Azerbaijan Journal of Mathematics
V. 9, No 1, 2019, January
ISSN 2218-6816

A New Approach for Unique Restoration of a Time-Dependent Matrix Potential in a Hyperbolic Scattering Problem on the Semi-Axis

M.I. Ismailov*, I. Tekin

Abstract. This paper considers the inverse scattering problem on the semi-axis for the matrix hyperbolic system with a special structure of matrix potential. The possible relationship between the matrix scattering operators for the first order systems with unshifted and space-shifted potentials is given. By using this relationship, the new approach for unique restoration of this potential from the matrix scattering operators on the semi-axis is given.

Key Words and Phrases: inverse scattering problem, scattering operator, first order hyperbolic system, transformation operator.

2010 Mathematics Subject Classifications: 34L25, 35L50, 35P25

1. Introduction

First order hyperbolic systems of equations describe many physical processes associated with wave propagation. For instance, there exist systems of equations for acoustics, electromagnetic oscillations and those of dynamic equations of theory of elasticity. One velocity transport equation is reduced to a first order hyperbolic system under the assumption that the velocity accepts the finite fixed number of directions and the transport process is plane symmetric [19].

The inverse problems for the first order hyperbolic system play important role in different areas of applied mathematics and mathematical physics. The inverse problem for the first order hyperbolic system is a problem of finding the coefficients from the some known functionals of its solutions. Several results on the inverse problems for the first order hyperbolic system which are observed in

*Corresponding author.

seismology, geophysics and gravitation and magnitude detection are considered in [15] and in scattering problems for the wave propagation in [10].

The traditional model for wave propagation is the wave equation with coefficients that are independent of time [16]. For the problems of wave propagation in the non-stationary medium, the coefficients of the relevant PDEs vary not only in space, but also in time. The model problems, where time-dependent coefficients occur, can be found in [1, 2, 20]. There are many papers dealing with inverse problems in wave propagation, but only a few of them deal with the solution of the inverse problem for a non-stationary medium. The inverse scattering problems (ISPs) for time dependent potentials have been studied by L. P. Nizhnik ([10]). The ISP for the multidimensional wave equations with time-dependent potential was also considered in [17]. In [6, 14], the time-dependent potentials are determined by Dirichlet to Neumann mappings for the hyperbolic type equations.

The literature on the inverse problems for two component hyperbolic system is vast, see [4, 5, 13, 21, 22], to name just a few. In contrast to this case, the inverse problems for hyperbolic systems with more than two equations are not so many. It is important to notice the reference [3] which deals with the inverse mixed problem for the first order hyperbolic system and the references [10, 18] and [8, 9, 11] which are dedicated to ISP for the first order hyperbolic system in wholeaxis and in semi-axis, respectively.

The ISP on wholeaxis for the first order hyperbolic system is solved in [9] by the Gelfand-Levitan-Marchenko equation and the scattering data for the ISP is given in [12]. The Riemann-Hilbert approach to the ISP for the first order hyperbolic system on whole-axis is studied in [18].

This paper is dedicated to the ISP for the firstorder M-canonical hyperbolic system of size $2n$ on the semi-axis, under a general boundary condition. In a standard manner, the Volterra-type integral representation of the solution is introduced, some properties of the scattering matrix are described and then the problem of reconstructing the potential from the scattering operator is discussed. The potential is determined uniquely by two scattering operators for the considered system subject to two different boundary conditions.

We consider the direct determination of coefficients from the scattering operator on the semi-axis. The determination is realized in the following way: we show that the matrix scattering operator admits two-sided factorization by using transformation operator at infinity. Then we consider the scattering problem for the space-shifted system and we show the relationship between scattering operator of the original system and the space-shifted system. By the help of this relationship we can uniquely recover the potential for every $x > 0$ from the scattering operator of original system. The alternative way to show the uniqueness of the solution of ISP is considered in [9] by converting the problem on semi-axis to the problem on wholeaxis. This way allows us to give the algorithm of the solution and the description of scattering data of ISP on the semi-axis in terms of scattering operator on the wholeaxis, but not in the terms of operators on the semi-axis.

The rest of the paper is organized as follows. In Section 2, the formulation of the ISP for the M-canonical system is given and some necessary results on transformation operator are transferred from [9]. In Section 3, the relationship between scattering operators of systems with unshifted and space-shifted potentials is presented and the uniqueness of the solution of ISP is proved.

2. Scattering problem on the semi-axis

We consider the scattering problem for the system of the following form on the half-plane $\mathbb{R}_+ \times \mathbb{R}$: in the following form

$$\sigma\partial_t\psi - \partial_x\psi = Q(x,t)\psi,\ (x,t) \in \mathbb{R}_+ \times \mathbb{R}, \tag{1}$$

where $\sigma = \begin{bmatrix} \sigma_1 & 0_n \\ 0_n & \sigma_2 \end{bmatrix}$ is a constant matrix with the block diagonal matrices $\sigma_1 = diag\{\xi_1, ..., \xi_n\}$, $\sigma_1 = diag\{\xi_{n+1}, ..., \xi_{2n}\}$ with $\xi_1 > ... > \xi_n > 0 > \xi_{n+1} > ... > \xi_{2n}$.

Definition 1. *We will call the system (1) an M-canonical system, if it has matrix potential* $Q(x,t) = \begin{bmatrix} q_{11}(x,t) & q_{12}(x,t) \\ q_{21}(x,t) & q_{22}(x,t) \end{bmatrix}$ *where* q_{11} *and* $Pq_{22}P$ *are strictly lower triangular,* $q_{12}P$ *and* Pq_{21} *are lower triangular* $n \times n$ *matrix blocks,* P *is a permutation matrix with* $[P]_{j,n+1-j} = 1,\ j = 1, ..., n,\ [P]_{jk}$ *denotes the* (j,k) *element of matrix* P.

Let us consider matrix blocks $q_{ij}(x,t)$, $i,j = 1,2$ whose measurable complex-valued entries belong to the Schwartz class.

The scattering problem for the M-canonical system (1) on the semi-axis consists of finding the solution of M-canonical system (1) with the boundary condition at $x = 0$

$$\psi_2(0,t) = H\psi_1(0,t), \quad \det H \neq 0, \tag{2}$$

and the asymptotic condition

$$\psi_1(x,t) = \Im_{\sigma_1 x} a(t) + o(1),\ x \to +\infty, \tag{3}$$

where $\Im_{\sigma_1 x} = diag(T_{\xi_1 x}, ..., T_{\xi_n x})$, $\Im_{\sigma_2 x} = diag(T_{\xi_{n+1} x}, ..., T_{\xi_{2n} x})$ are shift operators such that $T_{\xi_i x} = h(t + \xi_i x)$, $i = 1, ..., 2n$ and $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix}$ with the n-dimensional column vectors ψ_1 and ψ_2. H is the constant transmission matrix of order n with $\det H \neq 0$ and the vector function $a(t)$ denotes the profile of incident waves. We will denote by $C_b(\mathbb{R}, \mathbb{C}^n)$ the space of $\mathbb{C}^n$-valued bounded continuous functions on $\mathbb{R}$.

By combining Theorem 1, Lemma 1 and 2 of [9], we have the following lemma.

Lemma 1. *For an arbitrary incident wave vector $a(t) \in C_b(\mathbb{R}, \mathbb{C}^n)$ there exists a unique bounded continuous solution of the scattering problem and the second component of the solution satisfies the asymptotic relation*

$$\psi_2(x, t) = \Im_{\sigma_2 x} b(t) + o(1), \; x \to +\infty, \tag{4}$$

where $b(t) \in C_b(\mathbb{R}, \mathbb{C}^n)$ defines the profile of the scattered waves. Moreover, the following additional properties of the solution of the scattering problem hold for $detH \neq 0$:

1) If $a(s) = 0$ for $s \leq \lambda$, then for $t + \xi_1 x \leq \lambda$, the solution of the scattering problem is equal to zero, i.e. $\psi_1(x, t) = \psi_2(x, t) = 0$.

2) If $b(s) = 0$ for $s \geq \lambda$, then for $t + \xi_{2n} x \geq \lambda$, the solution of the scattering problem is equal to zero, i.e. $\psi_1(x, t) = \psi_2(x, t) = 0$.

In the view of Lemma 1, for every incident vector function $a(t) \in C_b(\mathbb{R}, \mathbb{C}^n)$, when the M-canonical system (1) satisfies conditions (2) and (3), there exists a unique solution $\psi = \begin{bmatrix} \psi_1(x, t) \\ \psi_2(x, t) \end{bmatrix}$. For this solution there exist scattered waves $b(t) \in C_b(\mathbb{R}, \mathbb{C}^n)$ according to (4). By comparing the incident and scattered waves, the scattering operator S_H is defined by

$$b = S_H H a. \tag{5}$$

Single scattering operator for the M-canonical system (1) on the semi-axis is not sufficient to uniquely determine the coefficients of the M-canonical system (1) by the scattering operator $\mathbf{S}_H$. This was shown in [9].

Consider two scattering problems on the semi-axis for the M-canonical system (1).

First scattering problem: find a solution $\psi_1^1(x, t)$ and ψ_2^1 of the M-canonical system (1) such that the asymptotic relation

$$\psi_1^1(x, t) = \Im_{\sigma_1 x} a(t) + o(1), \; x \to +\infty,$$

holds and the boundary condition

$$\psi_2^1(0,t) = H_1\psi_1^1(0,t), \quad \det H_1 \neq 0, \tag{6}$$

is satisfied.

Second scattering problem: find a solution $\psi_1^2(x,t)$ and ψ_2^2 of the M-canonical system (1) such that the asymptotic relation

$$\psi_1^2(x,t) = \Im_{\sigma_1 x}a(t) + o(1), \ x \to +\infty,$$

holds and the boundary condition

$$\psi_2^2(0,t) = H_2\psi_1^2(0,t) \quad \det H_2 \neq 0, \tag{7}$$

is satisfied.

We are going to investigate the solution of the inverse scattering problem considering the first and second scattering problems together under the assumption

$$\det(H_1 - H_2) \neq 0.$$

According to Lemma 1, for arbitrary $a(t) \in C_b(\mathbb{R}, \mathbb{C}^n)$ the first and second scattering problems have unique bounded solution. Moreover, these solutions satisfy the asymptotic relations

$$\psi_2^k(x,t) = \Im_{\sigma_2 x}b^k(t) + o(1), \ x \to +\infty, \quad k = 1, 2,$$

where $b^k(t) \in C_b(\mathbb{R}, \mathbb{C}^n)$ defines the profile of the scattered waves.

The scattering operators corresponding to the first and second scattering problems are denoted by $\mathbf{S}_{H_k}$:

$$\mathbf{S}_{H_k} : H_k a(t) \to b^k, \quad k = 1, 2. \tag{8}$$

In solving ISPs, the Volterra type integral representation of the solution plays an important role. Such representation can be derived from transformation operator as $x \to +\infty$.

Lemma 2. *(Theorem 2, [9]) For any $a(t)$, $b(t)$, $(-\infty < t < +\infty)$, there exists a unique solution of the M-canonical system (1). This solution admits the representation*

$$\begin{aligned} \psi_1(x,t) &= [\mathbf{I}_n + \mathbf{A}_{11-}(x)]\Im_{\sigma_1 x}a(t) + \mathbf{A}_{12+}(x)\Im_{\sigma_2 x}b(t), \\ \psi_2(x,t) &= \mathbf{A}_{21-}(x)\Im_{\sigma_1 x}a(t) + [\mathbf{I}_n + \mathbf{A}_{22+}(x)]\Im_{\sigma_2 x}b(t), \end{aligned} \tag{9}$$

where $\mathbf{A}_{ij-}(x)f(t) = \int\limits_{t}^{+\infty} A_{ij}(x,t,s)\, f(s)ds$, $\mathbf{A}_{ij+}(x)f(t) = \int\limits_{-\infty}^{t} A_{ij}(x,t,s)\, f(s)ds$. *The kernels* $A_{ij}(x,t,s)$ $(i,j = 1,2)$ *are determined uniquely by the coefficients of the M-canonical system (1) and for the fixed* x *these kernels are the Hilbert-Schmidt kernels.*

In addition, these kernels are related to the potentials by the formulas

$$[A_{11}(x,t,t)]_{i,j} = \begin{cases} \frac{1}{\xi_j-\xi_i}[q_{11}]_{i,j}(x,t), & i \neq j, \\ \frac{1}{\xi_i-\xi_{2n-j+1}} \int\limits_{x}^{+\infty} \left\{[q_{12}]_{i,n-j+1}\,[q_{21}]_{n-j+1,i}\right\}(s, t+\xi_i(x-s))ds, & i = j, \end{cases}$$

$$[A_{12}(x,t,t)]_{i,j} = -\frac{1}{\xi_{n+j}-\xi_i}[q_{12}]_{i,j}(x,t), \quad [A_{21}(x,t,t)]_{i,j} = \frac{1}{\xi_j-\xi_{n+i}}[q_{21}]_{i,j}(x,t), \tag{10}$$

$$[A_{22}(x,t,t)]_{i,j} =$$

$$= \begin{cases} -\frac{1}{\xi_{n+j}-\xi_{n+i}}[q_{22}]_{i,j}(x,t), & i \neq j, \\ -\frac{1}{\xi_{n+i}-\xi_{n-j+1}} \int\limits_{x}^{+\infty} \left\{[q_{21}]_{i,n-j+1}\,[q_{12}]_{n-j+1,i}\right\}(s, t+\xi_{n+i}(x-s))ds, & i = j, \end{cases}$$

$i,j \;=\; 1,...,r$

Using the representation (9) and the boundary conditions (6) and (7) we obtain

$$b^k(t) = (\mathbf{I}_n + \mathbf{A}_{22+} - H_k\mathbf{A}_{12+})^{-1}(\mathbf{I}_n + H_k\mathbf{A}_{11-}H_k^{-1} - \mathbf{A}_{21-}H_k^{-1})H_k a(t), \tag{11}$$

where $\mathbf{A}_{ij-} = \mathbf{A}_{ij-}(0)$, $\mathbf{A}_{ij+} = \mathbf{A}_{ij+}(0)$, $i,j = 1,2$.

By comparing the formula (11) to the definition of the scattering operator, we obtain the following Volterra structure of the matrix scattering operators for the first and second scattering problems on the semi-axis:

$$\mathbf{S}_{H_k} = (\mathbf{I}_n + \mathbf{A}_{22+} - H_k\mathbf{A}_{12+})^{-1}({}_n + H_k\mathbf{A}_{11-}H_k^{-1} - \mathbf{A}_{21-}H_k^{-1}). \tag{12}$$

The scattering operators $\mathbf{S}_{H_k}$ $(k = 1,2)$ admit two-sided factorization. The formula (12) means that the matrix operator $\mathbf{S}_{H_k}$ admits right factorization. Moreover, this operator is given by the structure $\mathbf{S}_{H_k} = \mathbf{I}_n + \mathbf{F}_k$, where $\mathbf{F}_n$ is a matrix Hilbert-Schmidt operator. From (12) we also obtain the inversion of the operator $\mathbf{S}_{H_k}$. In addition, this operator is given by the structure $\mathbf{S}_{H_k}^{-1} = \mathbf{I}_n + \mathbf{G}_k$, where $\mathbf{G}_k$ is a matrix Hilbert-Schmidt operator. Let us denote the kernels of the operators $\mathbf{F}_k$ and $\mathbf{G}_k$ by $F_k(t,s)$, $G_k(t,s)$. The left factorization of $\mathbf{S}_{H_k}$ $(k=1,2)$ is proved in [9] by using

$$\begin{array}{l} \mathbf{A}_{11-}+\mathbf{A}_{12+}\mathbf{F}_k H_k = \mathbf{C}_{k+}-\mathbf{A}_{12+}H_k, \\ \mathbf{F}_k H_k+\mathbf{A}_{12-}+\mathbf{A}_{22+}\mathbf{F}_k H_k = \mathbf{D}_{k+}-\mathbf{A}_{22+}H_k, \\ \mathbf{A}_{12+}+H_k^{-1}\mathbf{G}_k+\mathbf{A}_{11-}H_k^{-1}\mathbf{G}_k = \mathbf{C}_{k-}-\mathbf{A}_{11-}H_k^{-1}, \\ \mathbf{A}_{22+}+\mathbf{A}_{21-}H_k^{-1}\mathbf{G}_k = \mathbf{D}_{k-}-\mathbf{A}_{21-}H_k^{-1}, \end{array} \tag{13}$$

and $\mathbf{F}_k\mathbf{G}_k = \mathbf{G}_k\mathbf{F}_k = -\mathbf{F}_k - \mathbf{G}_k$ where $\mathbf{C}_{k+}$, $\mathbf{D}_{k+}$, $\mathbf{C}_{k-}$, $\mathbf{D}_{k-}$, $(k=1,2)$ are Hilbert-Schmidt Volterra operators with the kernels

$$C_{k+}(t,s) = \int_{-\infty}^{t} A_{12}(0,t,\tau)F_k(\tau,s)H_k d\tau + A_{12}(0,t,s)H_k, \ \ s\le t,$$

$$D_{k+}(t,s) = F_k(t,s)H_k + \int_{-\infty}^{t} A_{22}(0,t,\tau)F_k(\tau,s)H_k d\tau + A_{22}(0,t,s)H_k, \ \ s\le t,$$

$$C_{k-}(t,s) = H_k^{-1}G_k(t,s) + \int_{t}^{+\infty} A_{11}(0,t,\tau)H_k^{-1}G_k(\tau,s)d\tau + A_{11}(0,t,s)H_k^{-1}, \ \ s\ge t,$$

$$D_{k-}(t,s) = \int_{t}^{+\infty} A_{21}(0,t,\tau)H_k^{-1}G_k(\tau,s)d\tau + A_{21}(0,t,s)H_k^{-1}, \ \ s\ge t.$$

3. Uniqueness and algorithm of the solution of ISP on the semi-axis

The problem of finding matrix potential Q from the known scattering matrices $\mathbf{S}_{H_k}$, $k=1,2$ is called the ISP of the M-canonical system (1), (2). The following result is about the M. G. Krein factorization of the Hilbert-Schmidt integral operators of the second kind ([7, 10]) and it will be firmly used below.

Lemma 3. *([10], Theorem 4.2.2) Let* $\mathbf{F}$ *be a Hilbert-Schmidt integral operator in* $L_2(\mathbb{R},\mathbb{C}^n)$ *and* $\mathbf{S}=\mathbf{I}_n+\mathbf{F}$. *If the operator* $\mathbf{S}$ *admits right factorization as* $\mathbf{S}=(\mathbf{I_n}-\mathbf{A}_+)^{-1}(\mathbf{I_n}-\mathbf{A}_-)^{-1}$, *then for any* δ *there exists an integral operator* $\mathbf{\Gamma}_\delta=(\mathbf{I_n}+\mathbf{F}\mathbf{E}_\delta)^{-1}\mathbf{F}$ *such that operators* $\mathbf{A}_+$ *and* $\mathbf{A}_-$ *are uniquely determined with respect to* $\mathbf{S}$ *by the formulas*

$$\begin{array}{rcl} A(t,\tau) & = & \mathbf{\Gamma}_t(t,\tau), \quad t\ge\tau, \\ A(t,\tau) & = & \mathbf{\Gamma}_\tau(t,\tau), \quad t\le\tau, \end{array}$$

where $A(t,\tau)$ $t\ge\tau$ *and* $A(t,\tau)$, $t\le\tau$ *are the kernels of the operators* $\mathbf{A}_+$ *and* $\mathbf{A}_-$, *respectively,* $\Gamma_\delta(t,\tau)$ *is the kernel of the operator* $\mathbf{\Gamma}_\delta$ *and* $\mathbf{E}_\delta$ *is a projection operator for* $t<\delta$.

Lemma 4. *([10], Theorem 4.3.3) Let* $\mathbf{F}$ *be a Hilbert-Schmidt integral operator in* $L_2(\mathbb{R}, \mathbb{C}^n)$ *and* $\mathbf{S} = \mathbf{I}_n + \mathbf{F}$*. If this operator admits two-sided factorization, then it is invertible,* $\mathbf{S}^{-1} = \mathbf{I_n} + \mathbf{G}$*, and it can be uniquely determined by* $\mathbf{F}_-$ *and* $\mathbf{G}_+$ *(or* $\mathbf{F}_+$ *and* $\mathbf{G}_-$*), where* $\mathbf{F}_-$ *and* $\mathbf{G}_+$ *are negative and positive parts of* $\mathbf{F}$ *and* $\mathbf{G}$*, respectively:*

$$\mathbf{F}_- h(t) = \int_t^{+\infty} \mathbf{F}(t,\tau)h(\tau)d\tau, \quad \mathbf{G}_{+-}h(t) = \int_{-\infty}^{t} \mathbf{G}(t,\tau)h(\tau)d\tau.$$

Using Lemma 3, from the formulas (10) and (12) we can determine the value of the potential for $x = 0$ from the scattering operator (the uniqueness of this determination will be proved in Theorem 2). To find the potential for any values $x_0 \geq 0$ it is natural to consider the scattering problem for the system (1) with the shifted potential

$$Q(x + x_0, t) = \begin{bmatrix} q_{11}(x + x_0, t) & q_{12}(x + x_0, t) \\ q_{21}(x + x_0, t) & q_{22}(x + x_0, t) \end{bmatrix}, \; x_0 > 0. \tag{14}$$

Denote by $\mathbf{S}_{H_k}(x_0)$ the scattering operators of the scattering problem on the semi-axis with potential (14). Based on Lemma 2, we conclude that for any $x_0 \geq 0$ the operators $\mathbf{S}_{H_k}$ admit the right factorization

$$\mathbf{S}_{H_k}(x_0) =$$
$$= (\mathbf{I}_n + \mathbf{A}_{22+}(x_0) - H_k\mathbf{A}_{12+}(x_0))^{-1}(\mathbf{I}_n + H_k\mathbf{A}_{11-}(x_0)H_k^{-1} - \mathbf{A}_{21-}(x_0)H_k^{-1}), \; k = 1, 2. \tag{15}$$

Moreover, the operators

$$\mathbf{F}_k(x_0) = \mathbf{S}_{H_k}(x_0) - \mathbf{I}_n, \; \mathbf{G}_k(x_0) = \mathbf{S}_{H_k}^{-1}(x_0) - \mathbf{I}_n$$

are Hilbert-Schmidt integral operators with the kernels denoted by $F_k(x_0, t, s)$, $G(x_0, t, s)$. It is easy to see that, for $x_0 = 0$, $F_k(0, t, s) = F_k(t, s)$, $G(0, t, s) = G(t, s)$.

The following theorem gives the unique solvability of the ISP for the problem (1)-(2).

Theorem 1. *Let* $\mathbf{S}_{H_k} = \mathbf{I}_n - \mathbf{F}_k$*,* $k = 1, 2$ *be scattering operators for the M-canonical system (1) with the boundary condition (2). The scattering operators* $\mathbf{S}_{H_k}(x_0) = \mathbf{I}_n - \mathbf{F}_k(x_0)$*,* $k = 1, 2$ *for the M-canonical system (1) with the shifted potential (14) are related to scattering operators* $\mathbf{S}_{H_k}$*,* $k = 1, 2$ *as follows:*

$$\begin{aligned} F_k(x_0, t, s) - \widetilde{F}_k(x_0, t, s) = 0, \; t \leq s, \\ G_k(x_0, t, s) - \widetilde{G}_k(x_0, t, s) = 0, \; t \geq s, \end{aligned} \tag{16}$$

where

$$\widetilde{F}_k(x_0,t,s)=\Im_{\sigma_2 x_0}\left[\sum_{i=1}^{n}F_k(t,s+\xi_i x_0)H_k^i\right]H_k^{-1},\ t\le s,$$
$$\widetilde{G}_k(x_0,t,s)=H_k\left[\sum_{i=1}^{n}\left(H_k^{-1}\right)^i\left(\Im_{\sigma_2 x_0}G_k^T(t+\xi_i x_0,s)\right)^T\right],\ t\ge s,$$

and H_k^i is the constant matrix with ith column equal to ith column of H_k and the rest part equal to zero, and $\left(H_k^{-1}\right)^i$ is the constant matrix with ith raw equal to ith raw of H_k^{-1} and the rest part equal to zero.

Proof. From the definition of the scattering operator $\mathbf{S}_{H_k}$ $(k=1,2)$ it follows that

$$b^k(s)=(\mathbf{I}_n+\mathbf{F}_k)H_k a(s)=H_k a(s)+\int_{-\infty}^{+\infty}F_k(s,\tau)H_k a(\tau)d\tau, \tag{17}$$

$$a(s)=H_k^{-1}(\mathbf{I}_n+\mathbf{G}_k)b^k(s)=H_k^{-1}b^k(s)+\int_{-\infty}^{+\infty}H_k^{-1}G_k(s,\tau)b^k(\tau)d\tau. \tag{18}$$

By taking into account (17), from the representation (9) for the first and second scattering problems, we obtain that

$$\psi_1^k(x,t)=\Im_{\sigma_1 x}a(t)+\int_t^{+\infty}A_{11}(x,t,s)\Im_{\sigma_1 x}a(s)ds+\int_{-\infty}^{t}A_{12}(x,t,s)\Im_{\sigma_2 x}H_k a(s)ds$$
$$+\int_{-\infty}^{t}A_{12}(x,t,s)\Im_{\sigma_2 x}\left(\int_{-\infty}^{+\infty}F_k(s,\tau)H_k a(\tau)d\tau\right)ds,$$

$$\psi_2^k(x,t)=\Im_{\sigma_2 x}H_k a(t)+\int_{-\infty}^{+\infty}\Im_{\sigma_2 x}F_k(t,s)H_k a(s)ds$$
$$+\int_t^{+\infty}A_{21}(x,t,s)\Im_{\sigma_1 x}a(s)ds+\int_{-\infty}^{t}A_{22}(x,t,s)\Im_{\sigma_2 x}H_k a(s)ds$$
$$+\int_{-\infty}^{t}A_{22}(x,t,s)\Im_{\sigma_2 x}\left(\int_{-\infty}^{+\infty}F_k(s,\tau)H_k a(\tau)d\tau\right)ds,\ k=1,2.$$

If $a_1(s)=0$ for $s\le\lambda$, then by Lemma 1, $\psi_1^k(x,t)=\psi_2^k(x,t)=0$, $k=1,2$, for $t+\xi_1 x\le\lambda$. Then we obtain that for $x\ge 0$ and $t+\xi_1 x\le\lambda$

$$0=\int_{\lambda}^{+\infty}\left(\Im_{-\sigma_1 x}A_{11}^T(x,t,s)\right)^T a(s)ds+$$

$$\begin{aligned} &+ \int\limits_{-\infty}^{t} A_{12}(x,t,s)\Im_{\sigma_2 x}\left(\int\limits_{\lambda}^{+\infty} F_k(s,\tau)H_k a(\tau)d\tau\right)ds,\\ 0 &= \int\limits_{\lambda}^{+\infty} \Im_{\sigma_2 x}F_k(t,\tau)H_k a(\tau)d\tau + \int\limits_{\lambda}^{+\infty}(\Im_{-\sigma_1 x}A_{21}^T(x,t,s))^T a(s)ds\\ &+ \int\limits_{-\infty}^{t} A_{22}(x,t,s)\Im_{\sigma_2 x}\left(\int\limits_{\lambda}^{+\infty} F_k(s,\tau)H_k a(\tau)d\tau\right)ds, \end{aligned}$$

where A^T denotes the transpose of matrix A. Taking into account that the function $a_1(s)$ $(s \geq \lambda)$ is arbitrary, it follows from the last equalities that, for $t+\xi_1 x \leq \lambda$ and $s \geq \lambda$ (i.e. $t+\xi_1 x \leq s$),

$$\begin{gathered} (\Im_{-\sigma_1 x}A_{11}^T(x,t,s))^T + \int\limits_{-\infty}^{t} A_{12}(x,t,\tau)\Im_{\sigma_2 x}F_k(\tau,s)H_k d\tau = 0,\\ \Im_{\sigma_2 x}F_k(t,s)H_k + (\Im_{-\sigma_1 x}A_{21}^T(x,t,s))^T + \int\limits_{-\infty}^{t} A_{22}(x,t,\tau)\Im_{\sigma_2 x}F_k(\tau,s)H_k d\tau = 0. \end{gathered} \tag{19}$$

Similarly, due to Lemma 1, from (18) and (9) we obtain the following relations between transformation operator and scattering operators for the first and second scattering problems:

$$\begin{gathered} \Im_{\sigma_1 x}H_k^{-1}G_k(t,s) + (\Im_{-\sigma_2 x}A_{12}^T(x,t,s))^T + \int\limits_{t}^{+\infty} A_{11}(x,t,\tau)\Im_{\sigma_1 x}H_k^{-1}G_k(\tau,s)d\tau = 0,\\ (\Im_{-\sigma_2 x}A_{22}^T(x,t,s))^T + \int\limits_{t}^{+\infty} A_{21}(x,t,\tau)\Im_{\sigma_1 x}H_k^{-1}G_k(\tau,s)d\tau = 0, \quad t+\xi_{2n}x \geq s. \end{gathered} \tag{20}$$

From (19) by substraction we obtain for $t \leq s$

$$\begin{aligned} &H_k A_{11}(x_0,t,s)H_k^{-1} - A_{21}(x_0,t,s)H_k^{-1} - \widetilde{F}_k(x_0,t,s)\\ &+ \int\limits_{-\infty}^{t}\left[H_k A_{12}(x_0,t,\tau) - A_{22}(x_0,t,\tau)\right]\widetilde{F}_k(x_0,t,\tau)d\tau = 0. \end{aligned} \tag{21}$$

Considering for $t \geq s$ the left hand side of (21) as the kernel of some operator $\mathbf{R}_{k+}(x_0)$ and $\widetilde{F}_k(x_0,t,s)$ as the kernel of operator $\mathbf{F}_k^{x_0}$, we rewrite (21) in the operator form:

$$H_k\mathbf{A}_{11-}(x_0)H_k^{-1} - \mathbf{A}_{21-}(x_0)H_k^{-1} - \left[\mathbf{I}_n + \mathbf{A}_{22+}(x_0) - H_k\mathbf{A}_{12+}(x_0)\right]\mathbf{F}_k^{x_0} = \mathbf{R}_{k+}(x_0),$$

hence

$$\mathbf{F}_k(x_0) - \mathbf{F}_k^{x_0} = (\mathbf{I}_n + \mathbf{A}_{22+}(x_0) - H_k\mathbf{A}_{12+}(x_0))^{-1}(\mathbf{I}_n + \mathbf{R}_{k+}(x_0)) - \mathbf{I}_n.$$

Taking into account that the right hand side is Volterra operator, we obtain that for $t \leq s$

$$F_k(x_0,t,s) - \widetilde{F}_k(x_0,t,s) = 0, \ t \leq s.$$

By analogy, the rest of (16) is proved by using (20). ◄

In this way, the algorithm of recovering of the potential $Q(x,t)$ from the scattering operators $\mathbf{S}_{H_k}$ is as follows:

(1) construct the operator $\mathbf{S}_{H_k}(x_0)$, $x_0 > 0$, by Lemma 4 and formula (16);

(2) find the factorization factors $\mathbf{A}_{22+}(x_0) - H_k\mathbf{A}_{12+}(x_0)$ and $H_k\mathbf{A}_{11-}(x_0)H_k^{-1} - \mathbf{A}_{21-}(x_0)H_k^{-1}$ from (15), by Lemma 3;

(3) determine the matrix coefficients $q_{ij}(x_0,t)$, $i,j = 1,2$ with respect to kernels of the operators $A_{ij+}(x_0)$ and $A_{ij-}(x_0)$, $i,j = 1,2$, by the formulas (10).

The uniqueness of this determination is proved by the following theorem.

Theorem 2. *Let the coefficients of the M-canonical system (1) belong to Schwartz class and $\mathbf{S}_{H_k}$, $k = 1,2$ be the scattering operators for the M-canonical system (1) on the half-plane, where the matrices H_1 and H_2 satisfy* $\det(H_1 - H_2) \neq 0$. *The coefficients of the M-canonical system (1) are uniquely determined by the scattering operators $\mathbf{S}_{H_k}$, $k = 1,2$.*

Proof. Assume that there are two potentials $Q^{(1)}$ and $Q^{(2)}$. Since A_{ij+} and A_{ij-}, $i,j = 1,2$, are determined uniquely by $\mathbf{S}_{H_k}$, $k = 1,2$ (Theorem 4 [9]). According to (10) we get $[q_{11}^{(1)}]_{i,j}(x,t) = [q_{11}^{(2)}]_{i,j}(x,t)$, $[q_{22}^{(1)}]_{i,j}(x,t) = [q_{22}^{(2)}]_{i,j}(x,t)$ for $i \neq j$, and

$$[q_{12}^{(1)}]_{i,j}(x,t) = [q_{12}^{(2)}]_{i,j}(x,t), \ [q_{21}^{(1)}]_{i,j}(x,t) = [q_{21}^{(2)}]_{i,j}(x,t)$$

$$\left[q_{21}^{(1)} - q_{21}^{(2)}\right]_{n-j+1,i}(x,t) + \int\limits_x^{+\infty} \left\{ \left[q_{12}^{(1)}\right]_{i,n-j+1} \left[q_{21}^{(1)} - q_{21}^{(2)}\right]_{n-j+1,i} \right.$$
$$\left. + \left[q_{12}^{(1)} - q_{12}^{(2)}\right]_{i,n-j+1} \left[q_{21}^{(2)}\right]_{n-j+1,i} \right\} (s, t + \xi_i(x-s))ds = 0,$$

$$\left[q_{12}^{(1)} - q_{12}^{(2)}\right]_{n-j+1,i}(x,t) + \int\limits_x^{+\infty} \left\{ \left[q_{21}^{(1)}\right]_{i,n-j+1} \left[q_{12}^{(1)} - q_{12}^{(2)}\right]_{n-j+1,i} \right.$$
$$\left. + \left[q_{21}^{(1)} - q_{21}^{(2)}\right]_{i,n-j+1} \left[q_{12}^{(2)}\right]_{n-j+1,i} \right\} (s, t + \xi_{n+i}(x-s))ds = 0$$

for $i = j$. Let us denote $Z_{12}(x,t) = q_{12}^{(1)}(x,t) - q_{12}^{(2)}(x,t)$, $Z_{21}(x,t) = q_{21}^{(1)}(x,t) -$

$q_{21}^{(2)}(x,t)$. Then the above formulas can be written in the following form:

$$\begin{aligned}
&[Z_2(x,t)]_{ij}(x,t)=0, \quad [Z_3(x,t)]_{ij}(x,t)=0\\
&[Z_3(x,t)]_{n-j+1,i}(x,t)+\int\limits_x^{+\infty}\left\{\left[Q_2^{(1)}\right]_{i,n-j+1}[Z_3(x,t)]_{n-j+1,i}\right.\\
&\qquad\left.+[Z_2(x,t)]_{i,n-j+1}\left[Q_3^{(2)}\right]_{n-j+1,i}\right\}(s,t+\xi_i(x-s))ds=0,\\
&[Z_2(x,t)]_{n-j+1,i}(x,t)+\int\limits_x^{+\infty}\left\{\left[Q_3^{(1)}\right]_{i,n-j+1}[Z_2(x,t)]_{n-j+1,i}\right.\\
&\qquad\left.+[Z_3(x,t)]_{i,n-j+1}\left[Q_2^{(2)}\right]_{n-j+1,i}\right\}(s,t+\xi_{n+i}(x-s))ds=0.
\end{aligned} \tag{22}$$

The system (22) is the homogeneous system of Volterra integral equations with respect to x. Since its coefficients belong to Schwartz class, it has only a trivial solution ($Z_{12}(x,t)=Z_{21}(x,t)=0$). Thus, $q_{12}^{(1)}(x,t)=q_{12}^{(2)}(x,t)$, $q_{21}^{(1)}(x,t)=q_{21}^{(2)}(x,t)$. The theorem is proved. ◄

4. Conclusion

This paper is dedicated to the ISP for the first-order M-canonical hyperbolic system of size $2n$ on the semi-axis, under a general boundary condition. In a standard manner, the Volterra-type integral representation of the solution is introduced, some of properties of the scattering operator are described and then the problem of reconstructing the potential from the scattering operator is discussed. The main result (Theorem 2) claims that the potential is determined uniquely by two scattering matrices for the considered system subject to two different boundary conditions, $\psi_2(0,t)=H_k\psi_1(0,t)$, $\det H_k\neq 0$, $k=1,2$, provided that H_1-H_2 is non-singular. Examples given in [9] show that (a) one scattering matrix is insufficient for unambiguous reconstruction and (b) the condition $det(H_1-H_2)=0$ is crucial.

We also consider the direct determination of coefficients from the scattering operator on the semi-axis. The determination is realized in the following way: we show that the matrix scattering operator admits two-sided factorization by using transformation operator at infinity. Then we consider the scattering problem for the space-shifted system and we show the relationship between scattering operator of the original system and the space-shifted system (Theorem 1). By the help of this relationship we can uniquely recover the potential for every $x>0$ from the scattering operator of original system. The existing over-determination in recovering of the potential will be relieved by the relations between the components of scattering operator, which suggests a line for further investigation. The another way to show the uniqueness of the solution of ISP is presented in [9] by converting the problem on semi-axis to the problem on wholeaxis. Since the ISP on the wholeaxis has been sufficiently well studied (the uniqueness is shown, the algorithm of the recovering of coefficients and the description of scattering data are given) in [12, 23], the algorithm of the solution of ISP on the semi-axis is naturally extended and the description of scattering data is given in terms of matrix scattering operators on the wholeaxis.

References

[1] I. Åberg, A. Karlsson, *"The source problem"—Transient waves propagating from internal sources in non-stationary media*, Wave Motion, **26**, 1997, 43-68.

[2] I. Åberg, G. Kristensson, J.N. Wall, *Propagation of transient electromagnetic waves in time-varying media-direct and inverse scattering problems*, Inverse Problems, **11**, 1995, 29-49.

[3] I.S. Ayoubi, *The Riemann-Green function and the invariant imbedding equations for hyperbolic systems of first-order*, Appl. Math. Comput., **55(2-3)**, 1993, 101-129.

[4] D.D. Bui, P. Nelson, *Existence and uniqueness of an inverse problem for a hyperbolic system of lidar probing*, Inverse Problems, **8(6)**, 1992, 821-829.

[5] R. Burridge, *The Gelfand-Levitan, the Marchenko, and the Gopinath-Sondhi integral equations of inverse scattering theory, regarded in the context of inverse impulse-response problems*, Wave Motion, **2**, 1980, 305-323.

[6] G. Eskin, *Inverse hyperbolic problems with time-dependent coefficients*, Commun. Par. Differ. Eqns., **32**, 2007, 1737-1758.

[7] I. Gohberg, M. Krein, *Theory and Applications of Volterra Operators in Hilbert Space*, Providence, RI, USA: American Mathematical Society, 1970.

[8] N.Sh. Iskenderov, *Inverse non-stationary scattering problem for hyperbolic system of N equations of the first order on a semi-axis*, Ukr. Math. J., **43**, 1991, 1524-1532.

[9] M.I. Ismailov,*Inverse scattering problem for hyperbolic system on a semi-axis in the case of equal number of incident and scattered waves*, Inverse Problems, **22**, 2006, 955-974.

[10] L.P. Nizhnik, *Inverse scattering problems for hyperbolic equations*, Nauka Dumka, Kiev, 1991 (in Russian).

[11] L.P. Nizhnik, N.Sh. Iskenderov, *An inverse nonstationary scattering problem for a hyperbolic system of three first-order equations on the half-axis*, Ukrain. Mat. Zh., **42**, 1990, 931-938.

[12] L.P. Nizhnik, V.G. Tarasov, *The inverse nonstationary scattering problem for a hyperbolic system of equations*, Sov. Math. Dokl., **18**, 1977, 397-401.

[13] Rakesh, *A one-dimensional inverse problem for a hyperbolic system with complex coefficients*, Inverse Problems, **17**, 2001, 1401-1417.

[14] A. Ramm, J. Sjöstrand, *An inverse problem of the wave equation*, Math. Z., **206**, 1991, 119-130.

[15] V.G. Romanov, *Inverse problems of Mathematical Physics*, VNU Science Press BV, Utrecht, Netherlands, 1987.

[16] A.B. Shabat, *An one-dimensional scattering theory*, I. Diferencial'nye Uravnenija, **8(1)**, 1972, 164-178 (in Russian).

[17] P. Stefanov, *Uniqueness of the multi-dimensional inverse scattering problem for time dependent potentials*, Math. Z., **201**, 1989, 541-560.

[18] L.Y. Sung, A.S. Fokas, *Inverse problem for $N \times N$ hyperbolic systems on the plane and $N-$ wave interactions*, Comm. Pure Appl. Math., **44**, 1991, 535-571.

[19] V.S. Vladimrov, *Mathematical problems of the one-velocity theory of particle transport*, Proc. V. A. Steklov Inst. Math., **61**, 1961, 3-158 (in Russian).

[20] A.E. Yagle, *Inverse scattering for time-varying one-dimensional layered media: algorithms and applications*, Inverse Problems, **13**, 1997, 867-88.

[21] M. Yamamoto, M. Trooshin, *Identification problem for one dimensional vibrating system*, Math. Methods Appl. Sci., **28**, 2005, 2037-2059.

[22] B. Zhu, X. Dou, *On an inverse initial boundary value problem for a one dimensional hyperbolic system of differential equations*, Acta Math. Appl. Sinica, **13**, 1997, 33-44.

[23] V.G. Tarasov,*Scattering data in an inverse scattering problem for a hyperbolic system of first-order differential equations*, Dopov Nats Akad Nauk Ukr Mat. Prirodozn Tekh Nauki, **7**, 1997, 60-67.

Mansur I. Ismailov
Department of Mathematics, Gebze Technical University, Gebze-Kocaeli 41400, Turkey
E-mail: mismailov@gtu.edu.tr

Ibrahim Tekin
Department of Mathematics, Bursa Technical University, Yıldırım-Bursa, 16310, Turkey
E-mail: ibrahim.tekin@btu.edu.tr

Received 29 May 2018
Accepted 07 August 2018