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Fuzzy Soft Relations over Semigroups

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Abstract. In this paper we attempt to generalize the notion of soft relation on semigroups studied by Feng et al. in 2013 to fuzzy soft relation on semigroups. Moreover, we study certain involved algebraic properties of this notion which can be applied in mathematics and information science.
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1. Introduction and preliminaries

In 1999, Molodtsov [14] introduced the notion of soft set as a new tool for dealing with uncertainties which traditional mathematical tools can not handle. Later, the theory of soft sets was applied to solve some decision making problems (see [11, 13, 16]). In 2007, this notion was generalized to soft groups by Aktaş and Çağman [2]. Then the normal soft groups were studied by Liu [10] and Yin [19]. Some generalizations of soft sets were done by Acar et al. [1] in 2010, where they introduced the notions of soft rings and soft ideals. Atagün and Sezgin [3] studied the soft substructure of rings and modules. The recent attempt in this area of research was made by Feng et al. [8] who investigated the soft relations on semigroups.

Our main purpose in this paper is to introduce and study the notion of fuzzy soft relation on semigroups which will give us certain algebraic properties. Over the years, Zadeh's well-known concept of fuzzy set [20] was extended to some algebraic structures (see [4, 5, 7, 9, 12, 15, 17, 18]).

Here we give a brief account of some basic definitions about soft sets, fuzzy soft sets and soft relations needed in the sequel.

Let X be an initial universe of objects and E be a set of parameters in relation to the objects in X and $A \subseteq E$. Also let $P(X)$ denote the power set of X.

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Definition 1. *[2] A pair (F, A) is called a soft set over X, where $F : A \to P(X)$ is a set valued mapping.*

Definition 2. *[8] (i) Let (σ, A) be a soft set over $X \times X$. Then (σ, A) is called a soft relation over X. In fact, (σ, A) is a parameterized collection of relations over X. That is, we have a relation $\sigma(a)$ on X for each parameter $a \in A$. We denote the collection of all soft relations over X by $\xi_{Br}(X)$.*

(ii) A soft relation (σ, A) over a set X is called a soft equivalence relation over X if it is reflexive, symmetric and transitive.

These definitions together with Corollary 2.4 of [8] show that a soft relation (σ, A) over a set X is a soft equivalence relation over X if and only if $\sigma(a) \neq \emptyset$ is an equivalence relation on X, for all $a \in A$.

Obviously each equivalence relation on a set partitions the set into disjoint equivalence classes. Therefore, a soft equivalence relation over a set X provides us a parameterized collection of partitions of X. Let $[x]_{\sigma(a)}$ denote the equivalence class containing $x \in X$ determined by $\sigma(a)$ for $a \in A$. Then $y \in [x]_{\sigma(a)}$ if and only if $(x, y) \in \sigma(a)$.

Definition 3. *[12] Let $I = [0, 1]$ and I^X denote the set of all fuzzy sets on X. A pair (F, A) is called a fuzzy soft set over X, where F is a mapping from A into I^X. That is, for each $a \in A$, $F_a : X \to [0, 1]$ is a fuzzy set on X.*

Definition 4. *[12] Let (F, A) and (G, B) be two fuzzy soft sets over X.*

(i) We say that (F, A) is a fuzzy soft subset of (G, B) and write $(F, A) \subseteq (G, B)$ if $A \subseteq B$ and $F_a \leq G_a$ for all $a \in A$. Moreover, (F, A) and (G, B) are said to be fuzzy soft equal if (F, A) is a fuzzy soft subset of (G, B) and vise-versa.

(ii) Union of (F, A) and (G, B) is a fuzzy soft set (H, C), where $C = A \cup B$ and for all $c \in C$:

$$H_c = \begin{cases} F_c, & c \in A \setminus B, \\ G_c, & c \in B \setminus A, \\ F_c \cap G_c, & c \in A \cap B. \end{cases}$$

We write $(F, A) \cup (G, B) = (H, C)$.

(iii) Intersection of (F, A) and (G, B) is a fuzzy soft set (H, C), where $C = A \cap B \neq \emptyset$ and for all $c \in C$, $H_c = F_c \cap G_c$. We write $(F, A) \cap (G, B) = (H, C)$.

(iv) The complement of a (F, A) is denoted by $(F, A)^c$ and is defined by $(F, A)^c = (F^c, A)$, where F^c is a mapping given by $F^c_a = [F_a]^c$ for any $a \in A$.

2. Fuzzy soft relation

In this section we define the notion of fuzzy soft relation over a set X and then give some basic concepts. For details about the fuzzy relations one may see [6, 15].

Definition 5. *Let (μ, A) be a fuzzy soft set over $X \times X$. Then (μ, A) is called a fuzzy soft relation over X.*

In fact, (μ, A) is a parameterized collection of fuzzy relations over X. That is, we have a fuzzy relation μ_a on X for each parameter $a \in A$. We shall denote the collection of all fuzzy soft relations over X by $\xi_{fbr}(X)$.

Example 1. *Let $X = \{h_1, h_2, ..., h_6\}$ and $E = \{e_1, e_2, ..., e_5\}$ be a set of parameters. Assume $A = \{e_1, e_2\}$. Then (σ, A) over X by definition $\{\sigma_{(e_1)}(h_1, h_2) = 0.1, \sigma_{(e_1)}(h_3, h_2) = 0.8\}$, $\{\sigma_{(e_2)}(h_1, h_2) = 0.7, \sigma_{(e_2)}(h_2, h_3) = 0.5\}$ is a fuzzy soft relation which we denote by,*

	(h_1, h_2)	(h_3, h_2)
σ_{e_1}	0.1	0.8
σ_{e_2}	0.7	0.5

Definition 6. *Let (μ, A) and (ρ, B) be two fuzzy soft relations over X and $C = A \cap B \neq \emptyset$. Then we define:*
(1) $(\mu, A) \leq (\rho, B)$ if and only if $A \subseteq B$ and $\mu_a(x, y) \leq \rho_a(x, y)$, for all $a \in A$ and $x, y \in X$.
(2) $(\mu, A) \wedge (\rho, B) = (\delta, C)$, where $\delta_c(x, y) = \mu_c \wedge \rho_c(x, y) = min\{\mu_c(x, y), \rho_c(x, y)\}$ for all $c \in C$.
(3) $(\mu, A) \vee (\rho, B) = (\delta, C)$, where $\delta_c(x, y) = \mu_c \vee \rho_c(x, y) = max\{\mu_c(x, y), \rho_c(x, y)\}$ for all $c \in C$.
(4) $(\mu, A) \circ (\rho, B) = (\delta, C)$, where $\delta_c(x, y) = \mu_c \circ \rho_c(x, y) = \bigvee_{z \in X}\{\mu_c(x, z) \wedge \rho_c(z, y)\}$ for all $c \in C$.

Composition of fuzzy soft relations on a set X is associative, because the composition of fuzzy relations on X is associative ([15, Proposition 2.3]). Thus for fuzzy soft relations (δ, A), (ρ, B) and (μ, C), where $A \cap B \cap C \neq \emptyset$, we have

$$(\delta, A) \circ ((\rho, B) \circ (\mu, C)) = ((\delta, A) \circ (\rho, B)) \circ (\mu, C).$$

The converse of a fuzzy soft relation (μ, A) over X denoted by $(\mu, A)^{-1}$ is also a fuzzy soft relation, where $(\mu, A)^{-1} = (\mu^{-1}, A)$ is defined as $\mu_a^{-1}(x, y) = \mu_a(y, x)$ for all $a \in A$, $x, y \in X$.

Example 2. *Let $X = \{h_1, h_2, ..., h_6\}$ and $E = \{e_1, e_2, ..., e_5\}$ be a set of parameters. Assuming $A = \{e_1, e_2\}$ and $B = \{e_1, e_2, e_3\}$, we define two fuzzy soft relations (σ, A) and (ρ, B) over X such that*

	(h_1, h_2)	(h_3, h_2)
σ_{e_1}	0.1	0.8
σ_{e_2}	0.7	0.5

and

	(h_1, h_2)	(h_3, h_2)
σ_{e_1}	0.5	0.8
σ_{e_2}	0.7	0.7
σ_{e_3}	0.4	0.2

Then $(\sigma, A) \leq (\rho, B)$.

Example 3. *Let $X = \{h_1, h_2, ..., h_6\}$ and $E = \{e_1, e_2, ..., e_5\}$ be a set of parameters. Assuming $A = \{e_1, e_2\}$ and $B = \{e_1, e_3\}$, we define two fuzzy soft relations (σ, A) and (ρ, B) over X such that $\{\sigma_{e_1}(h_1, h_2) = 0.1, \sigma_{e_1}(h_3, h_2) = 0.8\}$, $\{\sigma_{e_2}(h_1, h_2) = 0.7, \sigma_{e_2}(h_2, h_3) = 0.5\}$ and also $\{\rho_{e_1}(h_1, h_2) = 0.8, \rho_{e_1}(h_2, h_3) = 0.9\}$, $\{\rho_{e_3}(h_1, h_2) = 0.6, \rho_{e_3}(h_3, h_3) = 0.5\}$. Then $C = A \cap B = \{e_1\}$, so $(\delta, C) = (\sigma, A) \circ (\rho, B) = \{\delta_{e_1}(h_1, h_3) = \sigma_{e_1}(h_1, h_2) \wedge \rho_{e_1}(h_2, h_3) = 0.1, \delta_{e_1}(h_3, h_3) = \sigma_{e_1}(h_3, h_2) \wedge \rho_{e_1}(h_2, h_3) = 0.8\}$.*

Definition 7. *Let (μ, A) be a fuzzy soft relation over X. Then (μ, A) is called*

(1) fuzzy soft reflexive if $\mu_a(x, x) = 1$ for all $a \in A, x \in X$, i.e. μ_a is a fuzzy reflexive relation on X for all $a \in A$;

(2) fuzzy soft symmetric if $(\mu, A)^{-1} = (\mu, A)$, i.e. μ_a is a fuzzy symmetric relation on X for all $a \in A$;

(3) fuzzy soft transitive if $(\mu, A) \circ (\mu, A) \leq (\mu, A)$, i.e. μ_a is a fuzzy transitive relation on X for all $a \in A$;

(4) fuzzy soft equivalence if it is fuzzy soft reflexive, fuzzy soft symmetric and fuzzy soft transitive, i.e. μ_a is a fuzzy equivalence relation on X for all $a \in A$.

We denote the collection of all fuzzy soft equivalence relations over X by $\xi_{fE_r}(X)$.

Clearly, if (μ, A) is a fuzzy soft equivalence relation on X, then $(\mu, A) \circ (\mu, A) = (\mu, A)$.

Example 4. *Let $X = \{h_1, h_2, h_3\}$ and $E = \{e_1, e_2, ..., e_5\}$ be a set of parameters. Suppose that $A = \{e_1, e_2\}$. Then (μ, A) is a fuzzy soft reflexive and symmetric relation by next definitions:*

μ_{e_1}	h_1	h_2	h_3
h_1	1	0.8	0.6
h_2	0.8	1	0.2
h_3	0.6	0.2	1

and

μ_{e_1}	h_1	h_2	h_3
h_1	1	0.6	0.9
h_2	0.6	1	0.1
h_3	0.9	0.1	1

Example 5. *Let $X = \{h_1, h_2\}$ and $E = \{e_1, e_2, ..., e_5\}$ be a set of parameters. Suppose that $A = \{e_1\}$. Then (μ, A) is a fuzzy soft transitive relation by next definition:*

μ_{e_1}	h_1	h_2
h_1	0.4	0.2
h_2	0.7	0.3

Definition 8. *A fuzzy soft identity relation $(\triangle, A)$ on X is defined as*

$$\triangle_a(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y, \end{cases}$$

for all $a \in A$ and $x, y \in X$. Also we define a fuzzy soft zero relation $(\bigtriangledown, A)$ on X as $\bigtriangledown_a(x, y) = 0$ for all $a \in A$ and $x, y \in X$.

One can easily check that if $\{(\mu_i, A_i)\}_{i \in I}$ is a non-empty family of fuzzy soft equivalence relations on X, then $\bigwedge_{i \in I}(\mu_i, A_i)$ is also a fuzzy soft equivalence relation on X.

Definition 9. *Let (μ, A) be a fuzzy soft relation on X.*

(1) The fuzzy soft transitive closure of (μ, A) denoted by $(\mu, A)^\infty$ is defined as

$$(\mu, A)^\infty = \bigvee_{n=1}^{\infty} (\mu, A)^n,$$

where $(\mu, A)^n = (\mu, A) \circ (\mu, A) \circ \cdots \circ (\mu, A)$ (n factors).

(2) Let $\{(\mu_i, A_i)\}_{i \in I}$ be the family of all fuzzy soft equivalence relations on X containing (μ, A). The fuzzy soft equivalence relation $(\mu, A)^e$ generated by (μ, A) is defined as

$$(\mu, A)^e = \bigwedge_{i \in I, (\mu, A) \leq (\mu_i, A_i)} (\mu_i, A_i).$$

It is clear that $(\mu, A)^e$ is the smallest fuzzy soft equivalence relation on X containing (μ, A).

Proposition 1. *Let (μ, A) be a fuzzy soft relation on X. Then $(\mu, A)^\infty$ is the smallest transitive fuzzy soft relation on X containing (μ, A).*

Proof. It is clear that (μ, A) is contained in $(\mu, A)^\infty$. On the other hand, we know that for all $a \in A$, μ_a^∞ is a fuzzy transitive relation over X. So $(\mu, A)^\infty$ is a fuzzy soft transitive relation. Now assume that (ρ, A) is a fuzzy soft transitive relation over X containing (μ, A). We have

$$(\mu, A)^n = (\mu, A) \circ \cdots \circ (\mu, A) \leq (\rho, A) \circ \cdots \circ (\rho, A) \leq (\rho, A).$$

Thus for all $n \in \mathbb{N}$, $(\mu, A)^n \leq (\rho, A)$ and hence $(\mu, A)^\infty \leq (\rho, A)$. ◄

Proposition 2. *If (μ, A) is fuzzy soft symmetric, then so is $(\mu, A)^\infty$.*

Proof. Suppose that (μ, A) is a fuzzy soft symmetric relation over X. For $n \geq 1$ and $x, y \in X$ we have

$$\begin{aligned}
\mu_a^n(x,y) &= \bigvee\nolimits_{z_1,z_2,\ldots,z_{n-1}} \{\mu_a(x,z_1) \wedge \mu_a(z_1,z_2) \wedge \cdots \wedge \mu_a(z_{n-1},y)\} \\
&= \bigvee\nolimits_{z_{n-1},z_{n-2},\ldots,z_1} \{\mu_a(y,z_{n-1}) \wedge \mu_a(z_{n-1},z_{n-2}) \wedge \cdots \wedge \mu_a(z_1,x)\} \\
&= \mu_a^n(y,x).
\end{aligned}$$

Therefore, μ^∞ is a fuzzy symmetric relation over X and the proof is complete. ◄

Proposition 3. *Let (μ, A), (ρ, B) and (δ, A) be fuzzy soft relations on X. Then*
(1) $(\mu, A) \circ ((\rho, B) \vee (\delta, C)) = ((\mu, A) \circ (\rho, B)) \vee ((\mu, A) \circ (\delta, C))$.
(2) $(\mu, A) \leq (\rho, B) \Rightarrow (\mu, A) \circ (\delta, C) \leq (\rho, B) \circ (\delta, C)$.
(3) $(\mu, A) \leq (\rho, B) \Rightarrow (\mu, A)^\infty \leq (\rho, B)^\infty$.
(4) If $(\mu, A) \circ (\rho, B) = (\rho, B) \circ (\mu, A)$ where $(\mu, A), (\rho, B) \in \xi_{fE_r}(X)$, then $((\mu, A) \circ (\rho, B))^\infty = (\mu, A) \circ (\rho, B)$.

Proof.
(1) Let $x, y \in X$. Take any $d \in A \cap (B \cap C)$. Then

$$\begin{aligned}
\mu_d \circ (\rho_d \vee \delta_d)(x,y) &= \bigvee\nolimits_{z \in X} \mu_d(x,z) \wedge (\rho_d(z,y) \vee \delta_d(z,y)) \\
&= \bigvee\nolimits_{z \in X} (\mu_d(x,z) \wedge \rho_d(z,y)) \vee (\mu_d(x,z) \wedge \delta_d(z,y)) \\
&= (\mu_d \circ \rho_d) \vee (\mu_d \circ \delta_d)(x,y).
\end{aligned}$$

(2) Let $x, y \in X$. For any $d \in A \cap C$ we have

$$\begin{aligned}(\mu_d \circ \delta_d)(x,y) &= \bigvee_z(\mu_d(x,z) \wedge \delta_d(z,y))\\ &\leq \bigvee_z(\rho_d(x,z) \wedge \delta_d(z,y))\\ &= (\rho_d \circ \delta_d)(x,y).\end{aligned}$$

(3) This is similar to (1) and (2).

(4) $((\mu,A)\circ(\rho,B))^n = \overbrace{((\mu,A)\circ(\rho,B))\circ((\mu,A)\circ(\rho,B))\circ\cdots\circ((\mu,A)\circ(\rho,B))}^{(n\ factors)} =$

$\overbrace{((\mu,A)\circ(\mu,A)\circ\cdots\circ(\mu,A))}^{(n\ factors)}\circ\overbrace{((\rho,B)\circ(\rho,B)\circ\cdots\circ(\rho,B))}^{(n\ factors)} = (\mu,A)\circ(\rho,B)$.
So $((\mu,A)\circ(\rho,B))^\infty = (\mu,A)\circ(\rho,B)$. ◄

Theorem 1. *If (μ,A) is a fuzzy soft relation on X, then $(\mu,A)^e = [(\mu,A) \vee (\mu,A)^{-1} \vee (\triangle,A)]^\infty$.*

Proof. Let $(\delta,A) = [(\mu,A) \vee (\mu,A)^{-1} \vee (\triangle,A)]$. We know that $(\delta,A)^\infty$ is a fuzzy soft transitive relation over X containing (δ,A). On the other hand, $(\triangle,A) \leq [(\mu,A) \vee (\mu,A)^{-1} \vee (\triangle,A)] \leq (\delta,A)^\infty$. Since $(\triangle,A) \leq (\delta,A)^\infty$ for all $a \in A$, $\delta_a^\infty(x,x) = 1$ which means that $(\delta,A)^\infty$ is reflexive. Certainly, (δ,A) is symmetric and hence so is $(\delta,A)^\infty$. Thus $(\delta,A)^\infty \in \xi_{fE_r}$. Let (γ,A) be a fuzzy soft equivalence relation such that $(\mu,A) \leq (\gamma,A)$. Since $(\triangle,A) \leq (\gamma,A)$ and $(\mu,A)^{-1} \leq (\gamma,A)^{-1} = (\gamma,A)$, $(\mu,A) \vee (\mu,A)^{-1} \vee (\triangle,A) \leq (\gamma,A)$. Hence, $(\delta,A)^\infty \leq (\gamma,A)$. Consequently,

$$(\mu,A)^e = [(\mu,A) \vee (\mu,A)^{-1} \vee (\triangle,A)]^\infty,$$

which gives the result. ◄

3. Fuzzy soft congruence relations over semigroups

In this section we want to define fuzzy soft congruence relation over a semigroup S. For the definition of fuzzy congruence relations on semigroups one may see [17].

Definition 10. *A soft fuzzy relation (σ,A) over a semigroup S is called a fuzzy soft left [right] compatible relation if σ_a is a fuzzy left [right] compatible relation over S for all $a \in A$. That is, $\sigma_a(x,y) \leq \sigma_a(tx,ty)$ $[\sigma_a(x,y) \leq \sigma_a(xt,yt)]$ for all $a \in A, t, x, y \in S$.*
Moreover, (σ,A) is called fuzzy soft compatible if $\sigma_a(x,y) \wedge \sigma_a(s,t) \leq \sigma_a(xs,yt)$ for all $a \in A$, $x,y,s,t \in S$.

Definition 11. *A fuzzy soft equivalence relation (σ, A) over a semigroup S is called left (right) fuzzy soft congruence relation if σ_a is fuzzy left (right) congruence relation over S for all $a \in A$. We denote the set of all fuzzy soft congruences on a semigroup S by $FSC(S)$.*

Lemma 1. *A fuzzy soft equivalence relation (σ, A) over a semigroup S is fuzzy soft congruence relation if and only if (σ, A) is fuzzy soft left and right compatible.*

Proof. If $(\sigma, A) \in FSC(S)$, then for all $a \in A, t, x, y \in S$, $\sigma_a(x,y) = \sigma_a(x,y) \wedge \sigma_a(t,t) \leq \sigma_a(xt, yt)$. Also $\sigma_a(x,y) = \sigma_a(t,t) \wedge \sigma_a(x,y) \leq \sigma_a(tx, ty)$. Hence, (σ, A) is left and right compatible.

Conversely, if (σ, A) is left and right compatible, then for all $x, y, s, t \in S$, $a \in A$ we have

$$\begin{aligned} \sigma_a(x,y) \wedge \sigma_a(s,t) &= \sigma_a(x,y) \wedge \sigma_a(s,s) \wedge \sigma_a(y,y) \wedge \sigma_a(s,t)) \\ &\leq \sigma_a(xs, ys) \wedge \sigma_a(ys, yt) \\ &\leq \sigma_a(xs, yt). \end{aligned}$$

This completes the proof. ◄

Let (σ, A) be a fuzzy soft relation on a semigroup S and $\{(\sigma_i, A_i) : (\sigma, A) \leq (\sigma_i, A_i)\}$ be the family of all fuzzy soft congruences on S containing (σ, A). Then the fuzzy soft relation defined by $\widehat{(\sigma, A)} = \bigwedge_{i \in I, (\sigma, A) \leq (\sigma_i, A_i)} (\sigma_i, A_i)$ is clearly the smallest fuzzy soft congruence on S containing (σ, A). $\widehat{(\sigma, A)}$ is called the fuzzy soft congruence relation on S generated by (σ, A).

Definition 12. *Let $S^1 = S \cup \{1\}$, and (σ, A) be a fuzzy soft relation on a semigroup S. Then we define the fuzzy soft relation $(\sigma, A)^* = \{\sigma_a^*(c,d) : for\ all\ a \in A\}$ on S as $\sigma_a^*(c,d) = \bigvee\{\sigma_a(e,f)\}$, where there exist $x, y \in S^1$ such that $xey = c,\ xfy = d$, for all $c, d, e, f \in S$.*

Proposition 4. *Let (σ, A) and (ρ, A) be fuzzy soft relations on a semigroup S. Then the following assertions hold:*
(1) $(\sigma, A) \leq (\sigma, A)^*$.
(2) $((\sigma, A)^*)^{-1} = ((\sigma, A)^{-1})^*$.
(3) $(\sigma, A) \leq (\rho, A) \Rightarrow (\sigma, A)^* \leq (\rho, A)^*$.
(4) $((\sigma, A)^*)^* = (\sigma, A)^*$.
(5) $((\sigma, A) \vee (\rho, A))^* = (\sigma, A)^* \vee (\rho, A)^*$.
(6) $(\sigma, A) = (\sigma, A)^*$ *if and only if* (σ, A) *is left and right fuzzy soft compatible.*

Proof. The proof is similar to that of Proposition 3. ◄

Proposition 5. *Let (σ, A) be a fuzzy soft relation on a semigroup S. If it is left and right fuzzy soft compatible, so is $(\sigma, A)^\infty$.*

Proof. Let $a, b, c \in S$, $e \in A$ and $n \geq 1$,

$$\begin{aligned}\sigma_e^n(a,b) &= \bigvee\nolimits_{z_1,z_2,\dots,z_{n-1}}[\sigma_e^n(a,z_1)\wedge\sigma_e^n(z_2,z_3)\wedge\dots\wedge\sigma_e^n(z_{n-1},b)]\\ &\leq \bigvee\nolimits_{z_1c,z_2c,\dots,z_{n-1}c}[\sigma_e^n(ac,z_1c)\wedge\sigma_e^n(z_2c,z_3c)\wedge\dots\wedge\sigma_e^n(z_{n-1}c,bc)]\\ &\leq \sigma_e^n(ac,bc).\end{aligned}$$

Similarly, we can show that $(\sigma, A)^n$ is left fuzzy soft compatible. Therefore, $(\sigma, A)^\infty$ is fuzzy soft compatible. ◀

Proposition 6. *For a fuzzy soft relation (σ, A) on a semigroup S, we have $\widehat{(\sigma, A)} = ((\sigma, A)^*)^e$.*

Proof. We know that $((\sigma, A)^*)^e$ is a fuzzy soft equivalence relation containing $(\sigma, A)^*$, so it contains (σ, A). On the other hand, $[(\sigma, A)^* \vee ((\sigma, A)^*)^{-1} \vee (\triangle, A)] = [(\sigma, A) \vee (\sigma, A)^{-1} \vee (\triangle, A)]^*$. Hence, $[(\sigma, A)^* \vee ((\sigma, A)^*)^{-1} \vee (\triangle, A)]$ is fuzzy soft left and right compatible. So $[(\sigma, A)^* \vee ((\sigma, A)^*)^{-1} \vee (\triangle, A)]^\infty$ is fuzzy soft left and right compatible and hence $((\sigma, A)^*)^e$ is fuzzy soft congruence on S. Now if $(\rho, A) \in FSC(S)$, such that $(\sigma, A) \leq (\rho, A)$, then $(\sigma, A)^* \leq (\rho, A)^* = (\rho, A)$. Therefore, $((\sigma, A)^*)^e \leq (\rho, A)$ and so $\widehat{(\sigma, A)} = ((\sigma, A)^*)^e$. ◀

Definition 13. *Let (σ, A) and (ρ, B) be two fuzzy soft equivalence relations on a semigroup S. We define*

$$(\sigma, A) + (\rho, B) = ((\sigma, A) \vee (\rho, B))^e, (\sigma, A) \cdot (\rho, B) = (\sigma, A) \wedge (\rho, B).$$

Similarly, if (σ, A) and $(\rho, B) \in FSC(S)$, we define the addition and multiplication on $\xi_{FSC}(S)$ as follows

$$(\sigma, A) + (\rho, B) = (\widehat{(\sigma, A) \vee (\rho, B)}), (\sigma, A) \cdot (\rho, B) = (\sigma, A) \wedge (\rho, B).$$

Remark 1. *Under the above notations, $< \xi_{fE_r}(S), +, \cdot >$ and $< FSC(S), +, \cdot >$ are lattices.*

Proposition 7. *Let (σ, A) and (ρ, B) be in $\xi_{fE_r}(S)(FSC(S))$ for a semigroup S. Then*

$$(\sigma, A) + (\rho, B) = ((\sigma, A) \circ (\rho, B))^\infty.$$

Proof. First we have

$$(\sigma, A) + (\rho, B) = ((\sigma, A) \vee (\rho, B))^e$$
$$= [(\sigma, A) \vee (\rho, B) \vee ((\sigma, A) \vee (\rho, B))^{-1} \vee (\triangle, A \times B)]^{\infty}.$$

By the assumption, (σ, A) and (ρ, B) are fuzzy soft equivalence relations. Thus

$$[(\sigma, A) \vee (\rho, B) \vee ((\sigma, A) \vee (\rho, B))^{-1} \vee (\triangle, A \times B)] = (\sigma, A) \vee (\rho, B).$$

On the other hand, (σ, A), $(\rho, B) \leq (\sigma, A) \vee (\rho, B)$ and (σ, A), $(\rho, B) \leq (\sigma, A) \circ (\rho, B)$. Now by using Proposition 3, we have $((\sigma, A) \circ (\rho, B))^{\infty} \leq ((\sigma, A) \vee (\rho, B))^{\infty}$ and $((\sigma, A) \vee (\rho, B))^{\infty} \leq ((\sigma, A) \circ (\rho, B))^{\infty}$. Therefore, we get

$$(\sigma, A) + (\rho, B) = ((\sigma, A) \vee (\rho, B))^{\infty} = ((\sigma, A) \circ (\rho, B))^{\infty}.$$

which gives the result. ◄

Proposition 8. *Let (σ, A) and (ρ, B) be in $\xi_{fE_r}(S)(FSC(S))$ for a semigroup S such that $(\sigma, A) \circ (\rho, B) = (\rho, B) \circ (\sigma, A)$. Then*

$$(\sigma, A) + (\rho, B) = (\sigma, A) \circ (\rho, B).$$

Proof. By using 3 and 7, we prove the assertion. ◄

4. Isomorphism considerations

In this section we first give the definition of fuzzy kernel as a fuzzy soft congruence. Then we prove the homomorphism theorem for soft semigroups based on fuzzy soft congruence relations. The reader is referred to [8] for more details about soft semigroups and soft homomorphisms.

Definition 14. *Let $a \in A$, (σ, A) be a fuzzy soft congruence relation on a semigroup S. Then we define*

$$S/\sigma_a = \{(\sigma_a)_s : s \in S\},$$

where for all $t \in S$, $(\sigma_a)_s(t) = \sigma_a(s, t)$.

Definition 15. *Let (f, g) be a soft homomorphism from soft semigroup (F, A) to (G, B) which are two soft semigroups over semigroups S and T, respectively. We define $Fker(f, g)$ as a fuzzy soft relation (σ, A) over S such that for all $a \in A$ and $(\alpha, \beta) \in F(a) \times F(a)$:*

$$\sigma_a(\alpha,\beta)=\begin{cases} 1, & f(\alpha)=f(\beta), \\ 0, & f(\alpha)\neq f(\beta). \end{cases}$$

Proposition 9. *Let (σ, A) be a fuzzy soft congruence relation on a semigroup S. Then S/σ_a, $a \in A$, is a semigroup under the binary operation $\odot$ given by*

$$(\sigma_a)_s \odot (\sigma_a)_t = (\sigma_a)_{st}.$$

Proof. First we show that the operation is well-defined.

Let $(\sigma_a)_s = (\sigma_a)_{s^*}$ and $(\sigma_a)_t = (\sigma_a)_{t^*}$ for all $a \in A, s, s^*, t, t^* \in S$. We know that σ_a is a fuzzy equivalence, so $(\sigma_a)(s,s^*) = (\sigma_a)_s(s^*) = (\sigma_a)_{s^*}(s^*) = (\sigma_a)(s^*,s^*) = 1$. Similarly, $(\sigma_a)(t,t^*) = 1$. On the other hand,

$$\begin{aligned} \sigma_a(st, s^*t^*) &\geq \sigma_a \circ \sigma_a(st, s^*t^*) \\ &\geq \sigma_a(st, s^*t) \wedge (s^*t, s^*t^*) \\ &\geq \sigma_a(s, s^*) \wedge (t, t^*) \\ &= 1 \wedge 1 \\ &= 1. \end{aligned}$$

So $(\sigma_a)(st, s^*t^*) = 1$. It follows from $(\sigma_a)(s^*t^*, st) = 1$ and the symmetry property that $(\sigma_a)_{st} \geq (\sigma_a)_{s^*t^*}$, $(\sigma_a)_{st} \leq (\sigma_a)_{s^*t^*}$. Consequently,

$$(\sigma_a)_s \odot (\sigma_a)_s = (\sigma_a)_{st} = (\sigma_a)_{s^*t^*} = (\sigma_a)_{s^*} \odot (\sigma_a)_{t^*}.$$

It is easy to see that $[(\sigma_a)_s \odot (\sigma_a)_t] \odot (\sigma_a)_l = (\sigma_a)_s \odot [(\sigma_a)_t \odot (\sigma_a)_l]$, for all $s, t, l \in S$. Therefore, S/σ_a is a semigroup for all $a \in A$. ◄

The above proposition yields that S is homomorphic to S/σ_a for each $a \in A$, because we may define $f_a : S \to S/\sigma_a$ as $f_a(s) = (\sigma_a)_s$.

Let (F, A) be a soft semigroup over a semigroup S and (σ, A) be a soft relation over S such that for all $a \in A$, σ_a is a fuzzy congruence relation on $F(a)$. Then we say that (σ, A) is a fuzzy soft congruence relation on the soft semigroup (F, A).

Proposition 10. *Let (F, A) and (G, B) be soft semigroups over S and T, respectively, and let (f, g) be a soft homomorphism from (F, A) to (G, B). Then $Fker(f, g)$ is a fuzzy soft congruence relation on (F, A).*

Proof. First note that $f : S \to T$ is a homomorphism and $g : A \to B$ is a surjective map such that $f(F(a)) = g(G(a))$ for all $a \in A$. Since the fuzzy kernel of f is a fuzzy congruence on S, $Fker(f, g)$ is a fuzzy soft congruence relation on (F, A).

It is easy to see that $Fker(f)$ is an equivalence. Note that $ker(f) = \{(s,t) : f(s) = f(t)\}$ for $s, t \in S$ is a congruence on S. So if $(s,t) \in ker(f)$, then (ss^*, ts^*)

and (s^*s, s^*t) belong to $ker(f)$. Therefore, $Fkerf(s,t) = 1 = F(kerf(ss^*, ts^*))$. On the other hand, if $(s,t) \notin ker(f)$, $Fkerf(s,t) = o \leq F(kerf(ss^*, ts^*))$ and $Fkerf(s,t) = o \leq F(kerf(s^*s, s^*t))$, for all $s^* \in S$. Thus $Fker(f)$ is a fuzzy congruence on S. ◄

Theorem 2. *Let (F,A) , (G,B) be two soft semigroups over S, T, respectively. Let $(f,g) : (F,A) \to (G,B)$ be a soft homomorphism such that $g : A \to B$ is an injective map. Then there exists a soft monomorphism $(h,g) : (F,A)/Fker(f) \to (G,B)$ such that $h(F(a)/Fker(f)) = G(g(a))$ for all $a \in A$.*

Proof. First let $h : S/Fker(f) \to T$ be a map defined by $h((Fker(f))_s) = f(s)$ for all $s \in S$. We show that h is well-defined. Assume that for any $a,b \in S$, $(Fker(f))_a = (Fker(f))_b$. So $Fker(f)(a,b) = 1$ and then $(a,b) \in ker(f)$. It follows that

$$h((Fker(f))_a) = f(a) = f(b) = (Fker(f))_b.$$

Now h is one to one because if $f(a) = f(b)$, then $(a,b) \in ker(f)$ and so $Fker(f)(a,b)$ $= 1$. Hence, $(Fker(f))_a = (Fker(f))_b$. We prove that h is a homomorphism. Let $a,b \in S$. Then

$$\begin{aligned} h((Fker(f)_a) \odot (Fker(f)_b)) &= h(Fker(f)_{ab}) \\ &= f(ab) = f(a)f(b) \\ &= h(Fker(f)_a) \odot h(Fker(f)_b). \end{aligned}$$

Denote the soft set $(F,A)/Fker(f)$ by (K,A), where $K(a) = F(a)/Fker(f)$ for all $a \in A$. Then we get

$$\begin{aligned} h(K(a)) &= h(F(a)/Fker(f)) \\ &= f(F(a) \\ &= G(g(a)). \end{aligned}$$

So (h,g) is a soft monomorphism from $(F,A)/Fker(f)$ to (G,B). ◄

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