liouville equation ($p\left(x\right) \equiv 0$) and for diffusion equation with separated boundary conditions ($a_{13} = a_{14} = a_{21} = a_{22} = 0$) have been fully solved in [9-13]. For $a_{12} = a_{14} = a_{21} = a_{23} = 0$, $a_{11} = a_{22} = 1$, $a_{13} = a_{24} = -1$ the boundary conditions (2) become periodic, for $a_{12} = a_{14} = a_{21} = a_{23} = 0$, $a_{11} = a_{13} = a_{22} = a_{24} = 1$ they become antiperiodic, and for $a_{12} = a_{14} = a_{21} = a_{23} = 0$, $a_{11} = 1$, $|a_{13}| = 1$, $a_{13} \neq \pm 1$, $a_{22} = \bar{a}_{13}$, $a_{24} = 1$ they become quasi-periodic boundary conditions. Inverse problems corresponding to these cases have been considered in [13-19]. Recovery matters for boundary value problems with other types of nonseparated boundary conditions have been studied in detail in [20-29]. In those works, the unknown coefficients of differential equation and the parameters of boundary conditions are recovered either from the spectra of two or three boundary value problems with different separated or nonseparated boundary conditions, or from two spectra and two or three eigenvalues, or from the spectra of two similar or non-similar boundary value problems, some sequence of signs and some number (the boundary value problems are said to be similar if their characteristic functions differ by a constant).

Many problems of the theory of oscillations in mathematical physics lead to the inverse problems of spectral analysis for differential operators with a spectral parameter in the equation and in the boundary conditions. In case of separated boundary conditions, some of these problems have been fully solved in [30-37], where the properties of spectral data have been studied, uniqueness theorems have been proved, solution algorithms have been constructed, and necessary and sufficient conditions for the solvability of inverse problems have been obtained. Inverse problems for the operators with nonseparated boundary conditions containing a spectral parameter in polynomial form have been studied in [38-42]. Note that the latter works provide uniqueness theorems and solution algorithms, but the solvability conditions for the corresponding inverse problems have not been obtained there.

Denote by $P(\alpha_j)$ the boundary value problem generated on $[0,\pi]$ by the diffusion equation (1) and the boundary conditions of the form

$$\begin{aligned} y'(0) + (\alpha_j\lambda + \beta)\, y(0) + \omega y(\pi) = 0\,, \\ y'(\pi) + \gamma\, y(\pi) - \omega y(0) = 0\,, \end{aligned} \tag{3}$$

where $\alpha_j \neq 0,\ \beta,\ \gamma,\ \omega \neq 0$ are real numbers. It is easy to see that the characteristic function of $P(\alpha_j)$ $(\alpha_1 \neq \alpha_2)$ is

$$\delta_j(\lambda) = 2\omega - \eta(\pi,\lambda) + \omega^2 s(\pi,\lambda) + (\alpha_j\lambda + \beta)\,\sigma(\pi,\lambda)\,, \tag{4}$$

where $\eta(\pi,\lambda) = c'(\pi,\lambda) + \gamma\, c(\pi,\lambda)$, $\sigma(\pi,\lambda) = s'(\pi,\lambda) + \gamma\, s(\pi,\lambda)$, $c(x,\lambda)$, $s(x,\lambda)$ is a fundamental system of solutions of the equation (1), defined by the initial conditions $c(0,\lambda)=s'(0,\lambda) = 1$, $c'(0,\lambda) = s(0,\lambda) = 0$. The zeros $\gamma_k^{(j)}$ $(k = \pm 0, \pm 1, \pm 2, ...)$ of this function are the eigenvalues of the problem $P(\alpha_j)$.

Let θ_k $(k = \pm 1, \pm 2, ...)$ (the zeros of the function $\sigma(\pi,\lambda)$) be the eigenvalues of the boundary value problem generated by the equation (1) and the boundary conditions

$$y(0) = y'(\pi) + \gamma\, y(\pi) = 0. \tag{5}$$

Denote $\sigma_k = \mathrm{sign}\,[1 - |\omega\, s(\pi,\theta_k)|]$. The inverse problem is stated as follows.

Inverse problem A. Given the spectral data $\left\{\gamma_k^{(1)}\right\}$, $\left\{\gamma_k^{(2)}\right\}$, $\{\sigma_k\}$ of boundary value problems, construct the coefficients $p(x)$, $q(x)$ of the equation (1) and the parameters $\alpha_j,\ \beta,\ \gamma, \omega$ of the boundary conditions (3).

In this work, we prove the uniqueness theorem, construct solution algorithm and obtain sufficient conditions for the solvability of inverse problem A, i.e. we go through all the steps 1), 2), 3) of treating inverse problems. Note that the similar results for Sturm-Liouville operator have been obtained in [42, 43].

2. Uniqueness theorem

Theorem 1. *Boundary value problems $P(\alpha_1)$ and $P(\alpha_2)$ are uniquely recoverable if their spectra and the sequence of signs $\{\sigma_k\}$ are known.*

Proof. It is known [27, 44] that the following asymptotic formulas are true for the eigenvalues of the boundary value problems $P(\alpha_j)$ and (1), (5) as $|k| \to \infty$:

$$\gamma_k^{(j)} = k + a_j + \frac{(-1)^{k+1} b_j \omega - B_j}{k\pi} + \frac{\tau_k^{(j)}}{k}, \tag{6}$$

$$\theta_k = k - \frac{1}{2}\mathrm{sign}\, k + a + \frac{Q\pi + \gamma}{k\pi} + \frac{\xi_k}{k}, \tag{7}$$

where

$$a_j = a - \frac{1}{\pi}\mathrm{arctg}\alpha_j, \tag{8}$$

$$B_j = \frac{\beta + \alpha_j\, p(0)}{1 + \alpha_j^2} - \gamma - \pi\, Q, \tag{9}$$

$$a = \frac{1}{\pi}\int_0^\pi p\ (x)\ dx, \quad Q = \frac{1}{2\pi}\int_0^\pi \left[q\ (x) + p^2\,(x)\right]\, dx,$$

$$b_j = \frac{2}{\sqrt{1+\alpha_j^2}}, \quad \left\{\tau_k^{(j)}\right\}, \ \{\xi_k\} \in l_2.$$

Using the formulas [13, 27]

$$c\ (\pi,\, \lambda) = \cos\pi\ (\lambda - a) - c_1\frac{\cos\pi\ (\lambda - a)}{\lambda} + \pi\, Q\frac{\sin\pi\,(\lambda - a)}{\lambda} + \frac{1}{\lambda}\int_{-\pi}^{\pi}\psi_1\,(t)\ e^{i\lambda t}dt,$$

$$c'\,(\pi,\, \lambda) = -\lambda\sin\pi\ (\lambda - a) + c_0\sin\pi\ (\lambda - a) + \pi\, Q\cos\pi\,(\lambda - a) + \int_{-\pi}^{\pi}\psi_2\,(t)\ e^{i\lambda t}dt,$$

$$s\ (\pi,\, \lambda) = \frac{\sin\pi\ (\lambda - a)}{\lambda} + c_0\frac{\sin\pi\ (\lambda - a)}{\lambda^2} - \pi Q\frac{\cos\pi\,(\lambda - a)}{\lambda^2} + \frac{1}{\lambda^2}\int_{-\pi}^{\pi}\psi_3\,(t)\ e^{i\lambda t}dt,$$

$$s'\,(\pi,\, \lambda) = \cos\pi\ (\lambda - a) + c_1\frac{\cos\pi\ (\lambda - a)}{\lambda} + \pi Q\frac{\sin\pi\ (\lambda - a)}{\lambda} + \frac{1}{\lambda}\int_{-\pi}^{\pi}\psi_4\,(t)\ e^{i\lambda t}dt,$$

where $c_0 = \frac{1}{2}\left[p\ (0) + p\ (\pi)\right]$, $c_1 = \frac{1}{2}\left[p\ (0) - p\ (\pi)\right]$, $\psi_m\,(t) \in L_2\,[-\pi, \pi]$, $m =$ 1, 2, 3, 4, we obtain from (4) the following representation for the function $\delta_j\,(\lambda)$:

$$\delta_j\,(\lambda) = 2\omega + \lambda\,(\sin\pi\,(\lambda - a) + \alpha_j\cos\pi\,(\lambda - a)) + (\alpha_j\pi Q + \alpha_j\gamma - c_0)\sin\pi\,(\lambda - a) +$$

$$+ (\beta + \alpha_j c_1 - \pi Q - \gamma)\cos\pi\,(\lambda - a) + f\,(\lambda)\,, \tag{10}$$

where $f\,(\lambda) = \int_{-\pi}^{\pi}\tilde{f}\,(t)\,e^{i\lambda t}dt$, $\tilde{f}\,(t) \in L_2\,[-\pi, \pi]$. Using the given sequence $\left\{\gamma_k^{(j)}\right\}$, the function $\delta_j\,(\lambda)$ (as an entire function of exponential type) can be recovered in the form of infinite product. By representation (10) and Lemma 1.4.3 of [1], we have

$$\lim_{k\to\infty}\frac{\delta_j(2k)}{2k}=\alpha_j\cos\pi a-\sin\pi a,\ \lim_{k\to\infty}\frac{\delta_j(2k-1)}{2k-1}=\sin\pi a-\alpha_j\cos\pi a,$$

$$\lim_{k\to\infty}\frac{\delta_j\left(2k+\frac{1}{2}\right)}{2k+\frac{1}{2}}=\cos\pi a+\alpha_j\sin\pi a,\ \lim_{k\to\infty}\frac{\delta_j\left(2k+\frac{1}{2}\right)}{2k+\frac{1}{2}}=-\cos\pi a-\alpha_j\sin\pi a.$$

Hence

$$\lim_{k\to\infty}\frac{\delta_1(2k)+\delta_2(2k-1)}{2k}=(\alpha_1-\alpha_2)\cos\pi a,$$

$$\lim_{k\to\infty}\frac{\delta_1\left(2k+\frac{1}{2}\right)+\delta_2\left(2k-\frac{1}{2}\right)}{2k}=(\alpha_1-\alpha_2)\sin\pi a,$$

and, consequently

$$\mathrm{tg}\pi a=\lim_{k\to\infty}\frac{\delta_1\left(2k+\frac{1}{2}\right)+\delta_2\left(2k-\frac{1}{2}\right)}{\delta_1(2k)+\delta_2(2k-1)}. \tag{11}$$

Using (6), (8) and (11), we successively calculate

$$a_j=\lim_{k\to\infty}\left(\gamma_k^{(j)}-k\right),$$

$$\mathrm{tg}\pi a_j=\mathrm{tg}\left(\pi a-\mathrm{arctg}\alpha_j\right)=\frac{\mathrm{tg}\pi a-\alpha_j}{1+\alpha_j\mathrm{tg}\pi a},$$

$$\alpha_j=\frac{\mathrm{tg}\pi a-\mathrm{tg}\pi a_j}{1+\mathrm{tg}\pi a\cdot\mathrm{tg}\pi a_j} \tag{12}$$

$$\omega=\frac{\pi\sqrt{1+\alpha_j^2}}{2}\lim_{k\to\infty}k\left(\gamma_{2k+1}^{(j)}-\gamma_{2k}^{(j)}-1\right). \tag{13}$$

It follows that the parameters α_j and ω of boundary conditions (3) are uniquely recovered from the spectra$\left\{\gamma_k^{(1)}\right\}$ and $\left\{\gamma_k^{(2)}\right\}$ of boundary value problems $P(\alpha_1)$ and $P(\alpha_2)$ by means of the formulas (12) and (13).

Construct the function

$$\sigma(\pi,\lambda)=\frac{\delta_1(\lambda)-\delta_2(\lambda)}{(\alpha_1-\alpha_2)\lambda}. \tag{14}$$

From here, we find the zeros θ_k, $k=\pm1,\pm2,...$ of the function $\sigma(\pi,\lambda)$ for which the asymptotic formula (7) holds as $|k|\to\infty$. Knowing $\gamma_k^{(j)}$ and θ_k, we can also recover the parameter β, because, by virtue of (6), (7) and (9)

$$\beta=\frac{\alpha_1 d_2-\alpha_2 d_1}{\alpha_1-\alpha_2}, \tag{15}$$

where

$$d_j = \beta + \alpha_j p(0) = \left(1 + \alpha_j^2\right) \lim_{k\to\infty} \left[b_j \omega - 2k\pi \left(\gamma_{2k+1}^{(j)} - \theta_{2k+1} + \frac{1}{\pi}\text{arctg}\alpha_j - \frac{1}{2} \right) \right].$$

Using (4), let's recover the function

$$h_+(\lambda) = \beta\sigma(\pi, \lambda) - \eta(\pi, \lambda) + \omega^2 s(\pi, \lambda) \tag{16}$$

by the formula

$$h_+(\lambda) = \frac{\alpha_2 \delta_1(\lambda) - \alpha_1 \delta_2(\lambda)}{\alpha_2 - \alpha_1} - 2\omega. \tag{17}$$

Consider the function

$$h_-(\lambda) = \beta\sigma(\pi, \lambda) - \eta(\pi, \lambda) - \omega^2 s(\pi, \lambda). \tag{18}$$

From (16), (18), considering the identity

$$c(\pi, \lambda)\, \sigma(\pi, \lambda) - s(\pi, \lambda)\, \eta(\pi, \lambda) = 1, \tag{19}$$

it is easy to obtain $h_-^2(\theta_k) - h_+^2(\theta_k) = -4\omega^2$, and therefore

$$h_-(\theta_k) = \text{sign} h_-(\theta_k) \sqrt{h_+^2(\theta_k) - 4\omega^2}.$$

Taking into account (16), (18) and the intermittency of zeros of the functions $s(\pi, \lambda)$ and $\sigma(\pi, \lambda)$, we have

$$\text{sign } h_-(\theta_k) = \text{sign}\left[-\eta(\pi, \theta_k) - \omega^2 s(\pi, \theta_k)\right] =$$

$$= \text{sign}\left[\frac{1}{s(\pi, \theta_k)} - \omega^2 s(\pi, \theta_k)\right] = \text{sign}\frac{1 - [\omega s(\pi, \theta_k)]^2}{s(\pi, \theta_k)} = (-1)^{k+1}\sigma_k.$$

Consequently

$$h_-(\theta_k) = (-1)^{k+1} \sigma_k \sqrt{h_+^2(\theta_k) - 4\omega^2}. \tag{20}$$

Let

$$g(\lambda) = h_+(\lambda) - h_-(\lambda) - 2\omega^2 \frac{\sin(\lambda - a)\pi}{\lambda}. \tag{21}$$

Using the relations (7), (16) and (18), Lemma 3.2 of [16] and Theorem 28 of [45], it is easy to conclude that the function $g(\lambda)$ is uniquely defined by the spectrum $\{\theta_k\}$ of the boundary value problem (1), (5) and the sequences $\{\sigma_k\}$, $\{h_+(\theta_k)\}$ via the formula

$$g(\lambda) = \sigma(\pi, \lambda) \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{g(\theta_k)}{(\lambda - \theta_k)\, \dot{\sigma}(\pi, \theta_k)}, \tag{22}$$

where $g\left(\theta_k\right) = h_+\left(\theta_k\right) - (-1)^{k+1}\sigma_k\sqrt{h_+^2\left(\theta_k\right) - 4\,\omega^2} - 2\omega^2\,\frac{\sin(\theta_k - a)\pi}{\theta_k}$, and the point over function means a differentiation in λ.

Define the characteristic function $s\left(\pi, \lambda\right)$ of the boundary value problem generated by the equation (1) and Dirichlet boundary conditions

$$y\left(0\right) = y\left(\pi\right) = 0 \tag{23}$$

as follows:

$$s\left(\pi, \lambda\right) = \frac{1}{2\omega^2}\left[h_+\left(\lambda\right) - h_-\left(\lambda\right)\right], \tag{24}$$

where the function $h_-\left(\lambda\right)$ is given by (21). The zeros $\lambda_k,\ k = \pm1, \pm2, ...$ of the function (24) are the eigenvalues of the boundary value problem (1), (23). They satisfy the asymptotic formula

$$\lambda_k = k + a + \frac{Q}{k} + \frac{\eta_k}{k},\ \{\eta_k\} \in l_2. \tag{25}$$

From this formula and (7) it follows that

$$\gamma = \pi \lim_{k\to\infty} k\left(\theta_k - \lambda_k + \frac{1}{2}\right). \tag{26}$$

Finally, using $\sigma\left(\pi,\ \lambda\right)$, $s\ \left(\lambda, \pi\right)$ and γ, we recover the characteristic function of the problem generated by the equation (1) and the boundary conditions

$$y\left(0\right) = y'\left(\pi\right) = 0 \tag{27}$$

via the following formula:

$$s'\ \left(\lambda, \pi\right) = \sigma\left(\pi,\ \lambda\right) - \gamma\, s\left(\lambda, \pi\right). \tag{28}$$

It is known [13] that the coefficients of the equation (1) are defined uniquely by using the zeros of this function and the sequence $\{\lambda_k\}$.

Thus, the boundary value problems $P\left(\alpha_1\right)$ and $P\left(\alpha_2\right)$ are fully recovered from the given sequences $\left\{\gamma_k^{(1)}\right\}$, $\left\{\gamma_k^{(2)}\right\}$ and $\{\sigma_k\}$. ◄

3. Recovery algorithms for boundary value problems

As mentioned above, the coefficients of the equation (1) are uniquely defined by the spectra of the boundary value problems (1), (23) and (1), (27). Based on the methods of [13], these coefficients can be recovered by the following algorithm.

Algorithm 1. *Let the two sequences of real numbers* $\{\lambda_k\}$ *and* $\{\nu_k\}$ $(k = \pm 1, \pm 2, ...)$ *(the spectra of boundary value problems* (1), (23) *and* (1), (27), *respectively) be given.*

1) Using the given sequences $\{\lambda_k\}$ *and* $\{\nu_k\}$, *construct the functions*

$$s(\lambda) = \pi \prod_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\lambda_k - \lambda}{k}, \quad s_1(\lambda) = \prod_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\nu_k - \lambda}{k - \frac{1}{2}\operatorname{sign} k}.$$

2) Set

$$\varphi(\lambda) = e^{i\lambda\pi}\left[s_1(\lambda) - i\lambda s(\lambda)\right]$$

and construct the function

$$S(\lambda) = \frac{\overline{\varphi(\lambda)}}{\varphi(\lambda)} \quad (-\infty < \lambda < \infty).$$

Note that $S(\lambda)$ *is a scattering function of some boundary value problem*

$$y'' + \left[\lambda^2 - 2\lambda p(x) - q(x)\right] y = 0 \qquad (0 \le x \le \infty), \quad y(0) = 0,$$

where the real functions $q(x)$ *and* $p(x)$ *possess the following properties:*

$$q(x) = p(x) = 0 \text{ for } x > \pi, \; q(x) \in L_2[0,\pi], \; p(x) \in W_2^1[0,\pi].$$

3) Define the function $F(x)$ *by the formula*

$$F(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\left[e^{-2i\pi a} - S(\lambda)\right] e^{i\lambda x} dx,$$

where $a = \lim\limits_{k\to\infty}(\lambda_k - k)$.

4) Solve the integral equations

$$F(x+y) + \overline{K_0(x,y)} + \int_x^{\infty} K_0(x,t)\, F(y+t)\, dt = 0,$$

$$iF(x+y) + \overline{K_1(x,y)} + \int_x^{\infty} K_1(x,t)\, F(y+t)\, dt = 0 \quad (x \le y < \infty)$$

with respect to $K_0(x,y) \in L_1(x,\infty)$, $K_1(x,y) \in L_1(x,\infty)$, *where* $x \in [0,\infty)$ *is considered as a parameter.*

5) Define the function $\alpha(x)$ *as a solution of nonlinear Volterra type equation*

$$\alpha(x) = \int_x^{\infty} V(t, \alpha(t))\, dt = 0 \quad (0 \le x < \infty),$$

where

$$V(t,z) = [\mathrm{ReK}_0(\mathrm{t,t}) - \mathrm{ImK}_1(\mathrm{t,t})]\sin 2z + 2[\mathrm{ReK}_1(\mathrm{t,t})]\sin^2 z - 2[\mathrm{ImK}_0(\mathrm{t,t})]\cos^2 z.$$

6) Calculate $p(x)$ and $q(x)$ by the formulas

$$p(x) = -\alpha'(x)\,,$$

$$q(x) = -p^2(x) - 2\frac{d}{dx}\{[\mathrm{ReK}(\mathrm{x,x})]\cos\alpha(x) + [\mathrm{ImK}(\mathrm{x,x})]\sin\alpha(x)\}\,,$$

where

$$K(x,y) = K_0(x,y)\cos\alpha(x) + K_1(x,y)\sin\alpha(x)\,.$$

Based on the proof of Theorem 1, let's state the solution algorithm for the inverse problem A.

Algorithm 2. *Let the sequences $\left\{\gamma_k^{(1)}\right\}$, $\left\{\gamma_k^{(2)}\right\}$, $\{\sigma_k\}$ (the spectral data of boundary value problems $P(\alpha_1)$ and $P(\alpha_2)$) be given.*

1) Using the sequence $\left\{\gamma_k^{(j)}\right\}$, construct the function $\delta_j(\lambda)$ in the form of infinite product.

2) Calculate $\mathrm{tg}\pi a$ by the formula (11) and define the parameters α_j and ω of boundary conditions (3) by (12) and (13).

3) Construct the function $\sigma(\pi,\lambda)$ using (14) and find the zeros θ_k of this function.

4) Find the parameter β from the formula (15).

5) Recover the function $h_+(\lambda) = \beta\sigma(\pi,\lambda) - \eta(\pi,\lambda) + \omega^2 s(\pi,\lambda)$ using (17).

6) Find the values of the function $h_-(\lambda) = \beta\sigma(\pi,\lambda) - \eta(\pi,\lambda) - \omega^2 s(\pi,\lambda)$ at the points θ_k using (20).

7) Using $\sigma(\pi,\lambda)$ and $h_+(\theta_k)$, recover the function $g(\lambda)$ (see (21)) by the interpolation formula (22).

8) Knowing $g(\lambda)$, define $h_-(\lambda)$ from (21).

9) Define the characteristic function $s(\pi,\lambda)$ of boundary value problem (1), (23) by the formula (24).

10) Using asymptotic formulas (7) and (25), find the parameter γ from (26).

11) Using $\sigma(\pi,\lambda)$, $s(\lambda,\pi)$ and γ, recover the characteristic function $s'(\lambda,\pi)$ of the problem (1), (27) by the formula (28).

12) Using the sequences $\{\lambda_k\}$ and $\{\nu_k\}$ of zeros of the functions $s(\pi,\lambda)$ and $s'(\lambda,\pi)$, respectively, construct the coefficients $p(x)$, $q(x)$ of the equation (1) following Algorithm 1.

4. Sufficient conditions for solvability of inverse problem A

Theorem 2. *In order for the sequences of real numbers* $\left\{\gamma_k^{(1)}\right\}$, $\left\{\gamma_k^{(2)}\right\}$ $(k = \pm 0, \pm 1, \pm 2, ...)$ *and* $\{\sigma_k\}$ $(\sigma_k = -1, 0, 1\ ;\ k = \pm 1, \pm 2, ...)$ *to be spectral data of the boundary value problems of the form* $P(\alpha_1)$ *and* $P(\alpha_2)$ $(\alpha_1 < \alpha_2)$, *it is sufficient that the following conditions hold:*

1) the asymptotic formula $\gamma_k^{(j)} = k + a + a_j + \frac{(-1)^{k+1} A_j - B_j}{k\pi} + \frac{\tau_k^{(j)}}{k}$ *is true, where* $A_j = 2\omega \cos \pi a_j$, ω, a, a_j, B_j *are real numbers,* $a < 0$, $0 < a_j < \frac{1}{2}$, $a_1 > a_2$, $\omega \neq 0$, $\sum_{k=-\infty}^{\infty} \left(\tau_k^{(j)}\right)^2 < \infty$;

2) the numbers $\gamma_k^{(1)}$ *and* $\gamma_k^{(2)}$ *satisfy the inequalities*

$$\begin{gathered} 0 < \gamma_{+0}^{(2)} \leq \gamma_{+0}^{(1)} \leq \gamma_1^{(2)} \leq \gamma_1^{(1)} < \gamma_2^{(2)} \leq \gamma_2^{(1)} \leq \gamma_3^{(2)} \leq \gamma_3^{(1)} < ..., \\ 0 > \gamma_{-0}^{(2)} \geq \gamma_{-0}^{(1)} \geq \gamma_{-1}^{(2)} \geq \gamma_{-1}^{(1)} > \gamma_{-2}^{(2)} \geq \gamma_{-2}^{(1)} \geq \gamma_{-3}^{(2)} \geq \gamma_{-3}^{(1)} > ... \end{gathered} \tag{29}$$

when $\omega < 0$, *and the inequalities*

$$\begin{gathered} 0 < \gamma_{+0}^{(2)} < \gamma_{+0}^{(1)} < \gamma_1^{(2)} \leq \gamma_1^{(1)} \leq \gamma_2^{(2)} \leq \gamma_2^{(1)} < \gamma_3^{(2)} \leq \gamma_3^{(1)} \leq ..., \\ 0 > \gamma_{-0}^{(2)} > \gamma_{-0}^{(1)} > \gamma_{-1}^{(2)} \geq \gamma_{-1}^{(1)} \geq \gamma_{-2}^{(2)} \geq \gamma_{-2}^{(1)} > \gamma_{-3}^{(2)} \geq \gamma_{-3}^{(1)} \geq ... \end{gathered} \tag{30}$$

when $\omega > 0$, *with* $\gamma_{k-1}^{(3-j)} < \gamma_k^{(3-j)} < \gamma_{k+1}^{(3-j)}$ *for* $\gamma_k^{(j)} = \gamma_{k+1}^{(j)}$;

3) the inequality $b_k \overset{def}{=} |\delta_j(\theta_k) - 2\omega| - 2|\omega| \geq 0$ *is true, where*

$$\delta_j(\lambda) = \frac{\pi\left(\gamma_{-0}^{(j)} - \lambda\right)\left(\gamma_{+0}^{(j)} - \lambda\right)}{\cos \pi a_j} \prod_{\substack{k=-\infty \\ k \neq 0}}^{\infty} \frac{\gamma_k^{(j)} - \lambda}{k} \tag{31}$$

and θ_k*'s are the zeros of the function* $\delta_1(\lambda) - \delta_2(\lambda)$;

4) σ_k *is equal to 0 if* $b_k = 0$, *and to 1 or -1 if* $b_k > 0$; *besides, there exists* $N > 0$ *such that* $\sigma_k = 1$ *for all* $|k| \geq N$.

Proof. Denote $\alpha_j = -tg\pi a_j$. Then it follows from the inequalities $a_1 > a_2$ and $0 < a_j < \frac{1}{2}$ that $\alpha_1 < \alpha_2 < 0$. By Lemma 1.3 of [27], for $\delta_j(\lambda)$ (see (31)) we have the representation

$$\delta_j(\lambda) = \frac{1}{\cos \pi a_j}[A_j + (\lambda - a)\sin \pi(\lambda - a - a_j) + B_j \cos \pi(\lambda - a - a_j) +$$

$$+M_j \sin \pi (\lambda - a - a_j) + f_j (\lambda - a - a_j)],$$

where $f_j(\lambda) = \int_{-\pi}^{\pi} \tilde{f}_j(t) e^{i\lambda t} dt$, $\tilde{f}_j(t) \in L_2[-\pi, \pi]$. Hence, considering relations $A_j = 2\omega \cos \pi a_j$ and $\alpha_j = -\mathrm{tg}\pi a_j$, we have

$$\delta_j(\lambda) = 2\omega + \frac{\lambda - a}{\cos \pi a_j}[\sin \pi (\lambda - a) \cos \pi a_j - \cos \pi (\lambda - a) \sin \pi a_j] +$$

$$+\frac{M_j}{\cos \pi a_j}[\sin \pi (\lambda - a) \cos \pi a_j - \cos \pi (\lambda - a) \sin \pi a_j] +$$

$$+\frac{B_j}{\cos \pi a_j}[\cos \pi (\lambda - a) \cos \pi a_j + \sin \pi (\lambda - a) \sin \pi a_j] +$$

$$+\frac{1}{\cos \pi a_j} \int_{-\pi}^{\pi} \tilde{f}_j(t) e^{i(\lambda - a - a_j)t} dt =$$

$$= 2\omega + (\lambda - a) \ [\sin \pi (\lambda - a) + \alpha_j \cos \pi (\lambda - a)] +$$

$$+ (B_j + \alpha_j M_j) \cos \pi (\lambda - a) +$$

$$+ (M_j - \alpha_j B_j) \sin \pi (\lambda - a) + \int_{-\pi}^{\pi} \tilde{g}_j(t) e^{i(\lambda - a)t} dt , \ \tilde{g}_j(t) \in L_2[-\pi, \pi] . \qquad (32)$$

Therefore

$$\sigma(\lambda) = \frac{\delta_1(\lambda) - \delta_2(\lambda)}{(\alpha_1 - \alpha_2)(\lambda - a)} = \cos \pi (\lambda - a) - B_3 \pi \frac{\sin \pi (\lambda - a)}{\lambda - a} +$$

$$+M_3 \frac{\cos \pi (\lambda - a)}{\lambda - a} + \int_{-\pi}^{\pi} \frac{[\tilde{g}_1(t) - \tilde{g}_2(t)]}{\alpha_1 - \alpha_2} \cdot \frac{e^{i(\lambda - a)t}}{\lambda - a} dt, \qquad (33)$$

where

$$B_3 = \frac{M_2 - M_1 + \alpha_1 B_1 - \alpha_2 B_2}{\pi (\alpha_1 - \alpha_2)}, \ M_3 = \frac{B_1 + \alpha_1 M_1 - B_2 - \alpha_2 M_2}{\alpha_1 - \alpha_2}.$$

By Lemma 1.2 of [27], the following asymptotic formula is true for the zeros θ_k of the function $\sigma(\lambda)$:

$$\theta_k = k - \frac{1}{2} \mathrm{sign}\, k + a - \frac{B_3}{n} + \frac{\eta_k}{k} , \ \{\eta_k\} \in l_2. \qquad (34)$$

Using (32), for the function

$$u_1(\lambda) = \frac{\alpha_1 \delta_2(\lambda) - \alpha_2 \delta_1(\lambda)}{\alpha_1 - \alpha_2} - 2\omega \qquad (35)$$

we obtain the following representation:

$$u_1(\lambda) = (\lambda - a)\sin\pi(\lambda - a) + D\cos\pi(\lambda - a) +$$

$$+M\sin\pi(\lambda - a) + \int_{-\pi}^{\pi} r(t)\, e^{i(\lambda-a)t}dt,$$

$$D = \frac{\alpha_1(B_2 + \alpha_2 M_2) - \alpha_2(B_1 + \alpha_1 M_1)}{\alpha_1 - \alpha_2},$$

$$M = \frac{\alpha_1(M_2 - \alpha_2 B_2) - \alpha_2(M_1 - \alpha_1 B_1)}{\alpha_1 - \alpha_2}, r(t) \in L_2[-\pi, \pi].$$

As in [27], we define the function $u_2(\lambda)$ satisfying the condition

$$u_2(\theta_k) = (-1)^{k+1}\sigma_k\sqrt{u_1^2(\theta_k) - 4\omega^2} \tag{36}$$

as follows:

$$u_2(\lambda) = u_1(z) - 2\omega^2\left[\frac{\sin\pi(\lambda - a)}{\lambda - a} + B_4\pi\frac{4\cos\pi(\lambda - a)}{4(\lambda - a)^2 - 1} - \right.$$

$$\left. - \frac{M\sin\pi(\lambda - a)}{(\lambda - a)^2} + \frac{m(\lambda - a)}{(\lambda - a)^2}\right], \tag{37}$$

where B_4 is an arbitrary constant satisfying the inequality $B_4 \le B_3$, and $m(\lambda) = \sigma(\lambda)\sum_{k=-\infty}^{\infty}\frac{m_k}{\sigma'(\theta_k)(\lambda-\theta_k)}$ is an entire function of exponential type not greater than π belonging to $L_2(-\infty, \infty)$, $\{m_k\} \in l_2$.

Denote

$$s(\lambda) = \frac{1}{2\omega^2}[u_1(\lambda) - u_2(\lambda)]. \tag{38}$$

In view of (37), we have

$$(\lambda - a)s(\lambda) = \sin\pi(\lambda - a) + B_4\pi\frac{4(\lambda - a)\cos\pi(\lambda - a)}{4(\lambda - a)^2 - 1} + \frac{\tilde{m}(\lambda - a)}{\lambda - a}, \tag{39}$$

where $\tilde{m}(\lambda) = -M\sin\lambda\pi + m(\lambda)$, $\tilde{m}(0) = \tilde{m}'(0) = 0$.
Therefore, according to Lemma 2.1 of [27], if we denote by λ_k $(k = \pm 1, \pm 2, ...)$ the zeros of the function $(\lambda - a)s(\lambda)$, then we will have the following asymptotic formula:

$$\lambda_k = k + a - \frac{B_4}{k} + \frac{\tau_k}{k}, \quad \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty}\tau_k^2 < \infty. \tag{40}$$

Taking into account the obvious equality $\delta_1(\theta_k) = \delta_2(\theta_k)$ (see (33)), from (35) we obtain

$$u_1(\theta_k) = \delta_j(\theta_k) - 2\omega. \tag{41}$$

Third condition of theorem and the relations (29)-(30), (41) yield

$$\ldots,\ u_1(\theta_{-2}) \leq -2|\omega|\ ,\ u_1(\theta_{-1}) \geq 2|\omega|\ ,\ u_1(\theta_1) \geq 2|\omega|\ ,\ u_1(\theta_2) \leq -2|\omega|\ ,\ \ldots,$$

Consequently, there is a number h_k such that

$$u_1(\theta_k) = 2|\omega|(-1)^{k+1}\operatorname{ch} h_k. \tag{42}$$

From (36), by virtue of (42), we obtain

$$u_2(\theta_k) = 2|\omega|(-1)^{k+1}\sigma_k|\operatorname{sh} h_k|. \tag{43}$$

By (38), (42) and (43) we have

$$s(\theta_k) = \frac{1}{2\omega^2}\left[u_1(\theta_k) - u_2(\theta_k)\right] = \frac{(-1)^{k+1}}{|\omega|}\left(\operatorname{ch} h_k - \sigma_k|\operatorname{sh} h_k|\right) =$$

$$= \frac{(-1)^{k+1}\operatorname{ch} h_k}{|\omega|}\left(1 - \sigma_k|\operatorname{th} h_k|\right).$$

Hence, by the obvious inequality $|\operatorname{th} h_k| < 1$ it follows that

$$\operatorname{sign} s(\theta_k) = (-1)^{k+1}. \tag{44}$$

Then, every interval $\ldots,\ (\theta_{-3}, \theta_{-2})\ ,\ (\theta_{-2}, \theta_{-1})\ ,\ (\theta_1, \theta_2)\ ,\ (\theta_2, \theta_3)\ ,\ \ldots$ contains one and, due to the asymptotic equality (40), only one zero of the function $s(\lambda)$. Consequently, the zeros $\ldots,\ \theta_{-2},\ \theta_{-1},\ \theta_1,\ \theta_2,\ \ldots$ of the function $\sigma(\lambda)$ and the zeros $\ldots,\ \lambda_{-2},\ \lambda_{-1},\ \lambda_1,\ \lambda_2,\ \ldots$ of the function $s(\lambda)$ satisfy the inequalities

$$\ldots < \theta_{-3} < \lambda_{-2} < \theta_{-2} < \lambda_{-1} < \theta_{-1} < 0 < \theta_1 < \lambda_1 < \theta_2 < \lambda_2 < \theta_3 < \ldots. \tag{45}$$

Consider the function

$$s_1(\lambda) = \sigma(\lambda) - \gamma s(\lambda), \tag{46}$$

where

$$\gamma = (B_4 - B_3)\pi. \tag{47}$$

The equality (46), combined with the relations (33), (39) and (47), implies

$$s_1(\lambda) = \cos\pi(\lambda - a) - B_4\pi\frac{\sin\pi(\lambda - a)}{\lambda - a} + \frac{m_1(\lambda - a)}{z - a} \tag{48}$$

(where $m_1(\lambda) = M_4 \cos\lambda\pi + \int_{-\pi}^{\pi} \tilde{m}_1(t)\, e^{i\lambda t} dt$, $\tilde{m}_1(t) \in L_2[-\pi,\pi]$), which, in turn, by Lemma 2.1 of [27] implies that the following asymptotic formula holds for the zeros ν_k , $k = \pm 1, \pm 2, \ldots$ of the functions $s_1(z)$:

$$\nu_k = k - \frac{1}{2}\operatorname{sign} k + a - \frac{B_4}{k} + \frac{l_k}{k}\ , \qquad \sum_{\substack{k=-\infty \\ k\neq 0}}^{\infty} l_k^2 < \infty. \tag{49}$$

By the formula (31), the inequality $\alpha_1 < \alpha_2 < 0$ and the second condition of theorem, we have

$$\frac{\delta_1(0)}{\delta_2(0)} = \sqrt{\frac{1+\alpha_1^2}{1+\alpha_2^2}} \cdot \frac{\gamma_{-0}^{(1)} \cdot \gamma_{+0}^{(1)}}{\gamma_{-0}^{(2)} \cdot \gamma_{+0}^{(2)}} \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\gamma_k^{(1)}}{\gamma_k^{(2)}} > 1.$$

Then, due to the equality (33) and the inequality $\delta_j(0) < 0$ which follows from (31) by (29)-(30), we have

$$\sigma(0) = \frac{\delta_1(0) - \delta_2(0)}{(\alpha_2-\alpha_1)\, a} = \frac{\delta_2(0)}{(\alpha_2-\alpha_1)\, a}\left[\frac{\delta_1(0)}{\delta_2(0)} - 1\right] > 0. \tag{50}$$

Further, by Lemma 2.1 of [27], the function $s(\lambda)$ can be represented in the form of infinite product $s(\lambda) = \pi \prod_{k=-\infty, k\neq 0}^{\infty} \frac{\lambda_k - \lambda}{k}$. Hence it follows by (45) that

$$s(0) = \pi \prod_{\substack{k=-\infty \\ k\neq 0}}^{\infty} \frac{\lambda_k}{k} > 0. \tag{51}$$

Consequently, from (46), taking into account (47), (50), (51) and the inequality $B_4 \le B_3$, we obtain

$$s_1(0) = \sigma(0) - \gamma s(0) > 0.$$

On the other hand, by (44), (50) we have

$$\operatorname{sign} s_1(\lambda_k) = \operatorname{sign} \sigma(\lambda_k) = (-1)^k.$$

Therefore, every interval

$$\ldots\ ,\ (\lambda_{-2}, \lambda_{-1})\ ,\ (\lambda_{-1}, 0)\ ,\ (0, \lambda_1)\ ,\ (\lambda_1, \lambda_2)\ ,\ \ldots$$

contains one zero of the function $s_1(\lambda)$. By virtue of the asymptotic formula (49), the function $s_1(\lambda)$ has no other zeros. Thus, for the zeros of the functions $s(\lambda)$ and $s_1(\lambda)$ we have the inequalities

$$... < \nu_{-3} < \lambda_{-2} < \nu_{-2} < \lambda_{-1} < \nu_{-1} < 0 < \nu_1 < \lambda_1 < \nu_2 < \lambda_2 < \nu_3 < ....$$

Hence, considering the asymptotic formulas (40) and (49), by virtue of [13] we conclude that there exists only one pair of functions $q(x) \in L_2[0,\pi]$, $p(x) \in W_2^1[0,\pi]$ such that the sequences $\{\lambda_k\}$ and $\{\nu_k\}$ are the spectra of boundary value problems generated on $[0,\pi]$ by the same equation (1) with the found coefficients ($q(x)$ and $p(x)$) and the boundary conditions (23) and (27), and the equalities $s(\lambda) = s(\pi, \lambda)$, $s_1(\lambda) = s'(\pi, \lambda)$ hold. Using these equalities, it is easily shown that the spectra of the recovered boundary value problems coincide with the sequences $\left\{\gamma_k^{(1)}\right\}$ and $\left\{\gamma_k^{(2)}\right\}$. ◄

It can be shown that if the operator L given by the equalities

$$Ly = -y'' + q(x)y,$$

$$D(L) = \left\{y \in W_2^2[0,\pi] : y'(0) + \beta y(0) + \omega y(\pi) = y'(\pi) + \gamma y(\pi) - \omega y(0) = 0\right\},$$

is positive, then the conditions of Theorem 2 are also necessary.

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References

[1] V.A. Marcenko, *Sturm-Liouville operators and their applications*, Birkhauser, 1986.

[2] B. M. Levitan, *Inverse Sturm-Liouville problems*, VNU Science Press, Utrecht, 1987.

[3] J. Pöschel, E. Trubowitz, *Inverse spectral theory*, Academic Press Boston, 1987.

[4] V.A. Yurko, *Introduction to the theory of inverse spectral problems*, Fizmatlit, Moscow, 2007 (in Russian).

[5] V.A. Sadovnichii, Ya.T. Sultanaev, AM. Akhtyamov, *An inverse Sturm-Liouville problems with nonseparated boundary conditions*, Izd-vo MGU, Moscow, 2009 (in Russian).

[6] A. M. Akhtyamov, *Identification theory of boundary value problems and its applications*, Fizmatlit, Moscow, 2009 (in Russian).

[7] I.M. Nabiev, *Reconstruction of pencil and system of differential equations on the segment*, LAP-Lambert Academic Publishing, 2012 (In Russian).

[8] M. Möller, V. Pivovarchik, *Spectral theory of operator pencils, Hermite-Biehler functions, and their applications*, Birkhauser, Cham, 2015.

[9] B.M. Levitan, M.G. Gasymov, *Determination of a differential equation by two of its spectra*, Russian Mathematical Surveys, **19(2)**, 1964, 1-63.

[10] M.G. Gasymov, G.Sh. Guseinov, *Determining the diffusion operator from spectral data*, Akad. Nauk Azerbaidzhan SSR Dokl., **37(2)**, 1981, 19-23 (in Russian).

[11] I.M. Guseinov, L.I. Mammadova, *Reconstruction of the diffusion equation with singular coefficients for two spectra*, Dokl. Math., **90(1)**, 2014, 401-404.

[12] H.M. Huseynov, F.Z. Dostuyev, *On determination of Sturm-Liouville operator with discontinuity conditions with respect to spectral data*, Proc. of Institute of Math. and Mech. of the Azerbaijan National Academy of Sci., **42(2)**, 2016, 143-153.

[13] G.Sh. Guseinov, *Inverse spectral problems for a quadratic pencil of Sturm-Liouville operators on a finite interval*, Spectral theory of operators and its applications, Elm, Baku, **7**, 51-101, 1986 (in Russian).

[14] I.V. Stankevich, *An inverse problem of spectral analysis for Hill's equation*, Sov. Math. Dokl., **11**, 1970, 582-586.

[15] V.A. Sadovnichii, *Uniqueness of a solution of the inverse problem of spectral analysis for a differential equation with periodic boundary values*, Differ. Equ., **9**, 1974, 207-211.

[16] V.A. Marchenko, I.V. Ostrovskii, *A characterization of the spectrum of the Hill operator*, Math. USSR-Sb., **26(4)**, 1975, 493-554.

[17] O.A. Plaksina, *Characterization of the spectra of quasiperiodic problems for a Sturm–Liouville operator*, Teor. Funktsii Funktsional Anal. Prilozhen (Kharkov), **31**, 1979, 129-134 (in Russian).

[18] P. Kargaev, E. Korotyaev, *The inverse problem for the Hill operator, a direct approach*, Invent. Math., **129**, 1997, 567-593.

[19] I.M. Nabiev,*Determination of the diffusion operator on an interval*, Colloq. Math., **134(2)**, 2014, 165-178.

[20] V.A. Sadovnichii, *Uniqueness of the solution of the inverse problem for second-order differential equation with nonseparated boundary conditions, regularized sums of eigenvalues. Factorization of the characteristic determinant*, Dokl. Akad. Nauk SSSR, **206(2)**, 1972, 293-296.

[21] V.A. Yurko, *The inverse problem for differential operators of second order with regular boundary conditions*, Math. Notes, **18(4)**, 1975, 928-932.

[22] O.A. Plaksina, *Inverse problems of spectral analysis for Sturm–Liouville operators with nonseparated boundary conditions. II,*, Math. USSR-Sb., **64(1)**, 1989, 141-160.

[23] I.M. Guseinov, I.M. Nabiev, *A class of inverse boundary value problems for Sturm-Liouville operators*, Differ. Equ., **25(7)**, 1989, 779-784.

[24] M.G. Gasymov, I.M. Guseinov, I.M. Nabiev, *An inverse problem for the Sturm-Liouville operator with nonseparable self-adjoint boundary conditions*, Sib. Math. J., **31(6)**, 1990, 910-918.

[25] I.M. Guseinov, I.M. Nabiev, *Solution of a class of inverse boundary-value Sturm-Liouville problems*, Sb. Math., **186(5)**, 1995, 661-674.

[26] V.A. Yurko, *The inverse spectral problem for differential operators with nonseparated boundary conditions*, J. Math. Anal. Appl., **250(1)**, 2000, 266-289.

[27] I.M. Guseinov, I.M. Nabiev,*An inverse spectral problem for pencils of differential operators*, Sb. Math., **198(11-12)**, 2007, 1579-1598.

[28] V. Yurko, *An inverse spectral problem for nonselfadjoint Sturm-Liouville operators with nonseparated boundary conditions*, Tamkang J. Math., **43(2)**, 2012, 289-299.

[29] V.A. Sadovnichii, Ya.T. Sultanaev, A.M. Akhtyamov, *General inverse Sturm-Liouville problem with symmetric potential*, Azerbaijan Journal of Mathematics, **5(2)**, 2015, 96-108.

[30] C. van der Mee, V.N. Pivovarchik, *A Sturm-Liouville inverse spectral problem with boundary conditions depending on the spectral parameter*, Funct. Anal. Appl., **36(4)**, 2002, 315-317.

[31] N.J. Guliyev, *Inverse eigenvalue problems for Sturm-Liouville equations with spectral parameter linearly contained in one of the boundary conditions*, Inverse Problems, **21(4)**, 2005, 1315-1330.

[32] S.A. Buterin, V.A. Yurko, *An inverse spectral problem for pencils of differential operators on a finite interval*, Vestnik Bashkir. Univ., **4**, 2006, 8-12 (in Russian).

[33] Kh. R. Mamedov, *On the inverse problem for Strum–Liouville operator with a nonlinear spectral parameter in the boundary condition*, J. Korean Math. Soc., **46(6)**, 2009, 1243-1254.

[34] E.S. Panakhov, H. Koyunbakan, Ic. Unal, *Reconstruction formula for the potential function of Sturm-Liouville problem with eigenparameter boundary condition*, Inverse Probl. Sci. Eng.,**18(1)**, 2010, 173–180.

[35] Y. Güldü, R.Kh. Amirov, N. Topsakal, *On impulsive Sturm–Liouville operators with singularity and spectral parameter in boundary conditions*, Ukr. Math. J., **64(12)**, 2013, 1816-1838.

[36] K.R. Mamedov, F.A. Cetinkaya, *A uniqueness theorem for a Sturm-Liouville equation with spectral parameter in boundary conditions*, Appl. Math. Inf. Sci., **9(2)**, 2015, 981-988.

[37] M. Fallahi, M. Shahriari, F. Shareghi, *Reconstruction of the Sturm-Liouville operators with a finite number of tranmission and parameter dependent boundary conditions*, Azerbaijan Journal of Mathematics, **8(2)**, 2018, 3-20.

[38] V.A. Yurko, *Inverse problems for nonselfadjoint quasi-periodic differential*, Anal. Math. Phys., **2**, 2012, 215-230.

[39] V.A. Sadovnichii, Ya.T. Sultanaev, A.M. Akhtyamov, *Inverse problem for an operator pencil with nonseparated boundary conditions*, Dokl. Math., **79(2)**, 2009, 169-171.

[40] I.M. Nabiev, A.Sh. Shukurov, *Properties of the spectrum and uniqueness of reconstruction of Sturm–Liouville operator with a spectral parameter in the boundary condition*, Proc. of Institute of Math. and Mech. of NAS of Azerbaijan, **40(Special Issue)**, 2014, 332-341.

[41] G. Freiling, V. Yurko ,*Recovering nonselfadjoint differential pencils with nonseparated boundary conditions*, Applicable Anal., **94(8)**, 2015, 1649-1661.

[42] Ch.G. Ibadzadeh, I.M. Nabiev,*An inverse problem for Sturm–Liouville operators with nonseparated boundary conditions containing the spectral parameter*, J. Inverse Ill-Posed Probl., **24(4)**, 2016, 407–411.

[43] Ch.G. Ibadzadeh, I.M. Nabiev, *Reconstruction of the Sturm–Liouville operator with nonseparated boundary conditions and a spectral parameter in the boundary condition*, Ukr. Math. J., **69(9)**, 2018, 1416-1423.

[44] Ch.G. Ibadzadeh, I.M. Nabiev, *Asymptotics and mutual location of eigenvalues of the diffusion operators with non-separated boundary conditions and with a spectral parameter in the boundary condition*, News of Baku University, Ser. of phys.-math. sci., **3**, 2017, 49-56 (in Russian).

[45] B.Ya. Levin, *Lectures on entire functions*, Transl. Math. Monogr., **150**, Amer. Math. Soc., Providence, RI, 1996.

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