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The Mappings on Classes of Fuzzy Soft Bi-ideals over Semigroups

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Abstract. In this paper, we study fuzzy soft bi-ideals over semigroups and give some properties of soft bi-ideals, fuzzy soft left [right] ideals and fuzzy soft bi-ideals over semigroups together with presented and supported examples. Moreover, we show that the images of fuzzy soft bi-ideals over semigroups are the fuzzy soft bi-ideals over semigroups under some conditions.

Key Words and Phrases: soft bi-ideals, fuzzy soft left [right] ideals, fuzzy soft bi-ideal.

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1. Introduction

The real life problems in computer sciences, economics, environment, medical science, engineering and many other fields are the complex problems. These problems have been solved by the classical tools but the uncertainties still appear in these problems. While fuzzy sets [1], intuitionistic fuzzy sets [2], vague sets [3], rough sets [4], interval mathematics [5] and other mathematic tools are often useful approaches to explaining uncertainty. Accordingly, in 1999, Molodtsov [6] initiated a new mathematical tool which is soft set theory for modeling uncertainty. He pointed out directions of soft sets for the applications like soft analysis and game theory. Later, Maji et al. [7] defined soft subsets and soft super sets, equality of two soft sets. They presented soft binary operations, such as AND, OR, intersection, union and studied De Morgan's Laws of soft sets. Maji et al. [8] , discussed the application in decision making problems and described the choice value and reduced-soft-sets in decision making problems. After that the soft set theory has been developed by many researchers [9, 10, 11].

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In 2001, Maji et al. [12] extended the soft sets to fuzzy soft sets. They initiated the concept of fuzzy soft sets and defined fuzzy soft subsets, the intersection and union of fuzzy soft sets over a common universe. In 2007, Roy and Maji [13] used the Comparison table of fuzzy soft sets in the algorithm with decision making problems. In 2009, Kharal and Ahmad [14] defined the mappings of fuzzy soft sets and studied some properties of fuzzy soft images and fuzzy soft inverse images of fuzzy soft sets. Moreover, they defined the fuzzy soft intersection and union of fuzzy soft images and fuzzy soft inverse images. In 2011, Neog and Sut [15] studied the intersection and union of fuzzy soft sets and presented some properties such as commutative property, associative property, idempotent property, absorption property, distributive property. The fuzzy soft sets were developed to fuzzy soft semigroups by Yang [16] (2011). He defined fuzzy soft [left, right] ideals over semigroups and fuzzy soft semigroups, and studied sufficient and necessary conditions for α-level sets, intersection and union of fuzzy soft [left, right] ideals. In 2013, Khan et al. [17] studied fuzzy soft interior ideals of ordered semigroups. They showed that an ordered semigroup is simple if and only if it is fuzzy soft simple. Naz et al. [18] defined a product of two fuzzy soft semigroups and presented some properties of fuzzy soft interior ideals [quasi-ideals, bi-ideals, generalized bi-ideals] over semigroups under some conditions. In 2015, Siripitukdet and Suebsan [19] defined prime, semiprime and strongly prime fuzzy soft bi-ideals over semigroups and presented their properties. In 2016, Ersoy and Özkiriçci [20] introduced intuitionistic fuzzy soft semi-ideals over semirings and presented their properties such as intersection, union, AND, OR operations.

In this paper, we study some properties of fuzzy soft left [right] ideals and soft bi-ideals over semigroups with supported examples. Moreover, we present some properties such as product, intersection and union of fuzzy soft bi-ideals over semigroups under other conditions of Naz et al. [18]. Finally, we show that the images of fuzzy soft bi-ideals over semigroups are the fuzzy soft bi-ideals over semigroups under some conditions.

2. Preliminaries

In this section, we give some basic definitions and lemmas of fuzzy sets, soft sets and fuzzy soft semigroups.

Let S be a semigroup. A non-empty subset T of a semigroup S is called a *subsemigroup* if $T^2 \subseteq T$. A subsemigroup B of a semigroup S is called a *bi-ideal* [21] of S if $BSB \subseteq B$. An element a of a semigroup S is called a *regular element* if there exists an element $x \in S$ such that $a = axa$. A semigroup S is called *regular* [22] if every element of S is regular.

Let $\emptyset \neq T \subseteq S$. The *characteristic function* χ_T on T is the function from S

to the unit interval $[0,1]$ defined by

$$\chi_T(x) = \begin{cases} 1 & \text{if} \quad x \in \mathrm{T}, \\ 0 & \text{if} \quad x \notin \mathrm{T}. \end{cases}$$

The concept of fuzzy semigroups have been introduced by Kuroki [23, 24].

For any $a, b \in [0,1]$, define $a \vee b = max\{a, b\}$ and $a \wedge b = min\{a, b\}$. Then $a \vee b$ and $a \wedge b$ are elements in $[0,1]$. A function f from a non-empty set S to the unit interval $[0,1]$ is called a *fuzzy set* on S. A fuzzy set f on a semigroup S is called a *fuzzy subsemigroup* on S if $f(xy) \geq f(x) \wedge f(y)$ for all $x, y \in S$. For a fuzzy set f on S and $\alpha \in [0,1]$, the set $f^\alpha = \{x \in S | f(x) \geq \alpha\}$ is called a *level subset* [23] of f.

A fuzzy set f on a semigroup S is called a *fuzzy left [right] ideal* on S if $f(xy) \geq f(y)[f(xy) \geq f(x)]$ for all $x, y \in S$. A fuzzy set f on a semigroup S is called *a fuzzy ideal* on S if it is both a fuzzy left and a fuzzy right ideal on S. We clearly see that any fuzzy left [right] ideal on S is a fuzzy subsemigroup on S. A fuzzy subsemigroup f on a semigroup S is called a *fuzzy bi-ideal* [24] on S if $f(xyz) \geq f(x) \wedge f(z)$ for all $x, y, z \in S$.

If f and g are fuzzy sets on a semigroup S, then $f \leq g$, $f \vee g$, $f \wedge g$ (some authors [21, 24] use notations $f \subseteq g$, $f \cup g$, $f \cap g$, respectively) and $f \circ g$, respectively, are defined as follows:

$$f \leq g \text{ if } f(x) \leq g(x) \text{ for all } x \in S,$$
$$(f \vee g)(x) = f(x) \vee g(x) \text{ for all } x \in S,$$
$$(f \wedge g)(x) = f(x) \wedge g(x) \text{ for all } x \in S \text{ and}$$
$$(f \circ g)(x) = \begin{cases} \bigvee_{x=yz}[f(y) \wedge g(z)] & \text{if} \quad x \text{ is expressible as } x = yz \\ 0 & \text{otherwise}, \end{cases}$$

where $\bigvee_{x=yz}[f(y) \wedge g(z)] = sup\{f(y) \wedge g(z) | x = yz\}$.

The following lemmas are due to Mordeson et al. [21].

Lemma 1. ([21]) *Let T be a non-empty subset of a semigroup S. Then the following statements hold.*

(i) *χ_T is a fuzzy subsemigroup on S if and only if T is a subsemigroup of S.*

(ii) *χ_T is a fuzzy [left, right] ideal on S if and only if T is an [left, right] ideal of S.*

Lemma 2. ([21]) *If f is any fuzzy set on a semigroup S and g is any fuzzy bi-ideal on S, then the products $f \circ g$ and $g \circ f$ are both fuzzy bi-ideals on S.*

Now, we give some definitions and lemmas of soft sets and fuzzy soft sets over semigroups.

Definition 1. [6] *Let U be an initial universe set and E be a set of parameters. Let $P(U)$ denote the power set of U and $\emptyset \neq A \subseteq E$. A pair (F, A) is called a soft set over U, where F is a mapping given by $F : A \to P(U)$.*

Definition 2. [16] *A soft set (F, A) over a semigroup S is said to be a soft semigroup over S if $F(p)$ is a subsemigroup of S for all $p \in A$.*

Definition 3. [25] *A soft set (F, A) over a semigroup S is said to be a soft bi-ideal over S if $F(p)$ is a bi-ideal of S for all $p \in A$.*

Definition 4. [12] *Let E be a set of parameters and $\emptyset \neq A \subseteq E$. A pair (F, A) is called a fuzzy soft set over a semigroup S, where $F : A \to Fuz(S)$ is a mapping and $Fuz(S)$ is the set of all fuzzy sets on S.*

Let (F, A) be a fuzzy soft set over a semigroup S. For $p \in A, F(p) \in Fuz(S)$. Set $F_p := F(p)$. Then $F_p \in Fuz(S)$.

Definition 5. [26] *Let (F, A) be a fuzzy soft set over a semigroup S. For each $\alpha \in [0, 1]$, the set $(F, A)^\alpha$ is called an α-level set of (F, A), where*

$$(F_p)^\alpha = \{x \in S \mid F_p(x) \geq \alpha\} \text{ for all } p \in A.$$

Definition 6. [16] *A fuzzy soft set (F, A) over a semigroup S is called a fuzzy soft subsemigroup if F_p is a fuzzy subsemigroup on S for all $p \in A$.*

Definition 7. [16] *A fuzzy soft set (F, A) over a semigroup S is called a fuzzy soft left [right] ideal if F_p is a fuzzy left [right] ideal on S for all $p \in A$.*

Definition 8. [16] *A fuzzy soft set (F, A) over a semigroup S is called a fuzzy soft ideal if (F, A) is both a fuzzy soft left and a fuzzy soft right ideal on S.*

Definition 9. [18] *A fuzzy soft set (F, A) over a semigroup S is called a fuzzy soft bi-ideal over S if F_p is a fuzzy bi-ideal on S for all $p \in A$.*

Definition 10. [19] *Let T be a subset of a semigroup S and let A be a set of parameters. A fuzzy soft characteristic function (θ^T, A) over S is defined by $(\theta^T)_p = \chi_T$ for all $p \in A$.*

Definition 11. [12] *Let (F, A) and (G, B) be two fuzzy soft sets over a semigroup S. Define $(F, A) \leq (G, B)$ if (i) $A \subseteq B$ and (ii) $F_p \leq G_p$ for all $p \in A$.*

Definition 12. [12] *Let (F, A) and (G, B) be two fuzzy soft sets over a semigroup S. Define $(F, A) = (G, B)$ if and only if $(F, A) \leq (G, B)$ and $(G, B) \leq (F, A)$.*

Definition 13. [12] *Let (F, A) and (G, B) be two fuzzy soft sets over a semigroup S with $A \cap B \neq \emptyset$. Define $(F, A)\tilde{\bigwedge}(G, B) := (F \wedge G, A \cap B)$.*

Then $(F, A)\tilde{\bigwedge}(G, B)$ is a fuzzy soft set over S. For all $p \in A \cap B$, $(F \wedge G)_p = F_p \wedge G_p$.

Remark 1. *Some authors* [12, 15] *call* $(F,A)\tilde{\bigwedge}(G,B)$*, the intersection of* (F,A) *and* (G,B)*.*

Definition 14. [12] *Let* (F,A) *and* (G,B) *be two fuzzy soft sets over a semigroup* S*. Define*

$$(F,A)\tilde{\bigvee}(G,B) := (F \vee G, A \cup B).$$

Then $(F,A)\tilde{\bigvee}(G,B)$ *is a fuzzy soft set over* S*, where for all* $p \in A \cup B$*,*

$$(F \vee G)_p = \begin{cases} F_p, & if \quad p \in A - B, \\ G_p, & if \quad p \in B - A, \\ F_p \vee G_p, & if \quad p \in A \cap B. \end{cases}$$

Remark 2. *Some authors* [12, 15] *call* $(F,A)\tilde{\bigvee}(G,B)$*, the union of* (F,A) *and* (G,B)*.*

Definition 15. [18] *The product of two fuzzy soft sets* (F,A) *and* (G,B) *over a semigroup* S *is denoted by*

$$(F,A) \odot (G,B) := (F \circ G, A \cup B).$$

Then $(F,A) \odot (G,B)$ *is a fuzzy soft set over* S*, where for all* $p \in A \cup B$

$$(F \circ G)_p = \begin{cases} F_p, & if \quad p \in A - B, \\ G_p, & if \quad p \in B - A, \\ F_p \circ G_p, & if \quad p \in A \cap B. \end{cases}$$

Lemma 3. [18] *Let* $(F,A),(G,B)$ *and* (H,C) *be fuzzy soft sets over a semigroup* S*. Then*

$$((F,A) \odot (G,B)) \odot (H,C) = (F,A) \odot ((G,B) \odot (H,C)).$$

Definition 16. [14] *Let* U *be an initial universe set and let* E *be a set of parameters. A collection of all fuzzy soft sets over* U *with parameters from* E *is called a fuzzy soft class and is denoted by* $FS(U,E)$*.*

That is, $FS(U,E) = \{(F,A) | A \subseteq E \text{ and } F : A \to Fuz(U)\}$*.*

Remark 3. *If* S *is a semigroup and* E *is a set of parameters, then the fuzzy soft class* $FS(S,E)$ *is a semigroup under the operation* $\odot$ *by Definition 15 and Lemma 3.*

3. The properties of fuzzy soft bi-ideals over semigroups

In this section, we present some properties of soft bi-ideals, fuzzy soft left [right] ideals and fuzzy soft bi-ideals over semigroups.

Proposition 1. *Let T be a non-empty subset of a semigroup S. Then the following statements hold.*

(i) (θ^T, A) is a fuzzy soft subsemigroup over S if and only if T is a subsemigroup of S.

(ii) (θ^T, A) is a fuzzy soft [left, right] ideal over S if and only if T is an [left, right] ideal of S.

Proof. It follows from Lemma 1.◄

In the following theorem, we show that a fuzzy soft set over a semigroup is a fuzzy soft bi-ideal over a semigroup if and only if an α-level set of fuzzy soft set over the semigroup is a soft bi-ideal over the semigroup for all $\alpha \in [0, 1]$.

Theorem 1. *Let (F, A) be a fuzzy soft set over a semigroup S. Then (F, A) is a fuzzy soft bi-ideal over S if and only if $(F, A)^\alpha$ is a soft bi-ideal over S for all $\alpha \in [0, 1]$.*

Proof. Assume that (F, A) is a fuzzy soft bi-ideal over S. Let $\alpha \in [0, 1]$, $p \in A, y, z \in (F_p)^\alpha$ and $x \in S$. Then $F_p(y) \geq \alpha$ and $F_p(z) \geq \alpha$. By the assumption, F_p is a fuzzy bi-ideal on S. We then have $F_p(y \cdot z) \geq F_p(y) \wedge F_p(z) \geq \alpha$. It follows that $y \cdot z \in (F_p)^\alpha$. Thus $(F_p)^\alpha$ is a subsemigroup of S. Also, we have $F_p(y \cdot x \cdot z) \geq F_p(y) \wedge F_p(z) \geq \alpha$.
This implies that $y \cdot x \cdot z \in (F_p)^\alpha$. Hence $(F_p)^\alpha$ is a bi-ideal of S. Therefore $(F, A)^\alpha$ is a soft bi-ideal over S.

Conversely, suppose that $(F, A)^\alpha$ is a soft bi-ideal over S for each $\alpha \in [0, 1]$. Let $p \in A$ and $y, z \in S$. Choose $\alpha := F_p(y) \wedge F_p(z)$. Then $y, z \in (F_p)^\alpha$. Since $(F_p)^\alpha$ is a subsemigroup of S, we then have $y \cdot z \in (F_p)^\alpha$.
This implies that $F_p(y \cdot z) \geq \alpha = F_p(y) \wedge F_p(z)$.
Thus F_p is a fuzzy subsemigroup on S.
By the assumption, $y \cdot x \cdot z \in (F_p)^\alpha$ for all $x \in S$. Thus

$$F_p(y \cdot x \cdot z) \geq \alpha = F_p(y) \wedge F_p(z).$$

Hence F_p is a fuzzy bi-ideal on S. Therefore (F, A) is a fuzzy soft bi-ideal over S.◄

The next theorem shows that $(F, A)\tilde{\bigvee}(G, B)$ is a fuzzy soft bi-ideal over a semigroup under some conditions.

Theorem 2. *Let (F, A) and (G, B) be two fuzzy soft bi-ideals over a semigroup S. If $A \subseteq B$ or $B \subseteq A$, then $(F, A)\tilde{\bigvee}(G, B)$ is a fuzzy soft bi-ideal over S.*

Proof. It suffices to prove the theorem for the case $A \subseteq B$.
Let $p \in A \cup B$. Since $A \subseteq B$, we have $A \cup B = B$.
If $p \in B - A$, then $(F \vee G)_p = G_p$.
If $p \in A \cap B = A$, then $(F \vee G)_p = F_p$.
Thus $(F \vee G)_p$ is a fuzzy bi-ideal on S. Hence $(F, A)\tilde{\bigvee}(G, B)$ is a fuzzy soft bi-ideal over S.◀

The converse of the theorem above is not true in general. It is shown by the following example.

Example 1. *Let $S = \{c_1(3.65), c_2(3.70), c_3(3.85)\}$ be a set of three candidates with grade point average (GPA) under consideration. We define the following binary operation $\bullet$:*

Table 1: Multiplication table of a semigroup S.

$\bullet$	c_1	c_2	c_3
c_1	c_1	c_2	c_3
c_2	c_2	c_2	c_3
c_3	c_3	c_3	c_3

*Let $E = \{e_1\{$good computer knowledge$\}$, $e_2\{$experience$\}$, $e_3\{$good speaking$\}$, $e_4\{$good English skill$\}\}$ be a set of parameters and $A = \{e_2, e_4\}$, $B = \{e_2, e_3\}$.
Let (F, A) and (G, B) be two fuzzy soft bi-ideals over S.*
$F_{e_2} = \{c_1/0.3, c_2/0.5, c_3/0.7\}, F_{e_4} = \{c_1/0.4, c_2/0.6, c_3/0.8\}$,
and $G_{e_2} = \{c_1/0.2, c_2/0.6, c_3/0.8\}, G_{e_3} = \{c_1/0.3, c_2/0.7, c_3/0.8\}$.
Thus $(F \vee G)_{e_2} = \{c_1/0.3, c_2/0.6, c_3/0.8\}$, $(F \vee G)_{e_3} = \{c_1/0.3, c_2/0.7, c_3/0.8\}$, $(F \vee G)_{e_4} = \{c_1/0.4, c_2/0.6, c_3/0.8\}$.
It is easy to verify that $(F, A)\tilde{\bigvee}(G, B)$ is a fuzzy soft bi-ideal over S, but $A \nsubseteq B$ and $B \nsubseteq A$.

The following theorem shows that $(F, A)\tilde{\bigwedge}(G, B)$ with $A \cap B \neq \emptyset$ is a fuzzy soft bi-ideal over a semigroup.

Theorem 3. *Let (F, A) and (G, B) be two fuzzy soft bi-ideals over a semigroup S. If $A \cap B \neq \emptyset$, then $(F, A)\tilde{\bigwedge}(G, B)$ is a fuzzy soft bi-ideal over S.*

Proof. Let $p \in A \cap B \neq \emptyset$. Thus $(F \wedge G)_p = F_p \wedge G_p$.
Let $x, y, z \in S$. Since F_p and G_p are both fuzzy subsemigroups on S, we have

$$(F \wedge G)_p(x \cdot y) = F_p(x \cdot y) \wedge G_p(x \cdot y)$$

$$\geq \{F_p(x) \wedge F_p(y)\} \wedge \{G_p(x) \wedge G_p(y)\}$$
$$= \{(F \wedge G)_p(x) \wedge (F \wedge G)_p(y)\}.$$

Thus $(F \wedge G)_p$ is a fuzzy subsemigroup on S.
Also, since F_p and G_p are both fuzzy bi-ideals on S, we have

$$(F \wedge G)_p(x \cdot y \cdot z) = F_p(x \cdot y \cdot z) \wedge G_p(x \cdot y \cdot z)$$

$$\geq \{F_p(x) \wedge F_p(z)\} \wedge \{G_p(x) \wedge G_p(z)\}$$
$$= \{(F \wedge G)_p(x) \wedge (F \wedge G)_p(z)\}.$$

Hence $(F \wedge G)_p$ is a fuzzy bi-ideal on S. Therefore $(F, A)\tilde{\bigwedge}(G, B)$ is a fuzzy soft bi-ideal over S.◄

By Lemma 2, if f is a fuzzy set on a semigroup S and g is a fuzzy bi-ideal on S, then $f \circ g$ and $g \circ f$ are both fuzzy bi-ideals on S, but in a fuzzy soft set over S if (F, A) is a fuzzy soft set over S and (G, B) is a fuzzy soft bi-ideal over S, then $(F, A) \odot (G, B)$ is not a fuzzy soft bi-ideal over S. This is shown by the following example.

Example 2. *Let $S = \{c_1(3.65), c_2(3.70), c_3(3.85)\}$ be a semigroup as in Example 1. Let $E = \{e_1\{good\ computer\ knowledge\},\ e_2\{experience\},\ e_3\{good\ speaking\},\ e_4\{good\ English\ skill\}\}$ be a set of parameters and $A = \{e_2, e_3\}$ and $B = \{e_3, e_4\}$. Let (F, A) be a fuzzy soft set over S, where*

$$F_{e_2} = \{c_1/0.8, c_2/0.4, c_3/0.6\}, F_{e_3} = \{c_1/0.7, c_2/0.5, c_3/0.4\},$$

and let (G, B) be a fuzzy soft bi-ideal over S, where

$$G_{e_3} = \{c_1/0.3, c_2/0.4, c_3/0.6\}, G_{e_4} = \{c_1/0.6, c_2/0.8, c_3/0.9\}.$$

Hence $(F \circ G)_{e_2} = \{c_1/0.8, c_2/0.4, c_3/0.6\}$, $(F \circ G)_{e_3} = \{c_1/0.3, c_2/0.4, c_3/0.6\}$, $(F \circ G)_{e_4} = \{c_1/0.6, c_2/0.8, c_3/0.9\}$.
Since $(F \circ G)_{e_2}(c_1 \bullet c_2 \bullet c_1) = (F \circ G)_{e_2}(c_2) = 0.4 < 0.8 = (F \circ G)_{e_2}(c_1) \wedge (F \circ G)_{e_2}(c_1)$, it follows that $(F, A) \odot (G, B)$ is not a fuzzy soft bi-ideal over S.

The next theorem shows that the product of two fuzzy soft bi-ideals over a semigroup is a fuzzy soft bi-ideal over the semigroup.

Theorem 4. *If (F, A) and (G, B) are two fuzzy soft bi-ideals over a semigroup S, then $(F, A) \odot (G, B)$ is a fuzzy soft bi-ideal over S.*

Proof. Let (F, A) and (G, B) be two fuzzy soft bi-ideals over S, and $p \in A \cup B$. Thus

$$(F \circ G)_p = \begin{cases} F_p & \text{if} \quad p \in A - B, \\ G_p & \text{if} \quad p \in B - A, \\ F_p \circ G_p & \text{if} \quad p \in A \cap B. \end{cases}$$

By the assumption and by Lemma 2, $(F \circ G)_p$ is a fuzzy bi-ideal on S.
Thus $(F, A) \odot (G, B)$ is a fuzzy soft bi-ideal over S.◄

The next theorem shows that a semigroup is regular if and only if $(F, A)\tilde{\bigwedge}(G, B) \leq (F, A) \odot (G, B)$ for every fuzzy soft bi-ideal (F, A) over S and for every fuzzy soft left [right] ideal (G, B) over S with $A \cap B \neq \emptyset$.

Theorem 5. *For a semigroup S, the following statements are equivalent:*
(i) *S is regular.*
(ii) $(F, A)\tilde{\bigwedge}(G, B) \leq (F, A) \odot (G, B)$ *for every fuzzy soft bi-ideal* (F, A) *over* S *and for every fuzzy soft left [right] ideal* (G, B) *over* S *with* $A \cap B \neq \emptyset$.

Proof. We prove only fuzzy soft left ideal [fuzzy soft right ideal is similar].
Assume that S is regular. Let (F, A) and (G, B) be any fuzzy soft bi-ideal and any fuzzy soft left ideal over S, respectively. Let $p \in A \cap B$. Then $A \cap B \subseteq A \cup B$. By Theorem 3.1.11(3) [21],

$$(F \wedge G)_p = F_p \wedge G_p \leq F_p \circ G_p = (F \circ G)_p.$$

Therefore $(F, A)\tilde{\bigwedge}(G, B) \leq (F, A) \odot (G, B)$.
On the other hand, suppose that statement (ii) holds. Let f and g be fuzzy bi-ideal and fuzzy left ideal on S, respectively. Let A be a non-empty set. Define $F_p = f$ and $G_p = g$ for all $p \in A$. Thus (F, A) and (G, B) are fuzzy soft bi-ideal and fuzzy soft left ideal over S, respectively. By the assumption,

$$(F, A)\tilde{\wedge}(G, A) \leq (F, A) \odot (G, A).$$

Fix $p \in A$. Then $F_p \wedge G_p = (F \wedge G)_p \leq (F \circ G)_p = F_p \circ G_p$.
Hence $f \wedge g \leq f \circ g$. By Theorem 3.1.11(3) [21], S is regular.◄

4. Mapping on classes of fuzzy soft bi-ideals over semigroups

In this section, we will rewrite notations and definitions of the images of functions introduced by Yang [16] so that the image of fuzzy soft bi-ideal is the fuzzy soft bi-ideal under some conditions.

Yang [16] gave the definitions of image of function and inverse image of function as follows.

Definition 17. [16] *Let $\widetilde{(X,E)}$ and $\widetilde{(Y,E')}$ be classes of fuzzy soft sets X and Y with parameters from E and E', respectively. Let $u : X \to Y$ and $p : E \to E'$ be two mappings. For a fuzzy soft set (Λ, Σ) in $\widetilde{(X,E)}$, where $\Sigma \subseteq E$, the image of (Λ, Σ) under the fuzzy soft function $f = (u,p)$, denoted by $f(\Lambda,\Sigma)$, is the fuzzy soft set over Y defined by $f(\Lambda,\Sigma) = (u(\Lambda), p(\Sigma))$, where*

$$u(\Lambda)(\beta)(y) = \begin{cases} \bigvee_{x \in u^{-1}(y)} \bigvee_{\alpha \in p^{-1}(\beta)\cap\Sigma} \Lambda(\alpha)(x) & \text{if} \quad u^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

for all $\beta \in p(\Sigma)$ and for all $y \in Y$.

Definition 18. [16] *Let $\widetilde{(X,E)}$ and $\widetilde{(Y,E')}$ be classes of fuzzy soft sets X and Y with parameters from E and E', respectively. Let $u : X \to Y$ and $p : E \to E'$ be two mappings, (Δ, Ω) be a fuzzy soft set in $\widetilde{(Y,E')}$, where $\Omega \subseteq E'$. The inverse image of (Δ,Ω) under the fuzzy soft function $f = (u,p)$, denoted by $f^{-1}(\Delta,\Omega)$, is the fuzzy soft set over X defined by $f^{-1}(\Delta,\Omega) = (u^{-1}(\Delta), v^{-1}(\Omega))$, where*

$$u^{-1}(\Delta)(\alpha)(x) = \Delta p(\alpha)(u(x)) \forall \alpha \in p^{-1}(\Omega), \forall x \in X.$$

In his definitions, we shall adapt some notation and rewrite the definitions as follows.

For any sets A and B, define $M(A,B) := \{f | f : A \to B\}$.
Let $FS(X,E)$ and $FS(Y,E')$ be fuzzy soft classes over the sets X and Y with the parameters from E and E', respectively. That is,

$FS(X,E) = \{(F,A) | A \subseteq E \text{ and } F : A \to Fuz(X)\}$,

$FS(Y,E') = \{(G,B) | B \subseteq E' \text{ and } G : B \to Fuz(Y)\}$.

Let $u : X \to Y$ and $v : E \to E'$ be two mappings.

We want to construct two functions

(i) $\Psi : FS(X,E) \to FS(Y,E')$ and

(ii) $\Sigma : FS(Y,E') \to FS(X,E)$.

(i) Let $(F,A) \in FS(X,E)$, where $\emptyset \neq A \subseteq E$.

Let $\Omega : M(A, Fuz(X)) \to M(v(A), Fuz(Y))$ be defined as follows:
Given $y \in Y$ and $\beta \in v(A)$, define

$$[\Omega(F)]_\beta(y) = \begin{cases} \bigvee_{x \in u^{-1}(y)} [\bigvee_{\delta \in v^{-1}(\beta)\cap A} F_\delta(x)] & \text{if} \quad u^{-1}(y) \neq \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

Let $\Psi(F,A) := (\Omega(F), v(A))$. Then $(\Omega(F), v(A))$ is an *image* of (F,A) under the function Ψ.

(ii) Let $(G,B) \in FS(Y,E')$, where $\emptyset \neq B \subseteq E'$.

Let $\Gamma : M(B, Fuz(Y)) \to M(v^{-1}(B), Fuz(X))$ be defined as follows:

Given $x \in X$ and $\delta \in v^{-1}(B)$, define

$$[\Gamma(G)]_{\delta}(x) = G_{v(\delta)}(u(x)).$$

Let $\Sigma(G, B) := (\Gamma(G), v^{-1}(B))$.

Remark 4. *Let $FS(X,E)$ and $FS(Y,E')$ be fuzzy soft classes over the sets X and Y, respectively. Suppose that $u : X \to Y$ and $v : E \to E'$ are two mappings and $\Psi : FS(X,E) \to FS(Y,E')$ and $\Sigma : FS(Y,E') \to FS(X,E)$ are also two mappings. If u and v are bijections, then $\Sigma \circ \Psi = I$ and $\Psi \circ \Sigma = I$, where $\circ$ is a composite mapping and I is an identity mapping.(Hence $\Psi$and Σ are bijections.)*

The following theorem shows that the image of a fuzzy soft bi-ideal over a semigroup is a fuzzy soft bi-ideal over the semigroup under some conditions.

Theorem 6. *Let $FS(S,E)$ and $FS(T,E')$ be fuzzy soft classes over semigroups S and T, respectively. Suppose that $\Psi : FS(S,E) \to FS(T,E')$ is a mapping, $v : E \to E'$ is an injection and u is an isomorphism from S to T. If (F,A) is a fuzzy soft bi-ideal over S, then $\Psi(F,A)$ is a fuzzy soft bi-ideal over T.*

Proof. Suppose that (F,A) is a fuzzy soft bi-ideal over S. By Theorem 1, $(F,A)^{\alpha}$ is a soft bi-ideal over S for all $\alpha \in [0,1]$. We shall show that $\Psi(F,A)^{\alpha} = (\Omega(F), v(A))^{\alpha}$ is a soft bi-ideal over T for all $\alpha \in [0,1]$. Let $p \in v(A)$. There exists $q \in A$ such that $p = v(q)$. Let $z_1, z_2 \in ([\Omega(F)]_p)^{\alpha}$. Thus

$$\alpha \leq [\Omega(F)]_p(z_1) = \bigvee_{s \in u^{-1}(z_1)} [\bigvee_{\beta \in v^{-1}(p) \cap A} F_{\beta}(s)] = \bigvee_{s \in u^{-1}(z_1)} F_q(s),$$

and

$$\alpha \leq [\Omega(F)]_p(z_2) = \bigvee_{s \in u^{-1}(z_2)} [\bigvee_{\beta \in v^{-1}(p) \cap A} F_{\beta}(s)] = \bigvee_{s \in u^{-1}(z_2)} F_q(s).$$

This implies that there exist $s_1, s_2 \in S$ such that $u(s_1) = z_1, u(s_2) = z_2$ and $F_q(s_1) \geq \alpha, F_q(s_2) \geq \alpha$. Then

$$\begin{aligned}[\Omega(F)]_p(z_1 \cdot z_2) &= \bigvee_{s \in u^{-1}(z_1 \cdot z_2)} [\bigvee_{\beta \in v^{-1}(p) \cap A} F_{\beta}(s)] \\ &= \textstyle\bigvee_{s \in u^{-1}(z_1 \cdot z_2)} F_q(s) \\ &\geq F_q(s_1 \cdot s_2) \\ &\geq F_q(s_1) \wedge F_q(s_2) \geq \alpha.\end{aligned}$$

Thus $z_1 \cdot z_2 \in ([\Omega(F)]_p)^{\alpha}$.

Let $y \in T$. Since u is surjective, there exists $x \in S$ such that $u(x) = y$ and since F_q is a fuzzy bi-ideal on S, we have

$$[\Omega(F)]_p(z_1 \cdot y \cdot z_2) = \bigvee_{s \in u^{-1}(z_1 \cdot y \cdot z_1)} [\bigvee_{\delta \in v^{-1}(p) \cap A} F_\delta(s)]$$

$$= \bigvee_{s \in u^{-1}(z_1 \cdot y \cdot z_1)} F_q(s)$$
$$= F_q(s_1 \cdot x \cdot s_1)$$
$$\geq F_q(s_1) \wedge F_q(s_1) \geq \alpha,$$

which implies that $z_1 \cdot y \cdot z_2 \in ([\Omega(F)]_p)^\alpha$.
Hence $([\Omega(F)]_p)^\alpha$ is a bi-ideal of T. Therefore $\Psi(F,A)^\alpha = (\Omega(F), v(A))^\alpha$ is a soft bi-ideal over T for all $\alpha \in [0,1]$. By Theorem 1, $\Psi(F,A)$ is a fuzzy soft bi-ideal over T.◀

The following example shows that if u is not an isomorphism from S to T, then the Theorem 6 is not true.

Example 3. *Let $S = \{c_1, c_2, c_3, c_4\}$ and $T = \{d_1, d_2, d_3\}$ be two semigroups having the multiplication tables, respectively,*

Table 2: Multiplication table of a semigroup S.

$\cdot$	c_1	c_2	c_3	c_4
c_1	c_1	c_2	c_3	c_4
c_2	c_2	c_2	c_3	c_4
c_3	c_3	c_3	c_3	c_4
c_4	c_4	c_4	c_4	c_4

Table 3: Multiplication table of a semigroup T.

$\bullet$	d_1	d_2	d_3
d_1	d_1	d_2	d_3
d_2	d_2	d_2	d_3
d_3	d_3	d_3	d_3

Let $E = \{e_1, e_2, e_3\}$, $E' = \{e'_1, e'_2, e'_3\}$, and let $FS(S,E)$ and $FS(T,E')$ be fuzzy soft classes over semigroups S and T, respectively.
Let $\Psi : FS(S,E) \to FS(T,E')$, $u : S \to T$ and $v : E \to E'$ be mappings defined by

$$u(c_1) = d_1,\ u(c_2) = d_2,\ u(c_3) = d_3,\ u(c_4) = d_1,\ v(e_1) = e'_1,\ v(e_2) = e'_2,\ v(e_3) = e'_3.$$

It is easy to check that u is not an isomorphism.
Let $A = \{e_1, e_2\} \subseteq E$ and let (F,A) be a fuzzy soft bi-ideal over S

$$F_{e_1} = \{c_1/0.2, c_2/0.4, c_3/0.6, c_4/0.8\}, F_{e_2} = \{c_1/0.3, c_2/0.5, c_3/0.7, c_4/0.9\}.$$

Then $[\Omega(F)]_{e'_1} = \{d_1/0.8, d_2/0.4, d_3/0.6\}$, *which implies that* $\Psi(F, A) = (\Omega(F), v(A))$. *Since* $[\Omega(F)]_{e'_1}(d_1 \bullet d_2 \bullet d_1) = [\Omega(F)]_{e'_1}(d_2) = 0.4 < 0.8 = [\Omega(F)]_{e'_1}(d_1) \wedge [\Omega(F)]_{e'_1}(d_1)$, *it follows that* $[\Omega(F)]_{e'_1}$ *is not a fuzzy bi-ideal on* T. *Therefore* $\Psi(F, A)$ *is not a fuzzy soft bi-ideal over* T.

The following theorem shows that $\Sigma(G, B)$ of a fuzzy soft bi-ideal over a semigroup is a fuzzy soft bi-ideal over the semigroup under some conditions.

Theorem 7. *Let* $FS(S, E)$ *and* $FS(T, E')$ *be fuzzy soft classes over semigroups* S *and* T, *respectively. Suppose that* $\Sigma : FS(T, E') \to FS(S, E)$ *is a mapping,* $v : E \to E'$ *is a mapping and* u *is a homomorphism from* S *to* T. *If* (G, B) *is a fuzzy soft bi-ideal over* T, *then* $\Sigma(G, B)$ *is a fuzzy soft bi-ideal over* S.

Proof. Assume that (G, B) is a fuzzy soft bi-ideal over T. Let $\delta \in v^{-1}(B)$. We then have $v(\delta) \in B$. This implies that $G_{v(\delta)}$ is a fuzzy bi-ideal on T. Let $x, y, z \in S$. Then

$$\begin{aligned}[\Gamma(G)]_\delta(x \cdot y) &= G_{v(\delta)}(u(x \cdot y)) \\ &= G_{v(\delta)}(u(x) \cdot u(y)) \\ &\geq G_{v(\delta)}(u(x)) \wedge G_{v(\delta)}(u(y)) \\ &= [\Gamma(G)]_\delta(x) \wedge [\Gamma(G)]_\delta(y).\end{aligned}$$

This implies that $[\Gamma(G)]_\delta$ is a fuzzy subsemigroup on S. Also, we have

$$\begin{aligned}[\Gamma(G)]_\delta(x \cdot y \cdot z) &= G_{v(\delta)}(u(x \cdot y \cdot z)) \\ &= G_{v(\delta)}(u(x) \cdot u(y) \cdot u(z)) \\ &\geq G_{v(\delta)}(u(x)) \wedge G_{v(\delta)}(u(z)) \\ &= [\Gamma(G)]_\delta(x) \wedge [\Gamma(G)]_\delta(z).\end{aligned}$$

Thus $[\Gamma(G)]_\delta$ is a fuzzy bi-ideal on S. Hence $\Sigma(G, B)$ is a fuzzy soft bi-ideal over S.◀

The following example satisfies the conditions of Theorem 7.

Example 4. *Let* $S = \{c_1, c_2, c_3\}$, $T = \{d_1, d_2, d_3\}$ *be two semigroups having the multiplication tables, respectively,*

Table 4: Multiplication table of a semigroup S.

$\cdot$	c_1	c_2	c_3
c_1	c_1	c_2	c_3
c_2	c_2	c_2	c_3
c_3	c_3	c_3	c_3

Table 5: Multiplication table of a semigroup T.

$\bullet$	d_1	d_2	d_3
d_1	d_1	d_2	d_3
d_2	d_2	d_2	d_3
d_3	d_3	d_3	d_3

Let $E = \{e_1, e_2, e_3\}$, $E' = \{e'_1, e'_2, e'_3, e'_4\}$ and let $FS(S,E)$ and $FS(T,E')$ be fuzzy soft classes over semigroups S and T, respectively. Suppose that $\Sigma : FS(T,E') \to FS(S,E)$, $u : S \to T$ and $v : E \to E'$ are mappings defined by

$$u(c_1) = d_2,\ u(c_2) = d_2,\ u(c_3) = d_3,\ v(e_1) = e'_1,\ v(e_2) = e'_1,\ v(e_3) = e'_3.$$

It is easy to check that u is a homomorphism from S to T.
Let $B = \{e'_1, e'_3\} \subseteq E'$ and let (G,B) be a fuzzy soft bi-ideal over T such that

$$G_{e'_1} = \{d_1/0.4, d_2/0.5, d_3/0.6\}, G_{e'_3} = \{d_1/0.6, d_2/0.7, d_3/0.8\}.$$

Then $[\Gamma(G)]_{e_1} = \{c_1/0.5, c_2/0.5, c_3/0.6\}, [\Gamma(G)]_{e_2} = \{c_1/0.5, c_2/0.5, c_3/0.6\}$, $[\Gamma(G)]_{e_3} = \{c_1/0.7, c_2/0.7, c_3/0.8\}$, which implies that $\Sigma(G,B) = (\Gamma(G), v^{-1}(B))$. It is easy to verify that $[\Gamma(G)]_{e_1}, [\Gamma(G)]_{e_2}$ and $[\Gamma(G)]_{e_3}$ are both fuzzy bi-ideals on S. Therefore $\Sigma(G,B)$ is a fuzzy soft bi-ideal over S.

The following corollary shows that $(\Sigma\circ\Psi)(F,A) = (F,A)$ and $(\Psi\circ\Sigma)(G,B) = (G,B)$ for every fuzzy soft bi-ideals (F,A) and (G,B), where $\circ$ is a composite mapping, under certain conditions.

Corollary 1. *Let $FS(S,E)$ and $FS(T,E')$ be fuzzy soft classes over semigroups S and T, respectively. Suppose that $u : S \to T$ and $v : E \to E'$ are two mappings and $\Psi : FS(S,E) \to FS(T,E')$ and $\Sigma : FS(T,E') \to FS(S,E)$ are also two mappings. If u and v are bijections, then $(\Sigma \circ \Psi)(F,A) = (F,A)$ and $(\Psi \circ \Sigma)(G,B) = (G,B)$ for every fuzzy soft bi-ideals (F,A) and (G,B), where $\circ$ is a composite mapping.*

Proof. It follows from Remark 4.◀

5. Conclusion

In this paper, we gave some properties of soft bi-ideals, fuzzy soft left [right] ideals and fuzzy soft bi-ideals over semigroups. We also proved that the images of fuzzy soft bi-ideals over semigroups are the fuzzy soft bi-ideals over semigroups under some conditions.

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