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Statistical Approximation by (p, q)-analogue of Bernstein-Stancu Operators

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Abstract. In this paper, some approximation properties of (p, q)-analogue of Bernstein-Stancu operators are studied. Rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated. Monotonicity of (p, q)-Bernstein-Stancu operators and a global approximation theorem by means of Ditzian-Totik modulus of smoothness is established. A quantitative Voronovskaja type theorem is developed for these operators. Furthermore, we show comparisons and some illustrative graphics for the convergence of operators to a function.

Key Words and Phrases: (p, q)-integers, (p, q)-Bernstein-Stancu operators, positive linear operators, Korovkin type approximation, statistical convergence, monotonicity for convex functions, Ditzian-Totik modulus of smoothness, Voronovskaja type theorem.

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1. Introduction and preliminaries

Mursaleen et al. [17] first applied the concept of (p, q)-calculus in approximation theory and introduced the (p, q)-analogue of Bernstein operators. Later, based on (p, q)-integers, some approximation results for Bernstein-Stancu operators, Bernstein-Kantorovich operators, Bleimann-Butzer and Hahn operators, (p, q)-Lorentz operators, Bernstein-Shurer operators, (p, q)-analogue of divided difference and Bernstein operators etc. have also been obtained by them in [18-20,22-24].

For similar works in approximation theory [37] based on q and (p, q)-integers, one can refer [1, 2, 3, 4, 7, 11, 13, 21, 25, 27, 30, 31, 38, 40, 41].

Very recently, Khalid et al. [32, 33, 34] has given a nice application in computer-aided geometric design and applied Bernstein basis for construction

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of (p, q)-Bézier curves and surfaces based on (p, q)-integers which is further generalization of q-Bézier curves and surfaces [28, 29, 37, 39]. For similar works, one can refer [6, 26]. Another advantage of using the parameter p has been discussed in [22].

Let us recall some notations of (p, q)-calculus .

For any $p > 0$ and $q > 0$, the (p, q) integers $[n]_{p,q}$ are defined by

$$[n]_{p,q} = p^{n-1}+p^{n-2}q+p^{n-3}q^2+...+pq^{n-2}+q^{n-1} = \begin{cases} \frac{p^n-q^n}{p-q}, & \text{when} \quad p \neq q \neq 1, \\ n\ p^{n-1}, & \text{when } p = q \neq 1, \\ [n]_q, & \text{when } p = 1, \\ n, & \text{when } p = q = 1, \end{cases}$$

where $[n]_q$ denotes the q-integers and $n = 0, 1, 2, \cdots$.

Obviously, it may be seen that $[n]_{p,q} = p^{n-1}[n]_{\frac{q}{p}}$.

The (p, q)-factorial is defined by

$$[0]_{p,q}! := 1 \quad \text{and} \quad [n]!_{p,q} = [1]_{p,q}[2]_{p,q} \cdots [n]_{p,q} \quad \text{if} \quad n \geq 1.$$

Also the (p, q)-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix}_{p,q} = \frac{[n]_{p,q}!}{[k]_{p,q}!\ [n-k]_{p,q}!} \quad \text{for all} \quad n, k \in \mathbb{N} \quad \text{with} \quad n \geq k.$$

The formula for (p, q)-binomial expansion is as follows:

$$(ax + by)^n_{p,q} := \sum_{k=0}^{n} p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_{p,q} a^{n-k}b^k x^{n-k}y^k,$$

$$(x + y)^n_{p,q} = (x + y)(px + qy)(p^2x + q^2y) \cdots (p^{n-1}x + q^{n-1}y),$$
$$(1 - x)^n_{p,q} = (1 - x)(p - qx)(p^2 - q^2x) \cdots (p^{n-1} - q^{n-1}x).$$

Details on (p, q)-calculus can be found in [9, 10, 17, 32, 33].

The (p,q)-Bernstein operators introduced by Mursaleen et al. for $0 < q < p \leq 1$ in [17] are defined as follows

$$B_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \times$$

$$\times \prod_{s=0}^{n-k-1} (p^s - q^s x) \;\; f\left(\frac{[k]_{p,q}}{p^{k-n}[n]_{p,q}} \right), \quad x \in [0,1]. \tag{1}$$

Note that when $p = 1$, (p,q)-Bernstein operators given by (1) turn out to be q-Bernstein operators.

Also, we have

$$\begin{aligned}(1-x)^n_{p,q} &= \prod_{s=0}^{n-1} (p^s - q^s x) = (1-x)(p-qx)(p^2-q^2x)...(p^{n-1}-q^{n-1}x) \\ &= \sum_{k=0}^{n} (-1)^k p^{\frac{(n-k)(n-k-1)}{2}} q^{\frac{k(k-1)}{2}} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k.\end{aligned}$$

Motivated by the above mentioned work on (p,q)-approximation and its applications, this paper is organized as follows: In Section 2, some basic results for (p,q)-analogue of Bernstein-Stancu operators as given in [19] have been recalled and based on it, second order moment is computed. In Section 3, Korovkin type statistical approximation properties have been studied for these operators. In Section 4, the rate of statistical convergence by means of modulus of continuity and Lipschitz type maximal functions has been investigated. Section 5 is based on monotonicity of (p,q)-Bernstein-Stancu operators. In Section 6, a global approximation theorem by means of Ditzian-Totik modulus of smoothness and a quantitative Voronovskaja type theorem is established.

The effects of the parameters p and q for the convergence of operators to a function is shown in Section 7.

2. (p,q)- Bernstein Stancu operators

Mursaleen et. al [19] introduced (p,q)-analogue of Bernstein-Stancu operators as follows:

$$S_{n,p,q}(f;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \times$$

$$\times \prod_{s=0}^{n-k-1} (p^s - q^s x) \;\; f\left(\frac{p^{n-k}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right), \quad x \in [0,1], \tag{2}$$

where α and β are real numbers which satisfy $0 \leq \alpha \leq \beta$.

Note that for $\alpha = \beta = 0$, (p,q)-Bernstein-Stancu operators given by (2) reduce to (p,q)-Bernstein operators as given in [17].

Also for $p = 1$, (p,q)-Bernstein-Stancu operators given by (2) turn out to be q-Bernstein-Stancu operators.

For $p = q = 1$, they reduce to classical Bernstein-Stancu operators.

We have the following auxiliary lemmas.

Lemma 1. *For $x \in [0,1]$, $0 < q < p \leq 1$, and $\alpha, \beta \in \mathbb{R}$ with $0 \leq \alpha \leq \beta$, we have*
(i) $S_{n,p,q}(1;x) = 1$,
(ii) $S_{n,p,q}(t;x) = \frac{[n]_{p,q}}{[n]_{p,q}+\beta} x + \frac{\alpha}{[n]_{p,q}+\beta}$,
(iii) $S_{n,p,q}(t^2;x) = \frac{q[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q}+\beta)^2} x^2 + \frac{[n]_{p,q}(2\alpha+p^{n-1})}{([n]_{p,q}+\beta)^2} x + \frac{\alpha^2}{([n]_{p,q}+\beta)^2}$.

Proof. Proof is given in [19] using the identity

$$\sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) = p^{\frac{n(n-1)}{2}}. \tag{3}$$

We give complete proof of Lemma 1 (iii) ◀

(iii)

$$S_{n,p,q}(t^2;x) = \frac{1}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \times$$

$$\times \prod_{s=0}^{n-k-1} (p^s - q^s x) \;\; \left(\frac{p^{n-k}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta}\right)^2$$

$$= \frac{1}{([n]_{p,q}+\beta)^2} \frac{1}{p^{\frac{n(n-1)}{2}}} \Bigg[p^{2n} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \; \frac{[k]_{p,q}^2}{p^{2k}}$$

$$+ 2\alpha \; p^n \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \; \frac{[k]_{p,q}}{p^k}$$

$$+\alpha^2 \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x) \Bigg],$$

$$S_{n,p,q}(t^2;x) = \frac{1}{([n]_p,q+\beta)^2}[(A)+(B)+(C)],$$

$$\begin{aligned}(A) &= \frac{1}{p^{\frac{n(n-1)}{2}}} p^{2n} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x)\ \frac{[k]_{p,q}^2}{p^{2k}} \\ &= \frac{p^{2n}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \frac{[n]}{[k]} \left[\begin{array}{c} n-1 \\ k-1 \end{array} \right] x^k (1-x)^{n-k}\ \frac{[k]^2}{p^{2k}}.\end{aligned}$$

On shifting the limits and using $[k+1]_{p,q} = p^k + q[k]_{p,q}$, we get our desired result.

$$\begin{aligned}(A) &= \frac{p^{2n}}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n-1} \left[\begin{array}{c} n-1 \\ k \end{array} \right] x^k (1-x)^{n-k-1}\ \frac{p^k + q[k]}{p^{2k+2}} \\ &= \frac{p^{2n-2}[n]x}{p^{\frac{n(n-1)}{2}}} \left[p^{\frac{(n-1)(n-2)}{2}} + \frac{q[n-1]x}{p} \sum_{k=0}^{n-2} \left[\begin{array}{c} n-2 \\ k \end{array} \right] x^k (1-x)^{n-k-2} \right] \\ &= p^n [n]x + q[n][n-1]x^2.\end{aligned}$$

Similarly

$$(B) = \frac{2\alpha\ p^n}{p^{\frac{n(n-1)}{2}}} \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k (1-x)^{n-k}\ \frac{[k]_{p,q}}{p^k} = 2\alpha[n]x$$

and

$$(C) = \frac{\alpha^2}{p^{\frac{n(n-1)}{2}}}\ \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k (1-x)^{n-k} = \alpha^2.$$

Lemma 2. *For $x \in [0,1]$, $0 < q < p \leq 1$ and $\alpha, \beta \in \mathbb{R}$ with $0 \leq \alpha \leq \beta$, let n be any given natural number. Then*

$$\begin{aligned}S_{n,p,q}\big((t-x)^2;x\big) &= \Big\{\tfrac{q[n]_{p,q}[n-1]_{p,q} - [n]_{p,q}^2 + \beta^2}{([n]_{p,q}+\beta)^2}\Big\}x^2 + \Big\{\tfrac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{([n]_{p,q}+\beta)^2}\Big\}x + \tfrac{\alpha^2}{([n]_{p,q}+\beta)^2} \\ &\leq \tfrac{[n]_{p,q}p^{n-1} - 2\alpha\beta}{2([n]_{p,q}+\beta)^2}\phi^2(x) \leq \tfrac{[n]_{p,q}}{[n]_{p,q}+\beta}\phi^2(x).\end{aligned}$$

3. Main results

3.1. Korovkin type approximation theorem

We know that $C[a, b]$ is a Banach space with norm

$$\|f\|_{C[a,b]} := \sup_{x\in[a,b]} |f(x)|, \ f \in C[a, b].$$

For typographical convenience, we will write $\|.\|$ instead of $\|.\|_{C[a,b]}$ if no confusion arises.

Definition 1. *Let $C[a, b]$ be the linear space of all real valued continuous functions f on $[a, b]$ and let T be a linear operator which maps $C[a, b]$ into itself. We say that T is positive if for every non-negative $f \in C[a, b]$, we have $T(f, x) \geq 0$ for all $x \in [a, b]$.*

The classical Korovkin type approximation theorem can be stated as follows [5, 12]:

Let $T_n : C[a, b] \to C[a, b]$ be a sequence of positive linear operators. Then $\lim_{n\to\infty} \|T_n(f; x)-f(x)\|_\infty = 0,$ for all $f \in C[a, b]$ if and only if $\lim_{n\to\infty} \|T_n(f_i; x)-f_i(x)\|_\infty = 0,$ for each $i = 0, 1, 2,$ where the test function $f_i(x) = x^i$.

In next subsection, we study statistical approximation properties of the operator $S_{n,p,q}$.

3.2. Statistical approximation

The statistical version of Korovkin theorem for a sequence of positive linear operators has been given by Gadjiev and Orhan [16].

Let K be a subset of the set $\mathbb{N}$ of natural numbers. Then, the asymptotic density $\delta(K)$ of K is defined as $\delta(K) = \lim_n \frac{1}{n}\big|\{k \leq n \ : \ k \in K\}\big|$ and $|.|$ represents the cardinality of the enclosed set. A sequence $x = (x_k)$ is said to be statistically convergent to the number L if for each $\varepsilon > 0$, the set $K(\varepsilon) = \{k \leq n : |x_k - L| > \varepsilon\}$ has asymptotic density zero (see [14, 15]), i.e.,

$$\lim_n \frac{1}{n}\big|\{k \leq n : |x_k - L| \geq \varepsilon\}\big| = 0.$$

In this case, we write $st - \lim x = L$.

Let us recall the following theorem:

Theorem 1. *[16] Let A_n be the sequence of linear positive operators from $C[0, 1]$ to $C[0, 1]$ which satisfy the conditions*

$st - \lim\limits_n \|S_{n,p,q}((t^\nu; x)) - (x)^\nu\|_C[0, 1] = 0$ *for* $\nu = 0, 1, 2.$ *Then for any function* $f \in C[0, 1]$, $st - \lim\limits_n \|S_{n,p,q}(f) - f\|_C[0, 1] = 0.$

3.3. Korovkin type statistical approximation properties

The main aim of this paper is to obtain the Korovkin type statistical approximation properties of operators defined in (2) with the help of Theorem 1.

Remark 1. *For $q \in (0,1)$ and $p \in (q,1]$, it is obvious that $\lim\limits_{n\to\infty}[n]_{p,q} = 0$ or $\frac{1}{p-q}$. In order to obtain the convergence results for the operator $L^n_{p,q}(f;x)$, we take a sequence $q_n \in (0,1)$ and $p_n \in (q_n,1]$ such that $\lim\limits_{n\to\infty} p_n = 1$, $\lim\limits_{n\to\infty} q_n = 1$ and $\lim\limits_{n\to\infty} p_n^n = 1$, $\lim\limits_{n\to\infty} q_n^n = 1$. So we get $\lim\limits_{n\to\infty}[n]_{p_n,q_n} = \infty$.*

Theorem 2. *Let $S_{n,p,q}$ be the sequence of operators and the sequences $p = p_n$ and $q = q_n$ satisfy Remark 1. Then for any function $f \in C[0,1]$*

$$st - \lim_n \| \; S_{n,p_n,q_n}(f,.) - f\| = 0.$$

Proof. Clearly for $\nu = 0$,

$$S_{n,p,q}(1,x) = 1,$$

which implies

$$st - \lim_n \|S_{n,p,q}(1;x) \; - 1 \;\| \quad = \quad 0.$$

For $\nu \; = \; 1$

$$\begin{aligned}\|S_{n,p,q}\;(t;x) \; - \; x \;\; \| &\leq \left| \frac{[n]_{p,q}}{[n]_{p,q}+\beta}x \;+\; \frac{\alpha}{[n]_{p,q}+\beta} \;-\; x\right| \\ &= \left|\left(\frac{[n]_{p,q}}{[n]_{p,q}+\beta} \;-\; 1\right)x \;+\; \frac{\alpha}{[n]_{p,q}+\beta}\right| \\ &\leq \left|\frac{[n]_{p,q}}{[n]_{p,q}+\beta} \;-\; 1\right| \;+\; \left|\frac{\alpha}{[n]_{p,q}+\beta}\right|.\end{aligned}$$

For a given $\epsilon > 0$, let us define the following sets

$$\begin{aligned} U &= \{n : \|S_{n,p,q}(t;x) - x\| \geq \epsilon\}, \\ U' &= \{n : 1 - \frac{[n]_{p,q}}{[n]_{p,q}+\beta} \geq \epsilon\}, \\ U'' &= \{n : \frac{\alpha}{[n]_{p,q}+\beta} \geq \epsilon\}. \end{aligned}$$

So using $\delta\{k \leq n : 1 - \frac{[n]_{p,q}}{[n]_{p,q}+\beta} \quad \geq \epsilon\}$, we get

$$st - \lim_{n} \|S_{n,p,q}(t; x) - x\| = 0.$$

Finally, for $\nu = 2$ we have

$$\|S_{n,p,q}(t^2 : x) - x^2\| \leq \left| \frac{q[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2} - 1 \right| + \left| \frac{[n]_{p,q}(2\alpha + p^{n-1})^2}{[n]_{p,q} + \beta} x \right| + \left| \frac{\alpha^2}{([n]_{p,q} + \beta)^2} \right|.$$

If we choose

$$\alpha_n = \frac{q[n]_{p,q}[n-1]_{p,q}}{([n]_{p,q} + \beta)^2} - 1,$$

$$\beta_n = \frac{[n]_{p,q}(2\alpha + p^{n-1})^2}{[n]_{p,q} + \beta},$$

$$\gamma_n = \frac{\alpha^2}{([n]_{p,q} + \beta)^2},$$

$st - \lim_{n} \alpha_n = st - \lim_{n} \beta_n = st - \lim_{n} \gamma_n = 0.$
Now given $\epsilon > 0$, we define the following four sets:

$$U_0 = \{n : \|S_{n,p,q}(t^2 : x) - x^2\| \geq \epsilon\},$$

$$U_1 = \{n : \alpha_n \geq \frac{\epsilon}{3}\},$$

$$U_2 = \{n : \beta_n \geq \frac{\epsilon}{3}\},$$

$$U_3 = \{n : \gamma_n \geq \frac{\epsilon}{3}\}.$$

It is obvious that $U_0 \subseteq U_1 \bigcup U_2 \bigcup U_3$. Thus we obtain
$\delta\{K \leq n : \|S_{n,p,q}(t^2 : x) - x^2\| \geq \epsilon\}$
$\leq \delta\{K \leq n : \alpha_n \geq \frac{\epsilon}{3}\} + \delta\{K \leq n : \beta_n \geq \frac{\epsilon}{3}\} + \delta\{K \leq n : \gamma_n \geq \frac{\epsilon}{3}\}.$
So the right hand side of the inequality is zero.
Then

$$st - \lim_{n} \|S_{n,p,q}(t; x) - x\| = 0,$$

holds and thus the proof is completed. ◀

4. The rate of convergence

In this part, rates of statistical convergence of the operators (2) by means of modulus of continuity and Lipschitz type maximal functions are introduced.

The modulus of continuity for the space of function $f \in C[0,1]$ is defined by

$$w(f;\delta) = \sup_{x,t \in C[0,1],\quad |t-x|<\delta} |f(t) - f(x)|,$$

where $w(f;\delta)$ satisfies the following conditions: for all $f \in C[0,1]$,

$$\lim_{\delta \to 0} \; w(f;\delta) = 0 \tag{4}$$

and

$$|f(t) - f(x)| \leq w(f;\delta)\left(\frac{|t-x|}{\delta} + 1\right). \tag{5}$$

Theorem 3. *Let the sequences $p = p_n$ and $q = q_n$ satisfy $0 < q_n < p_n \leq 1$. Then*

$$|S_{n,p,q}(t;x) - f(x)| \leq w(f; \sqrt{\delta_n(x)})(1 + q_n),$$

where

$$\delta_n(x) = \frac{1}{([n]_{p,q} + \beta)^2}[(q[n]_{p,q}[n-1]_{p,q} - [n]^2 + \beta^2)x^2 \; + \; ([n]_{p,q}p^{(n-1)} - 2\alpha\beta)x + \alpha^2]. \tag{6}$$

Proof. $|S_{n,p,q}(t;x) - f(x)| \leq S_{n,p,q}(|f(t) - f(x)| : x)$.

By using (5), we get

$$|S_{n,p,q}(t;x) - f(x)| \leq w(f;\delta)\{S_{n,p,q}(1;x) + \frac{1}{\delta} S_{n,p,q}(|t-x| : x)\}.$$

By using Cauchy-Schwarz inequality, we have

$$\begin{aligned}
|S_{n,p,q}(t;x) - f(x)| &\leq w(f;\delta_n)\left(1 + \frac{1}{\delta_n}[(S_{n,p,q}(t-x)^2;x)]^{\frac{1}{2}} \quad [S_{n,p,q}(1;x)]^{\frac{1}{2}}\right) \\
&\leq w(f;\delta_n)\left(1 + \frac{1}{\delta_n}\left\{\frac{1}{([n]_{p,q}+\beta)^2}[(q[n]_{p,q}[n-1]_{p,q} - [n]^2\right.\right. \\
&\left.\left. + \quad \beta^2)x^2 \quad + \quad ([n]_{p,q}p^{(n-1)} \quad - \quad 2\alpha\beta)x \quad + \alpha^2]\right\}\right),
\end{aligned}$$

so, obviously, choosing δ_n as in (6) finishes the proof. ◀

5. Monotonicity for convex functions

Oruç and Phillips proved that when the function f is convex on $[0,1]$, its q-Bernstein operators are monotonic decreasing. In this section we will study the monotonicity of (p,q)-Bernstein Stancu operators.

Theorem 4. *If f is a convex function on $[0,1]$, then $S_{n,p,q}(f;x) \geq f(x)$, $0 \leq x \leq 1$ for all $n \geq 1$ and $0 < q < p \leq 1$.*

Proof. We consider the knots $x_k = \frac{p^{n-k}[k]_{p,q}}{[n]_{p,q}}$,

$$\lambda_k = \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)-n(n-1)}{2}} x^k \prod_{s=0}^{n-k-1} (p^s - q^s x), \qquad 0 \leq k \leq n.$$

Using Lemma 1, we obtain

$$\lambda_0 + \lambda_1 + \lambda_2 + ... + \lambda_n = 1,$$

$$x_0\lambda_0 + x_1\lambda_1 + x_2\lambda_2 + ... + x_n\lambda_n = x.$$

From the convexity of the function f, we get
$S_{n,p,q}(f;x) = \sum\limits_{k=0}^{n} \lambda_k f(x_k) \geq f\left(\sum\limits_{k=0}^{n} \lambda_k x_k \right) = f(x)$. ◄

Theorem 5. *Let f be convex on $[0,1]$. Then $S_{n-1,p,q}(f;x) \geq S_{n,p,q}(f;x)$ for $0 < q < p \leq 1$, $0 \leq x \leq 1$, and $n \geq 2$. If $f \in C[0,1]$, the inequality holds strictly for $0 < x < 1$ unless f is linear in each of the intervals between consecutive knots $\frac{p^{n-k-1}[k]_{p,q}}{[n]_{p,q}}$, $0 \leq k \leq n-1$, in which case we have the equality.*

Proof. For $0 < q < p \leq 1$, we begin by writing

$$\prod_{s=0}^{n-1} (p^s - q^s x)^{-1} [S_{n-1,p,q}(f;x) - S_{n,p,q}(f;x)]$$

$$= \prod_{s=0}^{n-1} (p^s - q^s x)^{-1} \left[\sum_{k=0}^{n-1} \left[\begin{array}{c} n-1 \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)-(n-2)(n-1)}{2}} x^k \times \right.$$

$$\times \prod_{s=0}^{n-k-2} (p^s - q^s x) f\left(\frac{p^{n-k-1}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) - \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k p^{\frac{k(k-1)-n(n-1)}{2}} \times$$

$$\left. \times \prod_{s=0}^{n-k-1} (p^s - q^s x) f\left(\frac{p^{n-k}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) \right] =$$

$$= \sum_{k=0}^{n-1} \left[\begin{array}{c} n-1 \\ k \end{array} \right]_{p,q} p^{\frac{k(k-1)-(n-2)(n-1)}{2}} x^k \times$$

$$\times \prod_{s=n-k-2}^{n-1} (p^s - q^s x)^{-1} f\left(\frac{p^{n-k-1}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) - \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} x^k p^{\frac{k(k-1)-n(n-1)}{2}} \times$$

$$\times \prod_{s=n-k-1}^{n-1} (p^s - q^s x)^{-1} f\left(\frac{p^{n-k-1}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right).$$

Denote

$$\psi_k(x) = p^{\frac{k(k-1)}{2}} x^k \prod_{s=n-k-1}^{n-1} (p^s - q^s x)^{-1} \tag{7}$$

and use the following relation

$$p^{n-1} p^{\frac{k(k-1)}{2}} x^k \prod_{s=n-k-1}^{n-1} (p^s - q^s x)^{-1} = p^k \psi_k(x) + q^{n-k-1} \psi_{k+1}(x).$$

We find

$$\prod_{s=0}^{n-1} (p^s - q^s x)^{-1} [S_{n-1,p,q}(f;x) - S_{n,p,q}(f;x)]$$

$$= \sum_{k=0}^{n-1} [\ n-1k\ ]_{p,q} p^{\frac{-(n-2)(n-1)}{2}} p^{-(n-1)} (p^k \psi_k(x) + q^{n-k-1} \psi_{k+1}(x)) \times$$

$$\times f\left(\frac{p^{n-k-1}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right)$$

$$- \sum_{k=0}^{n} \left[\begin{array}{c} n \\ k \end{array} \right]_{p,q} p^{\frac{-n(n-1)}{2}} \psi_k(x) f\left(\frac{p^{n-k-1}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right)$$

$$= p^{\frac{-n(n-1)}{2}} \left[\sum_{k=0}^{n-1} \left[\begin{array}{c} n-1 \\ k \end{array} \right]_{p,q} p^k \psi_k(x) f\left(\frac{p^{n-k-1}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta} \right) \right.$$

$$+\sum_{k=1}^{n}\left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{p,q} q^{n-k}\psi_k(x) f\left(\frac{p^{n-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)-$$

$$-\sum_{k=0}^{n}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} \psi_k(x) f\left(\frac{p^{n-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)\Bigg]$$

$$= p^{\frac{-n(n-1)}{2}}\sum_{k=1}^{n-1}\Bigg\{\left[\begin{array}{c} n-1 \\ k \end{array}\right]_{p,q} p^k f\left(\frac{p^{n-k-1}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)$$

$$+\left[\begin{array}{c} n-1 \\ k-1 \end{array}\right]_{p,q} q^{n-k} f\left(\frac{p^{n-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)-\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} f\left(\frac{p^{n-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)\Bigg\}\psi_k(x)$$

$$= p^{\frac{-n(n-1)}{2}}\sum_{k=1}^{n-1}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q}\Bigg\{\frac{[n-k]_{p,q}}{[n]_{p,q}} p^k f\left(\frac{p^{n-k-1}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)$$
$$+\frac{[k]_{p,q}}{[n]_{p,q}} q^{n-k} f\left(\frac{p^{n-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)-f\left(\frac{p^{n-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)\Bigg\}\psi_k(x)$$

$$= p^{\frac{-n(n-1)}{2}}\sum_{k=1}^{n-1}\left[\begin{array}{c} n \\ k \end{array}\right]_{p,q} a_k\psi_k(x),$$

where
$a_k=\frac{[n-k]_{p,q}}{[n]_{p,q}}p^k f\left(\frac{p^{n-k-1}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)+\frac{[k]_{p,q}}{[n]_{p,q}}q^{n-k} f\left(\frac{p^{n-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right)-f\left(\frac{p^{n-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}\right).$

From (7) it is clear that each $\psi_k(x)$ is non-negative on $[0,1]$ for $0<q<p\leq 1$ and, thus, it suffices to show that each a_k is non-negative.
Since f is convex on $[0,1]$, for any t_0, t_1 and $\lambda\in[0,1]$ it follows that

$$f(\lambda t_0+(1-\lambda)t_1)\leq \lambda f(t_0)+(1-\lambda)f(t_1).$$

If we choose $t_0=\frac{p^{n-k}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}$, $t_1=\frac{p^{n-k-1}[k]_{p,q}+\alpha}{[n]_{p,q}+\beta}$, and

$\lambda=\frac{[k]_{p,q}}{[n]_{p,q}}q^{n-k}$, then $t_0,t_1\in[0,1]$ and $\lambda\in(0,1)$ for $1\leq k\leq n-1$, and we deduce that

$$a_k=\lambda f(t_0)+(1-\lambda)f(t_1)-f(\lambda t_0+(1-\lambda)t_1)\geq 0.$$

Thus $S_{n-1,p,q}(f;x) \geq S_{n,p,q}(f;x)$.

We have equality for $x = 0$ and $x = 1$, since the Bernstein Stancu polynomials interpolate f on these end points. The inequality will be strict for $0 < x < 1$ unless when f is linear in each of the intervals between consecutive knots

$$\frac{p^{n-k-1}[k]_{p,q} + \alpha}{[n]_{p,q} + \beta}, \quad 0 \leq k \leq n-1,$$

then we have

$$S_{n-1,p,q}(f;x) = S_{n,p,q}(f;x)$$

for $0 \leq x \leq 1$. ◄

6. A global approximation theorem

In this section, we establish a global approximation theorem by means of Ditzian-Totik modulus of smoothness and Voronovskaja type approximation result.

In order to prove our next result, we recall the definitions of the Ditzian-Totik first order modulus of smoothness and the K-functional. Let $\phi(x) = \sqrt{x(1-x)}$ and $f \in C[0,1]$. The first order modulus of smoothness is given by

$$\omega_\phi(f;t) = \sup_{0<h\leq t} \left\{ \Big| f(x + \frac{h\phi(x)}{2}) - f(x - \frac{h\phi(x)}{2}) \Big|, x \pm \frac{h\phi(x)}{2} \in [0,1] \right\}. \quad (8)$$

The corresponding k-functional to (10) is defined by

$k_\phi(f;t) = \inf_{g \in W_\phi[0,1]} \left\{ \|f - g\| + t\|\phi g'\| \right\}$ $(t > 0)$, where $W_\phi[0,1] = \{g : g \in AC_{loc}[0,1], \|\phi g'\| < \infty\}$ and $g \in AC_{loc}[0,1]$ means that g is absolutely continuous on every interval $[a,b] \subset [0,1]$. It is well known [35] that there exists a constant $C > 0$ such that

$$k_\phi(f;t) \leq C w_\phi(f;t). \quad (9)$$

Theorem 6. *Let $f \in C[0,1]$ and $\phi(x) = \sqrt{x(1-x)}$. Then for every $x \in [0,1]$ we have $\Big| S_{n,p,q}(f;x) - f(x) \Big| \leq C\omega_\phi\left(f; \frac{[n]_{p,q}}{\sqrt{([n]_{p,q}+\beta)}}\right)$, where C is a constant independent of n and x.*

Proof. Using the representation

$$g(t) = g(x) + \int_x^t g'(u)du,$$

we get

$$\Big|S_{n,p,q}(g;x)-g(x)\Big| = \Big|S_{n,p,q}\Big(\int_x^t g'(u)du;x\Big)\Big|. \tag{10}$$

For any $x\in(0,1)$ and $t\in[0,1]$, we find that

$$\Big|\int_x^t g'(u)du\Big| \le \|\phi g'\|\Big|\int_x^t \frac{1}{\phi(u)}du\Big|. \tag{11}$$

Further

$$\begin{aligned}\Big|\int_x^t \frac{1}{\phi(u)}du\Big| &= \Big|\int_x^t \frac{1}{\sqrt{u(1-u)}}du\Big| \\ &\le \Big|\int_x^t \Big(\frac{1}{\sqrt{u}}+\frac{1}{\sqrt{1-u}}\Big)du\Big| \\ &\le 2(|\sqrt{t}-\sqrt{x}|+|\sqrt{1-t}-\sqrt{1-x}|) \\ &= 2|t-x|\Big(\frac{1}{\sqrt{t}+\sqrt{x}}+\frac{1}{\sqrt{1-t}+\sqrt{1-x}}\Big) \\ &< 2|t-x|\Big(\frac{1}{\sqrt{x}}+\frac{1}{\sqrt{1-x}}\Big) \le \frac{2\sqrt{2}|t-x|}{\phi(x)}. \end{aligned} \tag{12}$$

From (10)-(12) and using the Cauchy-Schwarz inequality, we obtain

$$\begin{aligned}|S_{n,p,q}(g;x)-g(x)| &< 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)S_{n,p,q}(|t-x|;x) \\ &\le 2\sqrt{2}\|\phi g'\|\phi^{-1}(x)(S_{n,p,q}((t-x)^2;x))^{\frac{1}{2}}.\end{aligned}$$

Using Lemma 2, we get

$$|S_{n,p,q}(g;x)-g(x)| \le \frac{2\sqrt{2}[n]_{p,q}}{\sqrt{([n]_{p,q}+\beta}}\|\phi g'\|.$$

Now using the above inequality we can write

$$\begin{aligned}|S_{n,p,q}(f;x)-f(x)| &\le |S_{n,p,q}(f-g;x)|+|f(x)-g(x)|+|S_{n,p,q}(g;x)-g(x)| \\ &\le 2\sqrt{2}\Big(\|f-g\|+\frac{[n]_{p,q}}{\sqrt{([n]_{p,q}+\beta)}}\|\phi g'\|\Big).\end{aligned}$$

Taking the infimum on the right hand side of the above inequality over all $g\in W_\phi[0,1]$, we get

$$|S_{n,p,q}(f;x)-f(x)| \le CK_\phi\Big(f;\frac{[n]_{p,q}}{\sqrt{([n]_{p,q}+\beta)}}\Big). \tag{13}$$

Using (9), we complete the proof. ◄

On the other hand, for any $m = 1, 2, ...$ and $0 < q < p \leqslant 1$, there exists a constant $C_m > 0$ such that

$$|S_{n,p,q}((t-x)^m_{p,q}; x)| \leqslant C_m \frac{\phi^2(x)[n]_{p,q}}{([n]_{p,q}+\beta)^{\lfloor \frac{m+1}{2} \rfloor}}, \tag{14}$$

where $x \in [0,1]$ and $\lfloor a \rfloor$ is the integral part of $a \geq 0$.

Throughout this proof, C denotes a constant not necessarily the same at each occurrence.

Now combining (13)-(14) and applying Lemma 2, the Cauchy-Schwarz inequality, we get $\left| S_{n,p,q}(f;x) - f(x) \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{2([n]_{p,q}+\beta)^2} f''(x) \right|$

$$\leq 2\|f'' - g\| S_{n,p,q}((t-x)^2; x) + 2\|\phi g'\| \phi^{-1}(x) S_{n,p,q}(|t-x|^3; x)$$
$$\leq 2\|f'' - g\| \frac{\phi^2(x)[n]_{p,q}}{([n]_{p,q}+\beta)} + 2\|\phi g'\| \phi^{-1}(x) \{S_{n,p,q}((t-x)^2; x)\}^{\frac{1}{2}} \{S_{n,p,q}((t-x)^4; x)\}^{\frac{1}{2}}$$
$$\leq 2\|f'' - g\| \frac{\phi^2(x)[n]_{p,q}}{([n]_{p,q}+\beta)} + 2 \frac{C}{([n]_{p,q}+\beta)} \|\phi g'\| \frac{\phi(x)[n]_{p,q}}{([n]_{p,q}+\beta)^{\frac{1}{2}}}$$
$$\leq \frac{C[n]_{p,q}}{([n]_{p,q}+\beta)} \{\phi^2(x)\|f'' - g\| + ([n]_{p,q}+\beta)^{\frac{-1}{2}} \phi(x) \|\phi g'\|\}.$$

Since $\phi^2(x) \leq \phi(x) \leq 1, x \in [0,1]$, we obtain

$$\left| ([n]_{p,q}+\beta)^2 [S_{n,p,q}(f;x) - f(x)] - \frac{p^{n-1}[n]_{p,q} - 2\alpha\beta}{2} \phi^2(x) f''(x) \right| \leq C\{\|f'' - g\|$$
$$+ ([n]_{p,q}+\beta)^{\frac{-1}{2}} \phi(x) \|\phi g'\|\}.$$

◄

6.1. Voronovskaja type theorem

Using the first order Ditzian-Totik modulus of smoothnes, we prove a quantitative Voronovskaja type theorem for the (p,q)-Bernstein Stancu operators. For any $f \in C^2[0,1]$, the following inequalities hold:

$$|([n]_{p,q}+\beta)[S_{n,p,q}(f;x) - f(x)] - \frac{p^{n-1} - 2\alpha\beta}{2} \phi^2(x) f''(x)| \leqslant C\omega_\phi(f''\phi(x) n^{\frac{-1}{2}}), \tag{15}$$

$$|([n]_{p,q}+\beta)[S_{n,p,q}(f;x)-f(x)]-\frac{p^{n-1}-2\alpha\beta}{2}\phi^2(x)f''(x)|\leqslant C\phi(x)\omega_\phi(f'',n^{\frac{-1}{2}}), \tag{16}$$

where C is a positive constant.

Proof. Let $f\in C^2[0,1]$ be given and $t,x\in[0,1]$. Using Taylor's expansion, we have

$$f(t)-f(x)=(t-x)f'(x)+\int_x^t(t-u)f''(u)du. \tag{17}$$

Therefore

$$f(t)-f(x)-(t-x)f'(x)-\frac{1}{2}(t-x)^2f''(x)=\int_x^t(t-u)f''(u)du-\int_x^t(t-u)f''(x)dx$$

$$=\int_x^t(t-u)[f''(u)-f''(x)]du.$$

In view of Lemma 2, we get

$$\left|S_{n,p,q}(f;x)-f(x)-\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{2([n]_{p,q}+\beta)^2}\phi^2(x)f''(x)\right|\leq$$

$$\leq S_{n,p,q}\left(\left|\int_x^t|(t-u)||f''(u)-f''(x)|du\right|;x\right). \tag{18}$$

The quantity $|\int_x^t|f''(u)-f''(x)||(t-u)|du|$ was estimated in [36, p-337], as follows:

$$\left|\int_x^t f''(u)-f''(x)||t-u|du\right|\leq 2\|f''-g\|(t-x)^2+2\|\phi g'\|\phi^{-1}(x)|t-x|^3, \tag{19}$$

where $g\in W_\phi[0,1]$. On the other hand, for any $m=1,2,.......$ and $0<q<p\leqslant 1$, there exists a constant $C_m>0$ such that

$$|S_{n,p,q}((t-x)^m_{p,q};x)|\leqslant C_m\frac{\phi^2(x)[n]_{p,q}}{([n]_{p,q}+\beta)^{\lfloor\frac{m+1}{2}\rfloor}}, \tag{20}$$

where $x\in[0,1]$ and $\lfloor a\rfloor$ is the integral part of $a\geq 0$.

Throughout this proof, C denotes a constant not necessarily the same at each occurrence.

Now combining (8.4)-(8.5) and applying Lemma 2, the Cauchy-Schwarz inequality, we get $\left|S_{n,p,q}(f;x)-f(x)\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{2([n]_{p,q}+\beta)^2}f''(x)\right|$

$$\leq 2\|f''-g\|S_{n,p,q}((t-x)^2;x)+2\|\phi g'\|\phi^{-1}(x)S_{n,p,q}(|t-x|^3;x)$$

$$\leq 2\|f''-g\|\frac{\phi^2(x)[n]_{p,q}}{([n]_{p,q}+\beta)}+2\|\phi g'\|\phi^{-1}(x)\{S_{n,p,q}((t-x)^2;x)\}^{\frac{1}{2}}\{S_{n,p,q}((t-x)^4;x)\}^{\frac{1}{2}}$$
$$\leq 2\|f''-g\|\frac{\phi^2(x)[n]_{p,q}}{([n]_{p,q}+\beta)}+2\frac{C}{([n]_{p,q}+\beta)}\|\phi g'\|\frac{\phi(x)[n]_{p,q}}{([n]_{p,q}+\beta)^{\frac{1}{2}}}$$
$$\leq \frac{C[n]_{p,q}}{([n]_{p,q}+\beta)}\{\phi^2(x)\|f''-g\|+([n]_{p,q}+\beta)^{\frac{-1}{2}}\phi(x)\|\phi g'\|\}.$$

Since $\phi^2(x)\leq\phi(x)\leq 1, x\in[0,1]$, we obtain

$$\left|([n]_{p,q}+\beta)^2[S_{n,p,q}(f;x)-f(x)]-\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{2}\phi^2(x)f''(x)\right|\leq$$
$$\leq C\{\|f''-g\|+([n]_{p,q}+\beta)^{\frac{-1}{2}}\phi(x)\|\phi g'\|\}. \tag{21}$$

Also, the following inequality can be obtained

$$\left|([n]_{p,q}+\beta)^2[S_{n,p,q}(f;x)-f(x)]-\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{2}\phi^2(x)f''(x)\right|\leq$$
$$\leq C\phi(x)\{\|f''-g\|+([n]_{p,q}+\beta)^{\frac{-1}{2}}\|\phi g'\|\}. \tag{22}$$

Taking the infimum on the right-hand side of the above relations over $g\in W_\phi[0,1]$, we get

$$\left|([n]_{p,q}+\beta)^2[S_{n,p,q}(f;x)-f(x)]-\frac{p^{n-1}[n]_{p,q}-2\alpha\beta}{2}\phi^2(x)f''(x)\right|$$
$$\leq C\phi(x)K_\phi(f'';([n]_{p,q}+\beta)^{\frac{-1}{2}})CK_\phi(f'';\phi(x)([n]_{p,q}+\beta)^{\frac{-1}{2}}). \tag{23}$$

Using (8.9) and(7.2), we complete the proof. ◄

7. Graphical analysis

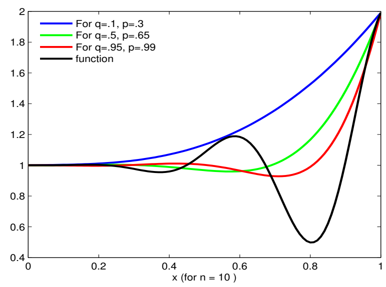
With the help of Matlab, we show comparisons and some illustrative graphics for the convergence of operators (2) to the function $f(x)=1+x^3\ sin(14x)$ under different parameters.

From figure 1(a), it can be observed that as q and p tend to 1 provided $0<q<p\leq 1$, (p,q)-Bernstein Stancu operators given by (2) converge to the function.

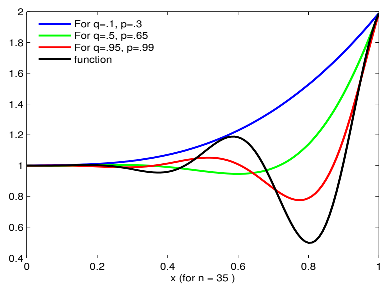
From figure 1(a) and (b), it can be observed that for $\alpha = \beta = 0$, as the value of n increases, (p, q)-Bernstein Stancu operators given by 2 converges toward the function $f(x) = 1 + x^3\ sin(14x)$.

Similarly from figure 2(a), it can be observed that for $\alpha = \beta = 3$, as q and p tend to 1 provided $0 < q < p \leq 1$, (p, q)-Bernstein Stancu operators given by 2 converge to the function.

From figure 2(a) and (b), it can be observed that as the value of n increases, (p, q)-Bernstein Stancu operators given by $f(x) = 1 + x^3\ sin(14x)$ converge towards the function.

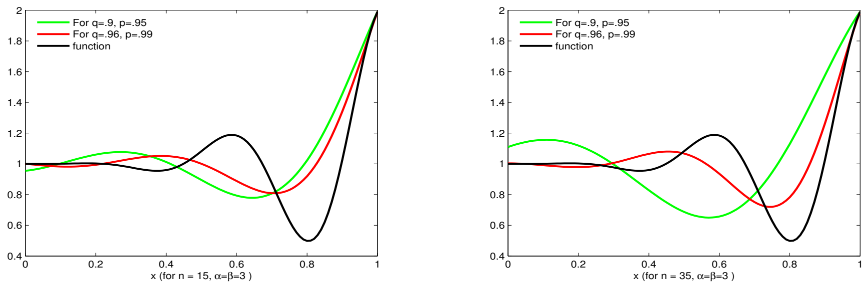


(a)



(b)

Figure 1: (p, q)-Bernstein Stancu operators



(a) (b)

Figure 2: (p,q)-Bernstein Stancu operators.

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