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On an Inverse Spectral Problem for a Perturbed Harmonic Oscillator

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Abstract. The inverse spectral problem for perturbed harmonic oscillators on a semi-axis with the same spectrum is investigated. The main equation of the inverse problem is obtained. The unique solvability of the main equation is proved.

Key Words and Phrases: perturbed harmonic oscillator, Schrödinger equation, transformation operator, inverse spectral problem, main equation.

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1. Introduction

Over the past few years, many papers have appeared dedicated to various problems of spectral analysis of a perturbed harmonic oscillator (see [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11], and references therein). McKean and Trubowitz [2] considered the problem of reconstruction for perturbed oscillator on the real line

$$T = \hat{T} + q\left(x\right), \ \widehat{T} = -\frac{d^2}{dx^2} + x^2.$$

They gave an algorithm for the reconstruction of $q\left(x\right)$ from norming constants for the class of real infinitely differentiable potentials, vanishing rapidly at $\pm\infty$, for fixed eigenvalues $\lambda_n\left(q\right) = \lambda_n\left(0\right)$ for all n and "norming constants" $\to 0$ rapidly as $n \to \infty$. Later on, B.M. Levitan [3] reproved some results of [11] without an exact definition of the class of potentials. It was also noted there that the perturbation potentials may be constructed by the standard procedure of the method of inverse problem.

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We consider the perturbed oscillator T_0, generated by the anharmonic equation

$$-y'' + x^2 y + q(x) y = \lambda y,\ 0 < x < \infty, \quad \lambda \in C, \tag{1}$$

and the boundary condition

$$y'(0) = 0, \tag{2}$$

where the real potential $q(x)$ satisfies the conditions

$$q(x) \in C^{(1)}[0,\infty), \int_0^\infty \left|x^j q(x)\right| dx < \infty,\ j = 0,1,2. \tag{3}$$

It is well known that the spectrum of T_0 is purely discrete and consists of simple eigenvalues (see, e.g., [1, 8]) $\lambda_n, n = 0,1,...,$ where $\lambda_n \to +\infty$ as $n \to \infty$. The corresponding normalized eigenfunctions $\left\{\frac{f(x,\lambda_n)}{\alpha_n}\right\}_{n=0}^\infty$, where $\alpha_n = \sqrt{\int_0^\infty |f(x,\lambda_n)|^2 dx}$, form an orthonormal basis for the space $L_2(0,\infty)$. Further, as in [2, 3], we assume that the perturbed oscillators have the same spectrum.

In present paper the inverse spectral problem for the perturbed oscillator T_0 is investigated by the method of transformation operators, i.e, the problem of reconstructing the perturbation potential $q(x)$ from spectral data $\{\lambda_n, \alpha_n > 0\}_{n=0}^\infty$. The obtained results can also be used to rigorously substantiate some formal statements of [3].

It should be noted that, in different statement, the inverse problems for perturbed harmonic oscillators have been studied in [6, 7, 8].

In the next section the transformation operator for the perturbed harmonic oscillator is constructed. The last section is dedicated to the solution of the inverse spectral problem. Note that inverse spectral problems for the Schródinger equation with some unbounded potentials were considered in [12, 13, 14].

2. The transformation operator

Consider the unperturbed equation

$$-y'' + x^2 y = \lambda y,\ 0 < x < \infty, \quad \lambda \in C. \tag{4}$$

It has [15] the solution$f_0(x,\lambda)$ in the form

$$f_0(x,\lambda) = D_{\frac{\lambda}{2}-\frac{1}{2}}\left(\sqrt{2}x\right),$$

where $D_\nu(x)$ is the Weber function. It is well known (see [7, 15]) that for each $x \in [0,\infty)$ the function $f_0(x,\lambda)$ is entire and the following asymptotic holds

$$f_0(x,\lambda) = \left(\sqrt{2}x\right)^{\frac{\lambda-1}{2}} e^{-\frac{x^2}{2}} \left(1 + O\left(x^{-2}\right)\right), \quad x \to \infty, \tag{5}$$

uniformly with respect to λ on bounded domains. It was shown in [1, 8], that the spectrum of $\hat{T}_0$ is purely discrete and consists of simple eigenvalues $\lambda_n^0 = 4n+1$, $n = 0, 1, \ldots$. The corresponding eigenfunctions $\left\{f_0\left(x, \lambda_n^0\right)\right\}_{n=0}^{\infty}$ form an orthogonal basis for the space $L_2(0,\infty)$. We have the equalities

$$f_0\left(x, \lambda_n^0\right) = D_{2n}\left(\sqrt{2}x\right) = 2^{-n} e^{-\frac{x^2}{2}} H_{2n}(x),$$

where $H_n(x)$ is the Hermite polynomial. From the well-known properties of Hermite polynomials it follows that

$$\left(\alpha_n^0\right)^2 = \int_0^{\infty} \left|f_0\left(x, \lambda_n^0\right)\right|^2 dx = (2n)! \frac{\sqrt{\pi}}{2}.$$

The functions $\left\{\frac{f_0\left(x,\lambda_n^0\right)}{\alpha_n^0}\right\}_{n=0}^{\infty}$ are normalized eigenfunctions of $\hat{T}_0$. Consequently,

$$\sum_{n=0}^{\infty} \frac{f_0\left(x, \lambda_n^0\right)}{\alpha_n^0} \frac{f_0\left(y, \lambda_n^0\right)}{\alpha_n^0} = \delta(x-y), \tag{6}$$

where $\delta(x)$ is Dirac's delta.

We now consider the perturbed equation (1). As is shown in [2, 7, 8], the equation (1) under condition (3) has a solution $f(x,\lambda)$ with asymptotic behavior $f(x,\lambda) = f_0(x,\lambda)(1+o(1))$, $x \to \infty$. We set

$$\sigma(x) = \int_x^{\infty} |q(t)|\, dt, \sigma_1(x) = \int_x^{\infty} \sigma(t)\, dt.$$

In the next theorem, by means of the transformation operator, a representation of the solution $f(x,\lambda)$ is obtained.

Theorem 1. *If $q(x)$ satisfies the condition (3) for $j = 1$, then for every λ the equation (1) has a solution $f(x,\lambda)$, representable in the form*

$$f(x,\lambda) = f_0(x,\lambda) + \int_x^{\infty} K(x,t)\; f_0(t,\lambda)\, dt, \tag{7}$$

where the kernel $K(x,t)$ is a continuous function and satisfies the following relations

$$|K(x,t)| \le \frac{1}{2}\sigma\left(\frac{x+t}{2}\right) e^{\sigma_1\left(\frac{x+t}{2}\right)}, \tag{8}$$

$$K(x,x) = \frac{1}{2}\int_x^{\infty} q\;(t)\; dt. \tag{9}$$

Proof. Substituting the representation (7) into equation (1), we find that the function (7) satisfies equation (1), if only the kernel $K(x,t)$ satisfies a hyperbolic equation of second order

$$\frac{\partial K\left(x,t\right)}{\partial x^{2}}-\frac{\partial K\left(x,t\right)}{\partial t^{2}}-\left(x^{2}-t^{2}-q\left(t\right)\right)\ K\left(x,t\right)=0,\ 0<x<t, \tag{10}$$

and the conditions

$$K\left(x,x\right)=\frac{1}{2}\int_{x}^{\infty}q\ (t)\ dt, \tag{11}$$

$$\lim_{x+t\to\infty}K\left(x,t\right)=0. \tag{12}$$

Reduce problem (10)-(12) to an integral equation. To this end, we reduce equation (10) to the canonical form. Assume

$$U\left(\xi,\eta\right)=U\left(\frac{t+x}{2},\frac{t-x}{2}\right)=K\left(x,t\right)=K\left(\xi-\eta,\xi+\eta\right).$$

For this function we find

$$L\left[U\right]\equiv\frac{\partial^{2}U\left(\xi,\eta\right)}{\partial\xi\partial\eta}-4\xi\eta U\left(\xi,\eta\right)=U\left(\xi,\eta\right)q\left(\xi+\eta\right) \tag{13}$$

with boundary conditions

$$U(\xi,0)=\frac{1}{2}\int_{\xi}^{\infty}q\left(\alpha\right)d\alpha, \tag{14}$$

$$\lim_{\xi\to\infty}U(\xi,\eta)=0,\ \eta>0. \tag{15}$$

Introduce the Riemann function $R(\xi,\eta;\xi_0,\eta_0)$ of the equation $L\left[U\right]=\psi(\xi,\eta)$, where $\psi(\xi,\eta)=U(\xi,\eta)q(\xi+\eta)$, i.e., the function satisfying the equation

$$L^{*}(R)\equiv\frac{\partial^{2}R}{\partial\xi\,\partial\eta}-4\xi\,\eta\,R=0\qquad\left\{\begin{array}{c}0<\eta<\eta_{0},\\ \xi_{0}<\xi<\infty,\\ 0<\eta<\xi,\end{array}\right.$$

and the conditions on the characteristics

$$R(\xi,\eta;\xi_0,\eta_0)\left.\right|_{\xi=\xi_0}=1,0\le\eta\le\eta_0,$$

$$R(\xi,\eta;\xi_0,\eta_0)\left.\right|_{\eta=\eta_0}=1,\ \xi_0\le\xi<\infty.$$

Let

$$R\left(\xi,\eta,\xi_{0},\eta_{0}\right)=J_{0}\left(z\right)=\sum_{n=0}^{\infty}\frac{\left(-1\right)^{n}}{\left(n!\right)^{2}}\left(\frac{z}{2}\right)^{2n},z=2\sqrt{\left(\xi^{2}-\xi_{0}^{2}\right)\left(\eta_{0}^{2}-\eta^{2}\right)}, \tag{16}$$

where $J_n(z)$ is the Bessel function of the first kind. It is easy to verify that this function satisfies the last three relations. In other world, $R(\xi,\eta,\,\xi_0,\eta_0)$ is the Riemann function of the equation (13) and has the symmetric property $R(\xi,\eta,\xi_0,\eta_0) = R(\xi_0,\eta_0,\xi,\eta)$. Using the well-known properties of the Bessel function, we find that the following relations hold

$$\begin{array}{l} \frac{\partial R}{\partial \xi} = O(\xi)\,,\ \frac{\partial R}{\partial \eta} = O(\xi)\,,\ \frac{\partial^2 R}{\partial \xi \partial \eta} = O(\xi)\,, \xi \to \infty, \\ \frac{\partial^2 R}{\partial \xi^2} = O\left(\xi^2\right),\ \frac{\partial^2 R}{\partial \eta^2} = O\left(\xi^2\right),\ \xi \to \infty. \end{array} \tag{17}$$

Now, apply the Riemann method (see [16]) to the equation (13). Then we obtain the following integral equation for $U(\xi_0,\eta_0)$:

$$U(\xi_0,\eta_0) = \frac{1}{2}\int_{\xi_0}^{\infty} R(\xi,0;\xi_0,\eta_0)q(\xi)d\xi - \int_{\xi_0}^{\infty} d\xi \int_0^{\eta_0} U(\xi,\eta)q(\xi+\eta)R(\xi,\eta;\xi_0,\eta_0)d\eta. \tag{18}$$

Thus, for solving problem (13)-(15) it is enough to solve integral equation (18) with respect to $U(\xi_0,\eta_0)$. Solving the above integral equation by the method of successive approximations with the relation $|R| \le 1$ taken into account, we obtain

$$|U(\xi_0,\eta_0)| \le \frac{1}{2}\sigma(\xi_0)\, e^{\sigma_1(\xi_0)}. \tag{19}$$

Differentiating equation (18) directly and using relations (17), we find that the function $U(\xi_0,\eta_0)$ and thus the function $K(x,t) = U\left(\frac{t+x}{2}, \frac{t-x}{2}\right)$ are twice continuously differentiable. Moreover, for each fixed x we have the relations

$$\begin{array}{l} \frac{\partial K(x,t)}{\partial x} = O\left(t^2\right),\ \frac{\partial K(x,t)}{\partial t} = O\left(t^2\right), \\ \frac{\partial^2 K(x,t)}{\partial x^2} = O\left(t^4\right),\ \frac{\partial^2 K(x,t)}{\partial t^2} = O\left(t^4\right), t \to \infty. \end{array}$$

From this and (19) it follows that the function $K(x,t) = U\left(\frac{t+x}{2}, \frac{t-x}{2}\right)$ satisfies the problem (10)-(12). This completes the proof of the theorem. ◄

3. Inverse problem

From the results of the previous section it follows that for each λ the function $f(x,\lambda)$ belongs to the space $L_2(0,\infty)$. Consequently, the spectrum of problem (1)-(2) coincides with the roots of the function $f(0,\lambda)$, i.e. the following relation holds: $f(0,\lambda_n) = 0, n = 0,1,....$ As is shown in [1], the following relations are true

$$\begin{array}{l} f_0(0,\lambda) = c_0 2^{\frac{\lambda}{4}}\Gamma\left(\frac{\lambda+1}{4}\right)\cos\left(\frac{\pi(\lambda-1)}{4}\right),\ c_0 = 2^{-\frac{1}{4}}\pi^{-\frac{1}{2}}, \\ f_0'(0,\lambda) = 2c_0 2^{\frac{\lambda}{4}}\Gamma\left(\frac{\lambda+3}{4}\right)\sin\left(\frac{\pi(\lambda-1)}{4}\right), \\ \dot{f}_0(0,\lambda) = -\frac{\pi c_0}{4} 2^{\frac{\lambda}{4}}\Gamma\left(\frac{\lambda+1}{4}\right)\sin\left(\frac{\pi(\lambda-1)}{4}\right), \end{array} \tag{20}$$

where $\dot{f} = \frac{\partial f}{\partial \lambda}$. Furthermore, if $0 \le x^2 \le \left(\frac{\lambda}{4}\right)^{\frac{1}{2}-\varepsilon}, \lambda \ge \lambda_0 > 0$, $0 < \varepsilon < \frac{1}{3}$, then we have the asymptotics expansions

$$f(x,\lambda) = c_0 2^{\frac{\lambda}{4}} \Gamma\left(\frac{\lambda+1}{4}\right)\left\{\cos\left[\pi\frac{\lambda-1}{4} - \sqrt{\lambda}x\right] + \lambda^{-\frac{1}{2}}\left(2x^3 + 2^{-\frac{1}{2}}\right) O(1)\right\}, \tag{21}$$

$$\dot{f}(x,\lambda) = c_0 2^{\frac{\lambda}{4}} \Gamma\left(\frac{\lambda+1}{4}\right) \times \left\{\frac{1}{4}\cos\left[\pi\frac{\lambda-1}{4} - \sqrt{\lambda}x\right] \ln\frac{\lambda+1}{2} - \right.$$
$$\left. -\frac{\pi}{4}\sin\left[\pi\frac{\lambda-1}{4} - \sqrt{\lambda}x\right] + \lambda^{-\frac{1}{2}}\left(2x^3 + 2^{-\frac{1}{2}}\right) O(1)\right\}. \tag{22}$$

The behavior of $f'(x,\lambda)$ as $\lambda \to \infty$ and $0 \le x \le x_0$, $x_0 > 0$ is determined [1] by the expansion

$$f'(x,\lambda) = 2c_0 2^{\frac{\lambda}{4}} \Gamma\left(\frac{\lambda+3}{4}\right)\left\{\sin\left[\pi\frac{\lambda-1}{4} - \sqrt{\lambda}x\right] + \lambda^{-\frac{1}{2}} O(1)\right\}. \tag{23}$$

Introduce the Wronskian

$$\{u,v\} = uv' - u'v.$$

The standard identity (see [8])

$$f^2 = \left\{\dot{f}, f\right\}'$$

yields

$$\alpha_n^2 = \int_0^\infty f^2(x,\lambda_n)\,dx = \left\{\dot{f}(x,\lambda_n), f(x,\lambda_n)\right\}\Big|_0^\infty = -\dot{f}(0,\lambda_n) f'(0,\lambda_n). \tag{24}$$

Using (20)-(23) and taking into account that $\lambda_n = 4n+1$, we obtain

$$f'(0,\lambda_n) = f_0'(0,\lambda_n)\left[1 + O\left(n^{-\frac{1}{2}}\right)\right],$$
$$\dot{f}(0,\lambda_n) = \dot{f}_0(0,\lambda_n)\left[1 + O\left(n^{-\frac{1}{2}}\right)\right].$$

Then it follows from (24) that

$$\alpha_n^{-2} = \left(\alpha_n^0\right)^{-2}\left[1 + O\left(n^{-\frac{1}{2}}\right)\right]. \tag{25}$$

Denote

$$F(x,y) = \sum_{n=0}^{\infty}\left\{(\alpha_n)^{-2} - \left(\alpha_n^0\right)^{-2}\right\} f_0(x,\lambda_n) f_0(y,\lambda_n). \tag{26}$$

Since $\left(\alpha_n^0\right)^2 = (2n)!\frac{\sqrt{\pi}}{2} = \frac{\sqrt{\pi}}{2}\Gamma(2n+1)$, by virtue of the well-known relations for the Gamma function [15]

$$\Gamma(az+b) = \sqrt{2\pi}e^{-az}(az)^{z+b-\frac{1}{2}}\left[1+O\left(z^{-1}\right)\right],\ z\to\infty,\ |\arg z|<\pi,$$

it follows from (21) that for each fixed x the following relation holds

$$\frac{f_0(x,\lambda_n)}{\alpha_n^0} = O\left(n^{-\frac{1}{4}}\right),\ n\to\infty.$$

From this and (25), (26) it follows that for each fixed x the series (26) converges in the metric of $L_2(0,\infty)$. Hence, for each fixed x the function $F(x,y)$ belongs to $L_2(0,\infty)$ as a function of y.

Theorem 2. *For each fixed $x\geq 0$ the kernel $K(x,y)$ appearing in representation (7) satisfies the linear integral equation*

$$F(x,y)+K(x,y)+\int_x^\infty K(x,t)F(t,y)\,dt = 0,\ \ y>x. \tag{27}$$

This equation is called the main equation or Gelfand-Levitan-Marchenko equation.

Proof. The functions $\left\{\frac{f(x,\lambda_n)}{\alpha_n}\right\}_{n=0}^{\infty}$ are normalized eigenfunctions of T_0. Consequently

$$\sum_{n=0}^{\infty}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f(y,\lambda_n)}{\alpha_n} = \delta(x-y), \tag{28}$$

where $\delta(x)$ is Dirac's delta. On the other hand, one can consider the relation (7) as a Volterra integral equation with respect to $f_0(x,\lambda)$. Solving this equation we obtain

$$f_0(y,\lambda) = f(y,\lambda)+\int_y^\infty \tilde{K}(y,t)f(t,\lambda)dt. \tag{29}$$

Moreover, from the well-known properties of the transformation operators [17] it follows that the kernel $\tilde{K}(y,t)$ satisfies an inequality analogous to (8). From (28), (29), we have

$$\begin{aligned}&\textstyle\sum_{n=0}^\infty \frac{f(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n} = \sum_{n=0}^\infty \frac{f(x,\lambda_n)}{\alpha_n}\frac{f(y,\lambda_n)}{\alpha_n}+\\ &\textstyle+\int_y^\infty \tilde{K}(y,t)\left\{\sum_{n=0}^\infty \frac{f(x,\lambda_n)}{\alpha_n}\frac{f(t,\lambda_n)}{\alpha_n}\right\}dt =\\ &\textstyle=\delta(x-y)+\int_y^\infty \tilde{K}(y,t)\,\delta(x-t)\,dt =\\ &=\delta(x-y)+\tilde{K}(y,x) = \delta(x-y),\end{aligned}$$

and hence with the help of (7)

$$\sum_{n=0}^{\infty}\frac{f(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}=\sum_{n=0}^{\infty}\frac{f_0(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}+$$
$$+\int_x^{\infty}K(x,t)\left\{\sum_{n=0}^{\infty}\frac{f_0(t,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}\right\}dt=$$
$$=\sum_{n=0}^{\infty}\frac{f_0\left(x,\lambda_n^0\right)}{\alpha_n^0}\frac{f_0\left(y,\lambda_n^0\right)}{\alpha_n^0}+\sum_{n=0}^{\infty}\left\{\frac{f_0(x,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}-\frac{f_0\left(x,\lambda_n^0\right)}{\alpha_n^0}\frac{f_0\left(y,\lambda_n^0\right)}{\alpha_n^0}\right\}+$$
$$+\int_x^{\infty}K(x,t)\left\{\sum_{n=0}^{\infty}\frac{f_0\left(t,\lambda_n^0\right)}{\alpha_n^0}\frac{f_0\left(y,\lambda_n^0\right)}{\alpha_n^0}\right\}dt+$$
$$+\int_x^{\infty}K(x,t)\left\{\sum_{n=0}^{\infty}\left\{\frac{f_0(t,\lambda_n)}{\alpha_n}\frac{f_0(y,\lambda_n)}{\alpha_n}-\frac{f_0\left(t,\lambda_n^0\right)}{\alpha_n^0}\frac{f_0\left(y,\lambda_n^0\right)}{\alpha_n^0}\right\}\right\}dt=$$
$$=\delta(x-y)+F(x,y)+K(x,y)+\int_x^{\infty}K(x,t)F(t,y)\,dt.$$

Comparing the last two equations, we arrive at (27).

If $q(x)$ satisfies condition (3) for $j=2$, then, as is shown in [18, see Lemma 6.3], the kernel $F(x,y)$ of the main equation (27) satisfies the inequality

$$|F(x,y)|\leq C\sigma\left(\frac{x+y}{2}\right). \tag{30}$$

In addition, function $F(x,y)$ is continuous in the set of arguments. It follows from (30) that

$$\int_0^{\infty}\sup_{x>0}\left|F^{\pm}(x,y)\right|dy<\infty. \tag{31}$$

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Theorem 3. *If function $F(x,y)$ satisfies condition* (31)*, then for each fixed $x\geq 0$ equation* (27) *has a unique solution $K(x,y)$ in $L_2(x,\infty)$.*

Proof. It is easy to check that for each fixed x, the operator

$$\Omega_x f(y)=\int_x^{\infty}F(y,t)\,f(t)\,dt$$

is compact in $L_2(x,\infty)$. Indeed, we have

$$\int_x^{\infty}dt\int_x^{\infty}|F(t,y)|^2\,dy\leq\int_x^{\infty}\sup_{y\geq x}|F(t,y)|\,dt\int_x^{\infty}|F(t,y)|\,dy\leq$$

$$\leq\int_x^{\infty}\sup_{y\geq x}|F(y,t)|\,dt\int_x^{\infty}\sup_{t\geq x}|F(t,y)|\,dt<\infty.$$

Hence, the operator Ω_x is a Hilbert-Schmidt type operator. Since (27) is a Fredholm equation, it is sufficient to prove that the homogeneous equation

$$h(y) + \int_x^\infty F(t,y)\,h(t)\,dt = 0, \tag{32}$$

has only the trivial solution $h(y) = 0$.

Let $h(y)$ be a solution of (32). Then

$$\int_x^\infty h^2(y)\,dy + \int_x^\infty \int_x^\infty F(t,y)\,h(t)\,h(y)\,dtdy = 0,$$

or

$$\int_x^\infty h^2(y)\,dy + \sum_{n=0}^\infty (\alpha_n)^{-2}\left(\int_x^\infty h(y)\,f_0(y,\lambda_n)\,dy\right)^2 -$$

$$-\sum_{n=0}^\infty \left(\alpha_n^0\right)^{-2}\left(\int_x^\infty h(y)\,f_0(y,\lambda_n)\,dy\right)^2 = 0.$$

Using Parseval's equality

$$\int_x^\infty h^2(y)\,dy = \sum_{n=0}^\infty \left(\alpha_n^0\right)^{-2}\left(\int_x^\infty h(y)\,f_0(y,\lambda_n)\,dy\right)^2,$$

for the function $h(y)$, extended by zero for $y < x$, we obtain

$$\sum_{n=0}^\infty (\alpha_n)^{-2}\left(\int_x^\infty h(y)\,f_0(y,\lambda_n)\,dy\right)^2 = 0.$$

Since $(\alpha_n)^{-2} > 0$, we have

$$\int_x^\infty h(y)\,f_0(y,\lambda_n)\,dy = 0, n \geq 0.$$

The system of functions $\{f_0(y,\lambda_n)\}_0^\infty$ is orthogonal basis in $L_2(x,\infty)$. This yields

$$h(y) = 0.$$

◀

Remark 1. *The solution of the inverse scattering problem can be constructed by the following algorithm. Calculate the function* $F(x,y)$ *by the spectral data* $\{\lambda_n, \alpha_n > 0\}_{n=0}^\infty$ *and* (26). *Find* $K(x,y)$ *by solving the main equation* (27). *Construct* $q(x)$ *by* (9). *Then, following the techniques of [13], in a narrower class of potentials one can achieve a complete solution to the inverse problem.*

Remark 2. *The obtained results also extend to the case when the spectra of perturbed harmonic oscillators are different. In this case we will have to use the asymptotic formula (see [1, 8])* $\lambda_n = 4n + 1 + O\left(n^{-\frac{1}{2}}\right)$, $n \to \infty$.

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