Azerbaijan Journal of Mathematics
V. 8, No 2, 2018, July
ISSN 2218-6816

On the Degree of Approximation by the Woronoi-Nörlund and Riesz Type Means in the *GHM*

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Abstract. The first results on approximation in the Hölder metric is based on the study of Prössdorf. In 1979, Leindler's paper on the generalizations of Prössdorf's theorems appeared. Later, in 2009, Leindler studied a similar problem on approximation by the Woronoi-Nörlund and the Riesz means which are more general than the Cesáro means with respect to the generalized Hölder metric (*GHM*) given by Das, Nath and Ray. In this paper, our aim is to give some results extending those of Leindler on the degree of approximation in *GHM* by using more general methods of means on the classes larger than classes of sequences used in Leindler's study.

Key Words and Phrases: Woronoi-Nörlund submethod, Riesz submethod, Hölder metric, degree of approximation.

2010 Mathematics Subject Classifications: 40G05, 41A25, 42A10

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1. Introduction and some notations

Let f be a $2\pi-$periodic function and $f \in L_p := L_p\,(0, 2\pi)$ for $p \geq 1$. Then by

$$s_n(f;x) = \frac{1}{2}a_0 + \sum_{k=1}^{n}(a_k \cos kx + b_k \sin kx) \equiv \sum_{k=1}^{n} A_k(f;x),$$

we denote a partial sum of the first $(n+1)$ terms of the Fourier series of $f \in L_p\,(p \geq 1)$ at a point x.

The degree of approximation of the sum $s_n(f;x)$ in different spaces has been studied by many authors. In [20], Quade investigated approximation properties of the partial sum of Fourier series in L_p norms. Chandra in [4] and Leindler in [13] generalized the results of Quade using the Woronoi-Nörlund and the Riesz

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means of Fourier series for some sequence classes. A similar problem was studied for more general means in [7] and [8].

The space

$$H_\alpha = \{f \in C_{2\pi} : |f(x) - f(y)| \leq K|x-y|^\alpha, 0 < \alpha \leq 1\},$$

where K is a positive constant, not necessarily the same at each occurrence, is a Banach space (see Prössdorf, [18]) with the norm $\|\cdot\|_\alpha$ defined by

$$\|f\|_\alpha = \|f\|_C + \sup_{x \neq y} \Delta^\alpha f(x,y), \tag{1}$$

where

$$\Delta^\alpha f(x,y) = \frac{|f(x) - f(y)|}{|x-y|^\alpha} \ (x \neq y),$$

by convention $\Delta^0 f(x,y) = 0$ and

$$\|f\|_C = \sup_{x \in [-\pi,\pi]} |f(x)|.$$

The metric generated by the norm (1) on H_α is called the Hölder metric. Prösdorff has studied the degree of approximation by Cesáro means in the Hölder metric and proved the following theorem.

Theorem 1 ([18]). *Let $f \in H_\alpha (0 < \alpha \leq 1)$ and $0 \leq \beta < \alpha \leq 1$. Then*

$$\|\sigma_n(f) - f\|_\beta = O(1) \begin{cases} n^{\beta-\alpha} & , 0 < \alpha < 1; \\ n^{\beta-1} \ln n & , \alpha = 1, \end{cases}$$

where $\sigma_n(f)$ is the Cesáro means of the Fourier series of f.

The case $\beta = 0$ in Theorem 1 has been considered by Alexits [1]. Leindler introduced more general classes than the Hölder classes of 2π-periodic continuous functions, generalized the results of Prössdorf [12]. Chandra obtained a generalization of Theorem 1 by considering the Woronoi-Nörlund transform [3]. In [16], Mohapatra and Chandra considered the problem by a matrix means of the Fourier series of $f \in H_\alpha$.

Further generalizations of the Hölder metric was given in [5] and [6]. In [5], Das *et al.* studied the degree of approximation by infinite matrix means involved in the deferred Cesàro means in a generalized Hölder metric. In [9], the degree of approximation of functions with respect to the norm given in [5] by the deferred Woronoi-Nörlund means and the deferred Riesz means of the Fourier series of the functions has been considered.

The modulus of continuity of $f \in C_{2\pi}$ is defined by

$$\omega(f,\delta) := \sup_{0<|h|\leq\delta} |f(x+h) - f(x)|,$$

where $C_{2\pi}$ is a space of all 2π-periodic and continuous functions defined on $[0, 2\pi]$ with the supremum norm. According to this, the class of functions H^{ω} is defined by

$$H^{\omega} := \{f \in C_{2\pi} : \omega(f,\delta) = O(\omega(\delta))\},$$

where $\omega(\delta)$ is a modulus of continuity.

In [15], Mazhar and Totik estimated, to the best possible extent, the degree of approximation of a function $f \in C_{2\pi}$ by the T-means of its Fourier series for the class H^{ω}. A generalization of H^{ω} space has been given by Das *et al.* in [6]: if $f \in L_p(0, 2\pi)$, $p \geq 1$, then

$$H_p^{(\omega)} := \{f \in L_p : A(f,\omega) < \infty\},$$

where ω is a modulus of continuity

$$A(f,\omega) := \sup_{t\neq 0} \frac{||f(\cdot + t) - f(\cdot)||_p}{\omega(|t|)},$$

$\|.\|_p$ denotes L_p-norm with respect to x and is defined by

$$\|f\|_p := \left\{ \frac{1}{2\pi} \int_0^{2\pi} |f(x)|^p \, dx \right\}^{\frac{1}{p}}.$$

The norm in the space $H_p^{(\omega)}$ is given by

$$||f||_p^{(\omega)} := ||f||_p + A(f,\omega), \tag{2}$$

and the metric generated by the norm (2) on $H_p^{(\omega)}$ is called the generalized Hölder metric (GHM). Das *et al.* proved the following theorem.

Theorem 2 ([6]). *Let v and w be moduli of continuity such that $\frac{w(t)}{v(t)}$ is nondecreasing. If $f \in H_p^{(\omega)}$, $p \geq 1$, then*

$$\|s_n - f\|_p^{(v)} = O\left(\frac{w(\pi/n)}{v(\pi/n)} \log n\right) + O(1)\frac{1}{n} \int_{\pi/n}^{\pi} \frac{w(t)}{t^2 v(t)} dt.$$

In [13], Leindler has established the following result improving Theorem 2.

Theorem 3 ([14]). *Let v and w be moduli of continuity such that $\frac{w(t)}{v(t)}$ is nondecreasing. Moreover, let the function*

$$\gamma(t) := \gamma(v, w, \varepsilon; t) := t^{-\varepsilon} \frac{w(t)}{v(t)},$$

be nonincreasing for some $0 < \varepsilon \leq 1$. If $f \in H_p^{(\omega)}$, $p \geq 1$. Then

$$||s_n - f||_p^{(v)} = O\left(\frac{w(1/n)}{v(1/n)} \log n\right) \quad \textit{for all} \quad n \geq 2.$$

According to this theorem, Leindler showed that if there exists an $\varepsilon > 0$ such that $t^{-\varepsilon}\frac{\omega(t)}{v(t)}$ is also nonincreasing, then the second term in Theorem 2 can be removed. In that paper, he also considered the degree of approximation of $f \in H_p^{(\omega)}$ by the Woronoi-Nörlund and the Riesz means defined as follows, respectively. Let $\{p_n\}$ be a positive sequence,

$$N_n(f; x) := \frac{1}{P_n} \sum_{m=0}^{n} p_{n-m} s_m(f; x),$$

$$R_n(f; x) := \frac{1}{P_n} \sum_{m=0}^{n} p_m s_m(f; x),$$

where $P_n = p_0 + p_1 + p_2 + ... + p_n \neq 0$ $(n \geq 0)$, and by convention $p_{-1} = P_{-1} = 0$.

Moreover, the results on the degree of approximation to functions by matrix means of Fourier series which are more general than the Woronoi-Nörlund and the Riesz means in *GHM* under some conditions can be found in [10]. Finally, let us recall some sequence classes. Assume that $u := (u_n)$ is a nonnegative sequence and $C := (C_n) = \frac{1}{n+1} \sum\limits_{m=0}^{n} u_m$. A sequence u is called almost monotone decreasing (briefly $u \in AMDS$) (increasing (briefly $u \in AMIS$)), if there exists a constant $K := K(u)$ which only depends on u such that

$$u_n \leq K u_m \qquad (K u_n \geq u_m)$$

for all $n \geq m$.

If $C \in AMDS$ $(C \in AMIS)$, then we say that the sequence u is almost monotone decreasing (increasing) mean sequence and denote $u \in AMDMS$ $(u \in AMIMS)$. Moreover, Mohapatra and Szal showed that the following embedding relations are true (see [17]):

$$AMDS \subset AMDMS$$

and

$$AMIS \subset AMIMS.$$

Throughout this paper, we shall use notations $D \ll R$ $(R \ll D)$ in inequalities if there exists a positive constant K such that $D \leq KR$ $(R \leq KD)$. However, K may be different in different occurrences of " $\ll$ ". We shall also use the notation

$$\Delta a_n = a_n - a_{n+1}.$$

2. Woronoi-Nörlund and Riesz type submethods

In this section we will give some definitions stated in [2] and [7], and also state results related to inclusions demonstrating the importance of these methods given in [7].

Let $\{\lambda(n)\}_{n=1}^{\infty}$ be a strictly increasing sequence of positive integers. The Cesáro submethod C_λ is defined as

$$(C_\lambda x)_n = \frac{1}{\lambda(n)+1} \sum_{k=0}^{\lambda(n)} x_k \quad , (n = 0, 1, 2, ...),$$

where x_k is a sequence of real or complex numbers. Therefore, the C_λ-method yields a subsequence of the Cesáro method C_1, and hence it is regular for any λ. C_λ is obtained by deleting a set of rows from Cesáro matrix.

Let $E = \{\lambda(n)\}_{n=1}^{\infty}$ and $F = \{\mu(n)\}_{n=1}^{\infty}$ be infinite subsets of $\mathbb{N}$. In [2] it was shown that $C_\lambda \subset C_\mu$ if and only if $F \setminus E$ is finite. Moreover, C_λ is equivalent to C_μ if and only if the symmetric difference $E \triangle F$ is finite. In particular, we see that $C_\lambda \subset C_1$ for any λ. The basic properties of C_λ-method can be found in [2] and [19].

In [7], the Woronoi-Nörlund submethod and Riesz submethod are given as follows, respectively:

$$N_n^\lambda(f; x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m} s_m(f; x),$$

$$R_n^\lambda(f; x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_m s_m(f; x),$$

where

$$P_{\lambda(n)} = p_0 + p_1 + p_2 + ... + p_{\lambda(n)} \neq 0 \quad (\lambda(n) \geq n \geq 0),$$

and by convention $p_{-1} = P_{-1} = 0$. The case $p_n = 1$ for all $(n \geq 0)$ of either $N_n^{\lambda}(f;x)$ or $R_n^{\lambda}(f;x)$ yields

$$\sigma_n^{\lambda}(f;x) = \frac{1}{\lambda(n)+1} \sum_{m=0}^{\lambda(n)} s_m(f;x).$$

Note that if we take $\lambda(n) = n$ in the last equality, then it coincides with the Cesàro means. Now let us give some inclusions stated in [11] due to these methods. Assume that $E = \{\lambda(n)\}_{n=1}^{\infty}$ and $F = \{\mu(n)\}_{n=1}^{\infty}$ are infinite subsets of $\mathbb{N}$.

Theorem 4 ([11]). *$N_n^{\lambda} \subseteq N_n^{\mu}$ if and only if $F \setminus E$ is finite, where $p_{\lambda(n)}/P_{\lambda(n)} \to 0$ and $p_{\mu(n)}/P_{\mu(n)} \to 0$ as $n \to \infty$.*

The next result is related to Theorem 4 taking into account $E \triangle F = (E \setminus F) \cup (F \setminus E)$.

Theorem 5 ([11]). *$N_n^{\lambda} \sim N_n^{\mu}$ if and only if $F \triangle E$ is finite, where $p_{\lambda(n)}/P_{\lambda(n)} \to 0$ and $p_{\mu(n)}/P_{\mu(n)} \to 0$ as $n \to \infty$.*

Remark 1. *In particular, we see that $N_n \subseteq N_n^{\mu}$ for any μ. Because the set $F \setminus \{0, 1, 2, \ldots\}$ is empty.*

Remark 2. *The similars of Theorem 4 and Theorem 5 can be also stated for the Riesz submethod in case $P_{\lambda(n)} \to \infty$ and $P_{\mu(n)} \to \infty$ as $n \to \infty$ instead of $p_{\lambda(n)}/P_{\lambda(n)} \to 0$ and $p_{\mu(n)}/P_{\mu(n)} \to 0$ as $n \to \infty$ in Theorem 4 and Theorem 5, respectively.*

Theorem 6 ([11]). *Let $\{p_n\}$ be a positive nonincreasing sequence. If*

$$\limsup_{n\to\infty} \frac{\lambda(n+1) - \lambda(n)}{P_{\lambda(n)}} = 0, \tag{2}$$

then the N_n^{λ}-method is equivalent to the N_n-method for bounded sequences.

Remark 3. *The similar theorem for Riesz method can be stated by taking*

$$\limsup_{n\to\infty} \frac{P_{\lambda(n+1)} - P_{\lambda(n)}}{P_{\lambda(n)}} = 0$$

instead of the condition (2) in Theorem 6. In this case, we don't need impose any monotonicity condition on the sequence $\{p_n\}$.

3. Approximation by Woronoi-Nörlund and Riesz submethods in *GHM*

In this part, we shall consider the degree of approximation to the functions belonging to the class $H_p^{(w)}$ by trigonometric polynomials given in [7] on large classes of sequences.

3.1. Some auxiliary results

Since $\gamma(t)$ is nonincreasing, we have

$$\frac{w(1/\lambda(n))}{v(1/\lambda(n))} \gg \lambda(n)^{-\varepsilon} \geq \frac{1}{\lambda(n)}. \tag{3}$$

Furthermore, we have

$$\frac{w(1/\tilde{\lambda}(n))}{v(1/\tilde{\lambda}(n))} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))}, \tag{4}$$

because $\frac{w(t)}{v(t)}$ is nondecreasing, where $\tilde{\lambda}(n)$ denotes the integer part of $\lambda(n)/2$. We need the following lemmas to prove main theorems in this section.

Lemma 1. *Let v and w be moduli of continuity such that $\frac{w(t)}{v(t)}$ is nondecreasing. Moreover, let the function*

$$\gamma(t) = t^{-\varepsilon}\frac{w(t)}{v(t)}$$

be nonincreasing for some $0 < \varepsilon \leq 1$. If $f \in H_p^{(\omega)}$, $p \geq 1$. Then

$$||s_n^\lambda - f||_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n) \quad \textit{for all} \quad \lambda(n) \geq n \geq 2. \tag{5}$$

Proof. By Theorem 2, we easily obtain

$$\|s_n^\lambda - f\|_p^{(v)} = O\left(\frac{w(\pi/\lambda(n))}{v(\pi/\lambda(n))}\log\lambda(n)\right) + O(1)\frac{1}{\lambda(n)}\int_{\pi/\lambda(n)}^{\pi}\frac{w(t)}{t^2 v(t)}dt. \tag{6}$$

Since $\gamma(t)$ is nonincreasing, the second term on the right-hand side of (6) is estimated as follows

$$\int_{\pi/\lambda(n)}^{\pi}\frac{w(t)}{t^2v(t)}dt = \int_{\pi/\lambda(n)}^{\pi}\frac{t^{-\varepsilon}w(t)}{v(t)}\frac{t^\varepsilon}{t^2}dt = \int_{\pi/\lambda(n)}^{\pi}\gamma(t)t^{\varepsilon-2}dt$$

$$\ll \gamma(1/\lambda(n))\int_{\pi/\lambda(n)}^{\pi}t^{\varepsilon-2}dt \ll \gamma(1/\lambda(n))\begin{cases}\log\lambda(n), & \varepsilon = 1;\\ \lambda(n)^{1-\varepsilon}, & 0 < \varepsilon < 1.\end{cases}$$

Therefore, if the two cases are taken into consideration, we have

$$\frac{1}{\lambda(n)}\int_{\pi/\lambda(n)}^{\pi}\frac{w(t)}{t^2v(t)}dt \ll \frac{1}{\lambda(n)}\lambda(n)^{\varepsilon}\frac{w(1/\lambda(n))}{v(1/\lambda(n))}max(\log\lambda(n),\lambda(n)^{1-\varepsilon})$$

$$\ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n). \tag{7}$$

Consequently (5) is obtained from (6) and (7). ◄

Lemma 2. *If the conditions of Lemma 1 are satisfied with* $0<\varepsilon<1$*, then*

$$\|\sigma_n^{\lambda}-f\|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n)}\log\lambda(n), \quad \lambda(n)\geq n\geq 2 \tag{8}$$

and

$$\|\sigma_n^{\lambda}-s_n^{\lambda}\|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n), \quad \lambda(n)\geq n\geq 2. \tag{9}$$

Proof. By using Theorem 3 and (3), we easily get

$$\|\sigma_n^{\lambda}-f\|_p^{(v)} \leq \frac{1}{\lambda(n)+1}\sum_{m=0}^{\lambda(n)}\|s_m-f\|_p^{(v)}$$

$$= \frac{1}{\lambda(n)+1}\left(\|s_0-f\|_p^{(v)}+\|s_1-f\|_p^{(v)}+\sum_{m=2}^{\lambda(n)}\|s_m-f\|_p^{(v)}\right)$$

$$\ll \frac{1}{\lambda(n)+1}\left(1+\sum_{m=2}^{\lambda(n)}\frac{w(1/m)}{v(1/m)}\log m\right)$$

$$\ll \frac{1}{\lambda(n)}\left(1+\lambda(n)^{\varepsilon}\frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n)\sum_{m=1}^{\lambda(n)}m^{-\varepsilon}\right)$$

$$\ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))}+\frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n) \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n).$$

This proves the first part of lemma. Now let us prove the second part. According to (5) and (8), we have

$$\|\sigma_n^{\lambda}-s_n^{\lambda}\|_p^{(v)} \leq \|\sigma_n^{\lambda}-f\|_p^{(v)}+\|f-s_n^{\lambda}\|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n).$$

Therefore, the proof of lemma is completed. ◀

If we take $\lambda(n)=n$ in (8), it will give us the result of Leindler in [14]. That is,

$$\|\sigma_n - f\|_p^{(v)} \ll \frac{w(1/n)}{v(1/n)} \log n, \quad n \geq 2. \tag{10}$$

Lemma 3 ([8])**.** *The following inequalities are valid:*

$$\Theta_{\lambda(n)} := \sum_{m=1}^{\lambda(n)} \left|\Delta_m(m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m}))\right| \ll \sum_{m=0}^{\lambda(n)-1} |\Delta p_m|,$$

and if

$$\sum_{m=1}^{\lambda(n)-1} m|\Delta p_m| \ll P_{\lambda(n)},$$

then

$$\Theta_{\lambda(n)} \ll P_{\lambda(n)}/\lambda(n).$$

Lemma 4. *If*

$$\sum_{k=0}^{\lambda(n)-1} k|\Delta p_k| \ll P_{\lambda(n)}$$

and the conditions of Lemma 1 with $0<\varepsilon<1$ are satisfied, then

$$\|N_n^\lambda - s_n^\lambda\|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n), \quad \lambda(n) \geq n \geq 2. \tag{11}$$

Proof. By Abel's transformation, we get

$$N_n^\lambda(f;x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} P_{\lambda(n)-m} A_m(f;x),$$

and thus

$$s_n^\lambda(f;x) - N_n^\lambda(f;x) = \sum_{m=0}^{\lambda(n)} A_m(f;x) P_{\lambda(n)} P_{\lambda(n)}^{-1} - P_{\lambda(n)}^{-1} \sum_{m=0}^{\lambda(n)} P_{\lambda(n)-m} A_m(f;x)$$

$$= P_{\lambda(n)}^{-1} \sum_{m=1}^{\lambda(n)} (P_{\lambda(n)} - P_{\lambda(n)-m}) A_m(f;x).$$

Hence, again by Abel's transformation we obtain

$$s_n^\lambda(f;x) - N_n^\lambda(f;x) =$$

$$= P_{\lambda(n)}^{-1} \sum_{m=1}^{\lambda(n)} \Delta_m(m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m})) \sum_{k=1}^{m} kA_k(f;x) + (\lambda(n)+1)^{-1} \sum_{k=1}^{\lambda(n)} kA_k(f;x).$$

Therefore, we have

$$\|s_n^\lambda - N_n^\lambda\|_p^{(v)} \le P_{\lambda(n)}^{-1} \sum_{m=1}^{\lambda(n)} |\Delta_m(m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m}))| \quad \|\sum_{k=1}^{m} kA_k(f;\cdot)\|_p^{(v)} +$$

$$+(\lambda(n)+1)^{-1} \|\sum_{k=1}^{\lambda(n)} kA_k(f;\cdot)\|_p^{(v)}.$$

As

$$\sigma_n^\lambda(f;x) - s_n^\lambda(f;x) = (\lambda(n)+1)^{-1} \sum_{k=1}^{\lambda(n)} kA_k(f;x),$$

by (9) we have

$$\|\sum_{k=1}^{\lambda(n)} kA_k(f;\cdot)\|_p^{(v)} = (\lambda(n)+1)\|\sigma_n^\lambda - s_n^\lambda\|_p^{(v)} \ll \lambda(n)\frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n).$$

Combining these results, we obtain

$$\|s_n^\lambda - N_n^\lambda\|_p^{(v)} \ll P_{\lambda(n)}^{-1} \lambda(n) \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n) \times$$

$$\times \sum_{m=1}^{\lambda(n)} |\Delta_m(m^{-1}(P_{\lambda(n)} - P_{\lambda(n)-m}))| + \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n).$$

By Lemma 3, we get

$$\|N_n^\lambda - s_n^\lambda\|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n).$$

◄

Lemma 5 ([8]). *Let*

$$(p_n) \in AMDMS$$

or

$$(p_n) \in AMIMS \quad \text{and satisfy} \quad (\lambda(n)+1)p_{\lambda(n)} \ll P_{\lambda(n)}.$$

Then, for $0 < \alpha < 1$,

$$\sum_{m=0}^{\lambda(n)} (m+1)^{-\alpha} p_{\lambda(n)-m} \ll (\lambda(n)+1)^{-\alpha} P_{\lambda(n)}.$$

Lemma 6 ([8]). *Let*

$$(p_n) \in AMIMS$$

or

$$(p_n) \in AMDMS \quad \text{and satisfy} \quad (\lambda(n)+1) \ll P_{\lambda(n)}.$$

Then, for $0 < \alpha < 1$,

$$\sum_{m=0}^{\lambda(n)} (m+1)^{-\alpha} p_m \ll (\lambda(n)+1)^{-\alpha} P_{\lambda(n)}.$$

3.2. Main results

Based on Section 2, we can state the following results, and we see that the results of Leindler are extended in the direction of improving the degree of approximation in *GHM* by using the methods of means on the classes larger than classes of sequences used in [14].

Theorem 7. *Let* v *and* w *be moduli of continuity such that* $\frac{w(t)}{v(t)}$ *is nondecreasing. Moreover, let the function*

$$\gamma(t) = t^{-\varepsilon} \frac{w(t)}{v(t)},$$

be nonincreasing for some $0 < \varepsilon \leq 1$. *If* $f \in H_p^{(\omega)}$, $p \geq 1$, *and one of the following additional conditions are satisfied:*

(*i*) $(p_n) \in AMDMS$,

(*ii*) $(p_n) \in AMIMS$ *and* $(\lambda(n)+1)p_{\lambda(n)} \ll P_{\lambda(n)}$, *then*

$$\| N_n^{\lambda} - f \|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n) \quad \text{for all} \quad \lambda(n) \geq n \geq 2. \tag{12}$$

Proof. Let us prove case (i). Since

$$N_n^{\lambda}(f;x)-f(x)=\frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)}p_{\lambda(n)-m}(s_m(f;x)-f(x)),$$

we can write the following inequality

$$\|N_n^{\lambda}-f\|_p^{(v)}\leq\frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)}p_{\lambda(n)-m}\|s_m-f\|_p^{(v)}.$$

Let us split the sum on the right-hand side into four parts

$$\frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)}p_{\lambda(n)-m}\|s_m-f\|_p^{(v)}=$$

$$=\frac{1}{P_{\lambda(n)}}p_{\lambda(n)}\|s_0-f\|_p^{(v)}+\frac{1}{P_{\lambda(n)}}p_{\lambda(n)-1}\|s_1-f\|_p^{(v)}+$$

$$+\frac{1}{P_{\lambda(n)}}\sum_{m=2}^{\tilde{\lambda}(n)}p_{\lambda(n)-m}\|s_m-f\|_p^{(v)}+\frac{1}{P_{\lambda(n)}}\sum_{m=\tilde{\lambda}(n)+1}^{\lambda(n)}p_{\lambda(n)-m}\|s_m-f\|_p^{(v)}$$

$$=:I_1+I_2+I_3+I_4.$$

First let us consider I_1 and I_2. If $(p_n)\in AMDMS$, then we know that $(C_{\lambda(n)})\in AMDS$, where

$$C_{\lambda(n)}=\frac{1}{\lambda(n)+1}\sum_{k=0}^{\lambda(n)}p_k.$$

Since $C_{\lambda(n)}\ll C_{\lambda(n)-1}\ll C_{\lambda(n)-2}$, we have

$$p_{\lambda(n)}<\frac{P_{\lambda(n)}}{\lambda(n)}\quad\text{and}\quad p_{\lambda(n)-1}<\frac{P_{\lambda(n)}}{\lambda(n)-1}.$$

Therefore we get

$$I_1\ll\frac{1}{\lambda(n)}\quad\text{and}\quad I_2\ll\frac{1}{\lambda(n)-1}.$$

According to (3), we have

$$I_1\ll\frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n)\quad\text{and}\quad I_2\ll\frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n). \tag{13}$$

Now, considering (4), Theorem 3 and Lemma 5, we obtain

$$\begin{aligned} I_3 &\ll \frac{1}{P_{\lambda(n)}} \sum_{m=2}^{\tilde{\lambda}(n)} p_{\lambda(n)-m} \frac{(m+1)^{\varepsilon}}{(m+1)^{\varepsilon}} \frac{w(1/m)}{v(1/m)} \log m \\ &\ll \frac{(\lambda(n)+1)^{\varepsilon}}{P_{\lambda(n)}} \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n) \sum_{m=2}^{\tilde{\lambda}(n)} p_{\lambda(n)-m} (m+1)^{-\varepsilon}, \end{aligned}$$

$$I_3 \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n). \tag{14}$$

Finally, let us estimate I_4. By Theorem 3 and (4), we see that

$$\begin{aligned} I_4 &\ll \frac{1}{P_{\lambda(n)}} \sum_{m=\tilde{\lambda}(n)+1}^{\lambda(n)} p_{\lambda(n)-m} \frac{w(1/m)}{v(1/m)} \log m \\ &\ll \frac{1}{P_{\lambda(n)}} \frac{w(1/\tilde{\lambda}(n))}{v(1/\tilde{\lambda}(n))} \log \lambda(n) \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}, \end{aligned}$$

$$I_4 \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n). \tag{15}$$

Therefore, according to (13)-(15), we have (12). In the case of (ii), the proof runs along the same lines as that of (i). The proof is completed. ◄

Remark 4. *Note that since $N_n \subseteq N_n^{\lambda}$ for any λ, and $AMDS \subset AMDMS$ and $AMIS \subset AMIMS$, by taking $\lambda(n) = n$ in Theorem 7, the cases (i) and (ii) of Theorem 2 in [14] are obtained, respectively. Also, $N_n^{\lambda}(f,x)$ gives the method of $\sigma_n^{\lambda}(f,x)$ in Theorem 7 in case $p_n = 1$. So we have*

$$\|\sigma_n^{\lambda} - f\|_p^{(v)} \ll \frac{\omega(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n), \quad \lambda(n) \geq n \geq 2.$$

Theorem 8. *If the conditions of Theorem 7 with $0 < \varepsilon < 1$ are satisfied and*

$$\sum_{m=1}^{\lambda(n)-1} m|\Delta p_m| \ll P_{\lambda(n)},$$

then

$$\| N_n^{\lambda} - f \|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n)$$

for all $\lambda(n) \geq n \geq 2.$

Proof. Taking into account (5) and (11), we obtain

$$\|N_n^\lambda - f\|_p^{(v)} \ll \|N_n^\lambda - s_n^\lambda\|_p^{(v)} + \|s_n^\lambda - f\|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n).$$

◀

Theorem 9. *If the conditions of Theorem 7 with $0 < \varepsilon < 1$ are satisfied and*

$$\sum_{m=0}^{\lambda(n)-1} |\Delta_m \left(\frac{1}{m+1} \sum_{k=0}^{m} p_{\lambda(n)-k} \right)| \ll \frac{P_{\lambda(n)}}{\lambda(n)}, \quad \lambda(n) \geq n \geq 2, \tag{16}$$

then

$$\| N_n^\lambda - f \|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n). \tag{17}$$

Proof. An elementary calculations yield

$$N_n^\lambda(f;x) - f(x) = \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)} p_{\lambda(n)-m}(s_m(f;x) - f(x)).$$

Using Abel's transformation, we get

$$\sum_{m=0}^{\lambda(n)-1} (s_m(f;x) - s_{m+1}(f;x)) \sum_{k=0}^{m} \frac{p_{\lambda(n)-k}}{P_{\lambda(n)}} + (s_n^\lambda(f;x) - f(x))$$

$$= (s_n^\lambda(f;x) - f(x)) - \frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)-1} (m+1) A_{m+1}(f;x) B_{\lambda(n),m},$$

where

$$B_{\lambda(n),m} := \frac{1}{(m+1)} \sum_{k=0}^{m} p_{\lambda(n)-k}.$$

By applying Abel's transformation on the sum at the right-hand side, we have

$$N_n^\lambda(f;x) - f(x) = (s_n^\lambda(f;x) - f(x)) -$$

$$-\frac{1}{P_{\lambda(n)}} \sum_{m=0}^{\lambda(n)-2} (B_{\lambda(n),m} - B_{\lambda(n),m+1}) \sum_{k=0}^{m} (k+1) A_{k+1}(f;x) +$$

$$+ \left(\sum_{m=0}^{\lambda(n)-1} (m+1) A_{m+1}(f;x) \right) \frac{1}{\lambda(n)} \sum_{k=0}^{\lambda(n)-1} \frac{p_{\lambda(n)-k}}{P_{\lambda(n)}}.$$

Hence

$$\|N_n^{\lambda}(f;.)-f(.)\|_p^{(v)} \le \|s_n^{\lambda}(f;.)-f(.)\|_p^{(v)}+$$

$$+\frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)-2} \mid \Delta_m(B_{\lambda(n),m}) \mid \| \sum_{k=1}^{m+1} kA_k(f;.) \|_p^{(v)} + \frac{1}{\lambda(n)} \| \sum_{m=1}^{\lambda(n)} mA_m(f;.) \|_p^{(v)}$$

$$\ll \|s_n^{\lambda}(f;.)-f(.)\|_p^{(v)} + \frac{1}{P_{\lambda(n)}}\{\mid B_{\lambda(n),0}-B_{\lambda(n),1}\mid + \mid B_{\lambda(n),1}-B_{\lambda(n),2}\mid\}$$

$$+\frac{1}{P_{\lambda(n)}}\sum_{m=2}^{\lambda(n)-2} \mid \Delta_m(B_{\lambda(n),m}) \mid \| \sum_{k=1}^{m+1} kA_k(f;.) \|_p^{(v)}$$

$$+\frac{1}{\lambda(n)} \| \sum_{m=1}^{\lambda(n)} mA_m(f;.) \|_p^{(v)} . \tag{18}$$

Since

$$s_n^{\lambda}(f;x)-\sigma_n^{\lambda}(f;x) = \frac{1}{\lambda(n)+1}\sum_{k=1}^{\lambda(n)} kA_k(f;x),$$

using (9), we see that

$$\| \sum_{k=1}^{\lambda(n)} kA_k(f;.)\|_p^{(v)} \ll \lambda(n)\frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n). \tag{19}$$

Therefore, using (5), (18) and (19) we get

$$\|N_n^{\lambda}(f;.)-f(.)\|_p^{(v)} \le \frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n) + \frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)-1} \mid \Delta_m(B_{\lambda(n),m}) \mid . \tag{20}$$

Finally, taking into account (16), (3) and (20) we obtain (17) for $\lambda(n) \ge n \ge 2$. ◄

Theorem 10. *Let the conditions of Theorem 7 with $0 < \varepsilon < 1$ and one of the following additional conditions be satisfied:*

(i) $(p_n) \in AMIMS$,

(ii) $(p_n) \in AMDMS$ and holds $(\lambda(n)+1) \ll P_{\lambda(n)}$.

Then

$$\| R_n^{\lambda} - f \|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n) \quad \textit{for all} \quad \lambda(n) \ge n \ge 2. \tag{21}$$

Proof. Simple calculation shows that

$$R_n^{\lambda}(f;x)-f(x)=\frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)}p_m(s_m(f;x)-f(x)).$$

Hence

$$\|R_n^{\lambda}-f\|_p^{(v)}\leq\frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)}p_m\|s_m-f\|_p^{(v)}$$

$$\leq\frac{1}{P_{\lambda(n)}}p_0\|s_0-f\|_p^{(v)}+\frac{1}{P_{\lambda(n)}}p_1\|s_1-f\|_p^{(v)}+$$

$$+\frac{1}{P_{\lambda(n)}}\sum_{m=2}^{\tilde{\lambda}(n)}p_m\|s_m-f\|_p^{(v)}+\frac{1}{P_{\lambda(n)}}\sum_{m=\tilde{\lambda}(n)+1}^{\lambda(n)}p_m\|s_m-f\|_p^{(v)}.$$

Proceeding similar to the proof of (13)-(15), we get the expected result (21). Moreover, the proof of the case (ii) is obtained by the similar way. ◄

Theorem 11. *If the conditions of Theorem 7 with* $0<\varepsilon<1$ *and the conditions*

$$(\lambda(n)+1)p_{\lambda(n)}\ll P_{\lambda(n)}\quad \textit{and}\quad \sum_{m=0}^{\lambda(n)-1}m^{1-\varepsilon}|\Delta p_m|\ll P_{\lambda(n)}\lambda(n)^{-\varepsilon}\tag{22}$$

hold, then

$$\| R_n^{\lambda}-f\|_p^{(v)}\ll\frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n)$$

for all $\lambda(n)\geq n\geq 2.$

Proof. We know that

$$R_n^{\lambda}(f;x)-f(x)=\frac{1}{P_{\lambda(n)}}\sum_{m=0}^{\lambda(n)}p_m(s_m(f;x)-f(x)).$$

By applying Abel's transformation, we have

$$\frac{1}{P_{\lambda(n)}}\left(\sum_{k=0}^{\lambda(n)-1}\left(\left(\sum_{m=0}^{k}(s_m-f)\right)(p_k-p_{k+1})\right)+\left(\sum_{k=0}^{\lambda(n)}(s_k-f)\right)p_{\lambda(n)}\right)$$

$$=\frac{1}{P_{\lambda(n)}}\left(\sum_{m=0}^{\lambda(n)-1}\Delta p_m\frac{m+1}{m+1}\sum_{k=0}^{m}(s_k-f)+\frac{\lambda(n)+1}{\lambda(n)+1}p_{\lambda(n)}\sum_{k=0}^{\lambda(n)}(s_k-f)\right)$$

$$= \frac{1}{P_{\lambda(n)}}\left(\sum_{m=0}^{\lambda(n)-1}(m+1)\Delta p_m(\sigma_m - f) + (\lambda(n)+1)p_{\lambda(n)}(\sigma_n^\lambda - f)\right).$$

By considering (8) and (10), we see that

$$\| R_n^\lambda - f \|_p^{(v)} \leq$$

$$\leq \frac{1}{P_{\lambda(n)}}\left(\sum_{m=0}^{\lambda(n)-1}(m+1)|\Delta p_m| \, \| \sigma_m - f \|_p^{(v)} + (\lambda(n)+1)p_{\lambda(n)} \, \| \sigma_n^\lambda - f \|_p^{(v)}\right)$$

$$\ll \frac{1}{P_{\lambda(n)}}\left(p_0\|\sigma_0 - f\|_p^{(v)} + 2|p_1 - p_0|\|\sigma_1 - f\|_p^{(v)}\right) +$$

$$+\frac{1}{P_{\lambda(n)}}\left(\sum_{m=2}^{\lambda(n)-1}(m+1)|\Delta p_m|\frac{w(1/m)}{v(1/m)}\log m + (\lambda(n)+1)p_{\lambda(n)}\frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n)\right)$$

$$\ll \frac{1}{P_{\lambda(n)}}\left(p_{\lambda(n)} + \sum_{m=2}^{\lambda(n)-1} m|\Delta p_m|\frac{w(1/m)}{v(1/m)}\log m + \lambda(n)p_{\lambda(n)}\frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n)\right).$$

Since $\gamma(t)$ is nonincreasing, we easily get

$$\sum_{m=2}^{\lambda(n)-1} m|\Delta p_m|\frac{w(1/m)}{v(1/m)}\log m \ll \sum_{m=2}^{\lambda(n)-1} m^{1-\varepsilon}|\Delta p_m| m^\varepsilon \frac{w(1/m)}{v(1/m)}\log m$$

$$\ll \quad \lambda(n)^\varepsilon \frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n)\sum_{m=0}^{\lambda(n)-1} m^{1-\varepsilon}|\Delta p_m| \ll P_{\lambda(n)}\frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n)$$

under the condition (22). Hence, combining these relations and using (3) we obtain

$$\begin{aligned}\| R_n^\lambda - f \|_p^{(v)} &\ll \frac{1}{P_{\lambda(n)}}\left(p_{\lambda(n)} + (P_{\lambda(n)} + \lambda(n)p_{\lambda(n)})\frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n)\right) \\ &\ll \frac{1}{\lambda(n)} + \frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n) \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n).\end{aligned}$$

◄

Remark 5. *Since $N_n \subseteq N_n^\mu$ for any μ, in case $\mu(n) = n$, the result in Theorem 11 is reduced to the result of Theorem 3 in [14]. Also, $R_n^\lambda(f,x)$ gives the method of $\sigma_n^\lambda(f,x)$ in Theorem 11 in the case $p_n = 1$. Therefore, we have*

$$\|\sigma_n^\lambda - f\|_p^{(v)} \ll \frac{\omega(1/\lambda(n))}{v(1/\lambda(n))}\log\lambda(n), \quad \lambda(n) \geq n \geq 2.$$

Theorem 12. *If the conditions of Theorem 7 with* $0 < \varepsilon < 1$ *are satisfied and*

$$\sum_{m=0}^{\lambda(n)-1} |\Delta_m(\frac{P_m}{m+1})| \ll \frac{P_{\lambda(n)}}{\lambda(n)}, \quad \lambda(n) \geq n \geq 2,$$

then

$$\| R_n^{\lambda} - f \|_p^{(v)} \ll \frac{w(1/\lambda(n))}{v(1/\lambda(n))} \log \lambda(n).$$

Since the proof of Theorem 12 is similar to the proof of Theorem 9, we will omit it.

Remark 6. *Note that the conditions in Theorem 9 and Theorem 12 are different from the conditions of results in [14].*

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Received 16 February 2017
Accepted 28 March 2018