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Approximation by Szász-Stancu-Durrmeyer type Operators Using Charlier Polynomials

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Abstract. In the present article, we introduce Szász-Stancu-Durremeyer type operators using Charlier polynomials. We discuss uniform convergence in compact interval in terms of Korovkin type theorem and order of approximation using simple modulus of continuity. Moreover, we study order of approximation in some functional spaces with the help of Peetre's K-functional, second order modulus of smoothness and Lipschitz class for these sequences of positive linear operators.

Key Words and Phrases: Szász operators, positive linear operators, modulus of continuity, Peetre's K-functional.

2010 Mathematics Subject Classifications: 41A10, 41A25, 41A36

1. Introduction

In 1950, Szász [1] gave an important generalization of Bernstein operators defined on non-negative semi axes which is known as classical Szász operators. These sequences of positive linear operators play a central role in the development of operator theory. Various extensions have been studied for Szász operators (see [2-5, 8] and references therein). Integral modification of Szász operators have been given by several mathematicians. One of them was introduced by Mazhar and Totik [9] which is known as Szász-Durrmeyer operators

$$S_n^*(f;x) = ne^{-nx}\sum_{k=0}^{\infty}\frac{(nx)^k}{k!}\int_0^{\infty} e^{-nt}\frac{(nt)^k}{k!}f(t). \tag{1}$$

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Varma and Taşdelen [10] defined Szász-type operators involving Charlier polynomials [11]. Recently, Wafi and Rao ([6, 7]) modified these sequences of positive linear operators and gave better approximation results in different functional spaces. Kajla and Agarwal [12] presented a Durrmeyer modification of Szász-type operators using Charlier polynomials as follows:

$$\begin{aligned} S_{n,a}(f;x) &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty} C_k^{(a)}(-(a-1)nx)\frac{1}{B(k+1,n)} \\ &\times \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} f(t)dt, \ n \in \mathbb{N}, \end{aligned} \tag{2}$$

where $f \in C_\gamma[0,\infty) := \{f \in C[0,\infty) : f(t) = O(t^\gamma) \text{ as } t \to \infty\}, n > \gamma, a > 1$ and $B(k+1,n)$ is the beta function defined by $B(x,y) = \int_0^{\infty} \frac{t^{x-1}}{(1+t)^{x+y}}dt = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$, $x, y > 0$. In view of the above, we introduce Szász-Stancu-Durrmeyer type operators $(\alpha \geq \beta \geq 0, n \in \mathbb{N})$

$$\begin{aligned} S_{n,a}^{\alpha,\beta}(f;x) &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty} C_k^{(a)}(-(a-1)nx)\frac{1}{B(k+1,n)} \\ &\times \int_0^{\infty} \frac{t^k}{(1+t)^{n+k+1}} f\left(\frac{nt+\alpha}{n+\beta}\right)dt. \end{aligned} \tag{3}$$

We notice that for $\alpha = \beta = 0$, operators (3) reduce to the operators (2). Several authors have discussed flexility in approximation results by means of these two shifted nodes α and β ([13, 14]). In this article, we study the uniform convergence, order of approximation, direct and local approximation using different functional spaces.

2. Basic lemmas

Lemma 1. *From [10], we have*

$$\begin{aligned} \sum_{k=0}^{\infty} \frac{C_k^{(u)}(-(a-1)nx)}{k!} &= e\left(1-\frac{1}{a}\right)^{-(a-1)nx}, \\ \sum_{k=0}^{\infty} k\frac{C_k^{(u)}(-(a-1)nx)}{k!} &= e\left(1-\frac{1}{a}\right)^{-(a-1)nx}(1+nx), \end{aligned}$$

$$\sum_{k=0}^{\infty} k^2 \frac{C_k^{(u)}(-(a-1)nx)}{k!} = e\left(1-\frac{1}{a}\right)^{-(a-1)nx}\left(2+\left(3+\frac{1}{a-1}\right)nx+n^2x^2\right).$$

Lemma 2. *From [12], we have*

$$\begin{aligned} S_{n,a}(1;x) &= 1, \\ S_{n,a}(t;x) &= \left(\frac{nx+2}{n-1}\right), n>1, \\ S_{n,a}(t^2;x) &= \frac{1}{(n-1)(n-2)}\left(n^2x^2+\left(6+\frac{1}{a-1}\right)nx+7\right), n>2. \end{aligned}$$

Lemma 3. *For the operators $S_{n,a}^{\alpha,\beta}$ defined by (3), we have*

$$S_{n.a}^{\alpha,\beta}(t^m;x) = \frac{n^m}{(n+\beta)^m}\sum_{k=0}^{m}\binom{m}{i}\left(\frac{\alpha}{m}\right)^{m-i} S_{n,a}(t^i;x).$$

Proof. From the operators (3), we have

$$\begin{aligned} S_{n.a}^{\alpha,\beta}(t^m;x) &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{1}{B(k+1,n)} \\ &\times \int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}\left(\frac{nt+\alpha}{n+\beta}\right)^m dt \\ &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}} \\ &\times \frac{n^m}{(n+\beta)^m}\left(\frac{nt+\alpha}{n}\right)^m dt \\ &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}} \\ &\times \frac{n^m}{(n+\beta)^m}\left(t+\frac{\alpha}{n}\right)^m dt \\ &= e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{1}{B(k+1,n)} \end{aligned}$$

$$\times \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} \frac{n^m}{(n+\beta)^m} \sum_{i=0}^{m} \binom{m}{i} t^i \left(\frac{\alpha}{n}\right)^{m-i} dt$$

$$= \frac{n^m}{(n+\beta)^m} \sum_{i=0}^{m} \binom{m}{i} \left(\frac{\alpha}{n}\right)^{m-i} e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!}$$

$$\times \frac{1}{B(k+1,n)} \int_0^\infty \frac{t^{k+i}}{(1+t)^{n+k+1}} dt$$

$$= \frac{n^m}{(n+\beta)^m} \sum_{i=0}^{m} \binom{m}{i} \left(\frac{\alpha}{n}\right)^{m-i} S_{n,a}(t^i;x).$$

◀

Lemma 4. *For the test functions $e_i(x) = t^i$, $i \in \{0,1,2\}$, we have*

$$S_{n,a}^{\alpha,\beta}(1;x) = 1,$$

$$S_{n,a}^{\alpha,\beta}(t;x) = \frac{\alpha}{n+\beta} + \frac{n}{n+\beta}\left(\frac{nx+2}{n-1}\right), n > 1,$$

$$S_{n,a}^{\alpha,\beta}(t^2;x) =$$

$$= \frac{1}{(n+\beta)^2(n-1)(n-2)}\left(n^4x^2 + (2n^2(n-2)\alpha + n^3\left(6+\frac{1}{a-1}\right))\right)x$$

$$+(n-1)(n-2)\alpha^2 + 4n(n-2)\alpha + 7n^2\Bigg), n > 2.$$

Proof. Using Lemma 2 and Lemma 3, we have

$$S_{n.a}^{\alpha,\beta}(t^m;x) = \frac{n^m}{(n+\beta)^m} \sum_{k=0}^{m} \binom{m}{i} \left(\frac{\alpha}{m}\right)^{m-i} S_{n,a}(t^i;x).$$

For $m = 0$,

$$S_{n.a}^{\alpha,\beta}(1;x) \quad = \quad S_{n,a}(1;x) = 1.$$

For $m = 1$,

$$\begin{aligned} S_{n.a}^{\alpha,\beta}(t;x) &= \frac{n}{(n+\beta)}\sum_{k=0}^{1}\binom{1}{i}\left(\frac{\alpha}{n}\right)^{1-i} S_{n,a}(t^i;x) \\ &= \frac{n}{n+\beta}\left[\frac{\alpha}{n}S_{n.a}(1;x)+S_{n.a}(t;x)\right] \\ &= \frac{n}{n+\beta}\left[\frac{\alpha}{n}+\left(\frac{nx+2}{n-1}\right)\right] \\ &= \frac{\alpha}{n+\beta}+\frac{n}{n+\beta}\left(\frac{nx+2}{n-1}\right). \end{aligned}$$

Similarly, we can prove it for $m=2$ also. ◀

Lemma 5. *Let $\psi_x^i(t)=(t-x)^i, i=0,1,2$. For the operators (3), we have*

$$S_{n,a}^{\alpha,\beta}(\psi_x^0;x)=1,$$

$$S_{n,a}^{\alpha,\beta}(\psi_x^1;x)=\frac{n(1-\beta)+\beta}{(n+\beta)(n-1)}x+\frac{(2+\alpha)n-\alpha}{(n+\beta)(n-1)}$$

$$S_{n,a}^{\alpha,\beta}(\psi_x^2;x)=$$

$$=\frac{1}{(n+\beta)^2(n-2)(n-1)}\Bigg(n^3+(n^2(2-2\beta+\beta^2)+n(4\beta-3\beta^2)+2\beta^2)x^2\Bigg).$$

Proof. Using Lemma 4 and linearity property, Lemma 5 can easily be proved. ◀

3. Rate of convergence of the operators $S_{n,a}^{\alpha,\beta}$

For $f\in C_B[0,\infty)$, where $C_B[0,\infty)$ is the set of all continuous and bounded functions on $[0,\infty)$, the modulus of continuity $\omega(f;\delta)$ is defined as

$$\omega(f;\delta):=\sup_{|t-y|\le\delta}|f(t)-f(y)|, \qquad t,y\in[0,\infty)$$

and

$$|f(t)-f(y)|\le\left(1+\frac{(t-y)^2}{\delta^2}\right)\omega(f;\delta). \tag{4}$$

Theorem 1. *Let $f \in C[0,\infty) \bigcap E$. Then $S_{n,a}^{\alpha,\beta}(f;x) \to f$ uniformly on each compact subset of $[0,\infty)$, where $C[0,\infty)$ is the space of continuous functions and* $E := \left\{ f : x \geq 0, \frac{f(x)}{1+x^2} \text{ is convergent as } x \to \infty \right\}$.

Proof. We have from Lemma 4, $S_{n,a}^{\alpha,\beta}(1;x) \to 1$, $S_{n,a}^{\alpha,\beta}(t;x) \to x$ and $S_{n,a}^{\alpha,\beta}(t^2;x) \to x^2$ as $n \to \infty$. Using universal Korovkin-type property (vi) of Theorem 4.1.4 in [15], we arrive at the desired result. ◄

Theorem 2. *(See [16]) Let $L : C([a,b]) \to B([a,b])$ be a linear and positive operator and let φ_x be the function defined by*

$$\varphi(x)(t) = |t - x|, \ (x,t) \in [a,b] \times [a,b].$$

If $f \in C_B([a,b])$ for any $x \in [a,b]$ and any $\delta > 0$, then the operator L satisfies the relation

$$|(Lf)(x) - f(x)| \leq |f(x)||(Le_0)(x) - L|\{(Le_0)(x) + \delta^{-1}\sqrt{(Le_0)(x)(L\varphi_x^2)(x)}\}\omega_f(\delta).$$

Theorem 3. *For $f \in C_B[0,\infty)$, the relation*

$$|S_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq 2\omega(f;\delta),$$

holds uniformly, where $\delta = \sqrt{S_{n,a}^{\alpha,\beta}(\psi_x^2(t);x)}$.

Proof. From Lemma 4, Lemma 5 and Theorem 2, we have

$$|S_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq \{1 + \delta^{-1}\sqrt{S_{n,a}^{\alpha,\beta}(\varphi(x)^2;x)}\}\omega(f;\delta),$$

which proves Theorem 3. ◄

4. Local approximation results

In this section, we deal with the order of approximation locally in $C_B[0,\infty)$ (space of real valued continuous and bounded functions f on $[0,\infty)$) with the norm $\|f\| = \sup\limits_{0 \leq x < \infty} |f(x)|$. Let, for any $f \in C_B[0,\infty)$ and $\delta > 0$, Peetre's K-functional be defined as

$$K_2(f,\delta) = inf\{\|f - g\| + \delta\|g''\| : g \in C_B^2[0,\infty)\},$$

where $C_B^2[0,\infty)=\{g\in C_B[0,\infty):g',g''\in C_B[0,\infty)\}$. By DeVore and Lorentz [17, p. 177, Theorem 2.4], there exists an absolute constant $C>0$ such that

$$K_2(f;\delta)\leq C\omega_2(f;\sqrt{\delta}),$$

where $\omega_2(f;\delta)$ is the second order modulus of continuity defined as

$$\omega_2(f;\sqrt{\delta})=\sup_{0<h\leq\sqrt{\delta}}\ \sup_{x\in[0,\infty)}|f(x+2h)-2f(x+h)+f(x)|.$$

Theorem 4. *Let $f\in C_B^2[0,\infty)$. Then, there exists a constant $C>0$ such that*

$$\mid S_{n,a}^{\alpha,\beta}(f;x)-f(x)\mid\leq C\omega_2\big(f;\sqrt{\xi_{n,a}(x)}\big)+\omega(f;S_{n,a}^{\alpha,\beta}(\psi_x;x)),$$

where $\xi_{n,a}^{\alpha,\beta}(x)=S_{n,a}^{\alpha,\beta}(\psi_x^2;x)+\big(S_{n,a}^{\alpha,\beta}(\psi_x;x)\big)^2$.

Proof. Consider the auxiliary operators defined as follows:

$$\widehat{S}_{n,a}^{\alpha,\beta}(f;x)=S_{n,a}^{\alpha,\beta}(f;x)+f(x)-f(\eta_{n,a}(x)),$$

where $\eta_{n,a}(x)=S_{n,a}(\psi_x;x)+x$. Using Lemma 5, we have

$$\begin{aligned}\widehat{S}_{n,a}^{\alpha,\beta}(1;x)&=1,\\ \widehat{S}_{n,a}^{\alpha,\beta}(\psi_x(t);x)&=0,\\ |\widehat{S}_{n,a}^{\alpha,\beta}(f;x)|&\leq 3\|f\|.\end{aligned}\tag{5}$$

For any $g\in C_B^2[0,\infty)$, by the Taylor's theorem, we get

$$g(t)=g(x)+(t-x)g'(x)+\int_x^t(t-v)g''(v)dv,\tag{6}$$

$$\begin{aligned}\widehat{S}_{n,a}^{\alpha,\beta}(g;x)-g(x)&=g'(x)\widehat{S}_{n,a}^{\alpha,\beta}(t-x;x)+\widehat{S}_{n,a}^{\alpha,\beta}\Big(\int_x^t(t-v)g''(v)dv;x\Big)\\ &=\widehat{S}_{n,a}^{\alpha,\beta}\Big(\int_x^t(t-v)g''(v)dv;x\Big)\\ &=S_{n,a}^{\alpha,\beta}\Big(\int_x^t(t-v)g''(v)dv;x\Big)-\int_x^{\eta_{n,a}(x)}(\eta_{n,a}(x)-v)g''(v)dv,\end{aligned}$$

$$|\widehat{S}_{n,a}^{\alpha,\beta}(g;x)-g(x)|\leq$$

$$\leq\left|S_{n,a}^{\alpha,\beta}\Big(\int\limits_x^t(t-v)g''(v)dv;x\Big)\right|+\left|\int\limits_x^{\eta_{n,a}(x)}(\eta_{n,a}(x)-v)g''(v)dv\right|. \tag{7}$$

Since

$$\left|\int\limits_x^t(t-v)g''(v)dv\right|\leq(t-x)^2\parallel g''\parallel, \tag{8}$$

we have

$$\left|\int\limits_x^{\eta_{n,a}(x)}(\eta_{n,a}(x)-v)g''(v)dv\right|\leq(\eta_{n,a}(x)-x)^2\parallel g''\parallel, \tag{9}$$

$$\begin{aligned}|\widehat{S}_{n,a}^{\alpha,\beta}(g;x)-g(x)| &\leq \left\{S_{n,a}^{\alpha,\beta}((t-x)^2;x)+(\eta_{n,a}(x)-x)^2\right\}\|g''\|\\ &= \gamma_{n,a}(x)\|g''\|. \end{aligned}\tag{10}$$

Now, we have

$$|S_{n,a}^{\alpha,\beta}(f;x)-f(x)|\leq|\widehat{S}_{n,a}^{\alpha,\beta}(f-g;x)|+|(f-g)(x)|+$$

$$+|\widehat{S}_{n,a}^{\alpha,\beta}(g;x)-g(x)|+\big|f(\eta_{n,a}(x))-f(x)\big|,$$

$$\begin{aligned}|S_{n,a}^{\alpha,\beta}(f;x)-f(x)| &\leq 4\|f-g\|+|\widehat{S}_{n,a}^{\alpha,\beta}(g;x)-g(x)|+\big|f(\eta_{n,a}(x))-f(x)\big|\\ &\leq 4\|f-g\|+\gamma_{n,a}(x)\|g''\|+\omega\Big(f;S_{n,a}^{\alpha,\beta}(\psi_x;x)\Big),\end{aligned}$$

$$|S_{n,a}^{\alpha,\beta}(f;x)-f(x)|\leq C\omega_2\big(f;\sqrt{\gamma_{n,a}(x)}\big)+\omega(f;S_{n,a}^{\alpha,\beta}(\psi_x;x)).$$

◀

Now, we shall discuss the rate of convergence of the operators defined by (3) in terms of the functions which belong to Lipschitz class

$$Lip_M^*(\gamma)=\{f\in C[0,\infty):|f(t)-f(x)|\leq M\frac{|t-x|^\gamma}{(t+x)^{\frac{\gamma}{2}}}:x,t\in(0,\infty)\}, \tag{11}$$

where M is a constant and $0<\gamma\leq 1$.

Theorem 5. *For $x \geq 0$ and $f \in Lip_M^*(\gamma)$, we have*

$$|S_{n,a}^{\alpha,\beta}(f;x) - f(x)| \leq M \left[\frac{\Theta_{n,a}(x)}{x} \right]^{\frac{\gamma}{2}},$$

where $\Theta_{n,a}(x) = S_{n,a}^{\gamma,\beta}((t-x)^2;x)$.

Proof. Let $\gamma = 1$ and $x \in (0,\infty)$. Then, for $f \in Lip_M^*(1)$, we have

$$\begin{aligned}
|S_{n,a}^{\alpha,\beta}(f;x) - f(x)| &\leq e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!} \frac{1}{B(k+1,n)} \\
&\times \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} \left(f\left(\frac{nt+\alpha}{n+\beta}\right) - f(t) \right) dt \\
&\leq Me^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!} \frac{1}{B(k+1,n)} \\
&\times \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} \frac{\left|\frac{nt+\alpha}{n+\beta} - x\right|}{\sqrt{\frac{nt+\alpha}{n+\beta} + x}} dt \\
&\leq \frac{M}{\sqrt{x}} e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx} \sum_{k=0}^{\infty} \frac{C_k^{(a)}(-(a-1)nx)}{k!} \frac{1}{B(k+1,n)} \\
&\times \int_0^\infty \frac{t^k}{(1+t)^{n+k+1}} \left|\frac{nt+\alpha}{n+\beta} - x\right| dt \\
&\leq \frac{M}{\sqrt{x}} S_{n,a}^{\alpha,\beta}(|t-x|;x) \\
&\leq M \frac{\sqrt{S_{n,a}^{\alpha,\beta}((t-x)^2;x)}}{\sqrt{x}} \\
&= M\left(\frac{\Theta_{n,a}(x)}{x}\right)^{\frac{1}{2}}.
\end{aligned}$$

Thus, the assertion holds for $\gamma = 1$. Now, let's prove it for $\gamma \in (0,1)$. From the Hölder's inequality with $p = \frac{1}{\gamma}$ and $q = \frac{1}{1-\gamma}$, we have

$$|S_{n,a}^{\alpha,\beta}(f;x) - f(x)| =$$

$$
\begin{aligned}
&= \left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}\right. \\
&\times \left.\left(\left|f\left(\frac{nt+\alpha}{n+\beta}\right)-f(x)\right|\right)^{\frac{1}{\gamma}}\right)^{\gamma}dt \\
&\times \left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}\right)^{1-\gamma} \\
&\leq \left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{1}{B(k+1,n)}\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}\right. \\
&\times \left.\left(\left|f\left(\frac{nt+\alpha}{n+\beta}\right)-f(x)\right|\right)^{\frac{1}{\gamma}}\right)^{\gamma},
\end{aligned}
$$

$$
\begin{aligned}
|S_{n,a}^{\alpha,\beta}(f;x)-f(x)| &\leq M\left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{1}{B(k+1,n)}\right. \\
&\times \left.\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}\frac{\left|\frac{nt+\alpha}{n+\beta}-x\right|}{\sqrt{\frac{nt+\alpha}{n+\beta}+x}}dt\right)^{\gamma} \\
&\leq \frac{M}{x^{\frac{\gamma}{2}}}\left(e^{-1}\left(1-\frac{1}{a}\right)^{(a-1)nx}\sum_{k=0}^{\infty}\frac{C_k^{(a)}(-(a-1)nx)}{k!}\frac{1}{B(k+1,n)}\right. \\
&\times \left.\int_0^{\infty}\frac{t^k}{(1+t)^{n+k+1}}\left|\frac{nt+\alpha}{n+\beta}-x\right|dt\right)^{\gamma} \\
&= \frac{M}{x^{\frac{\gamma}{2}}}\left(S_{n,a}^{\alpha,\beta}(f;x)(|t-x|;x)\right)^{\gamma} \\
&\leq M\left(\frac{\Theta_{n,a}(x)}{x}\right)^{\frac{\gamma}{2}}.
\end{aligned}
$$

This completes the proof of Theorem 5. ◄

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