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On Non-Existence of Positive Periodic Solution for Second Order Semilinear Parabolic Equation

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Abstract. Second order semilinear parabolic equation with time-periodic coefficients is considered in the domain $\{x;|x|>R\}\times(-\infty,+\infty)$.The absence of global positive periodic solutions is studied. The exact conditions are found under which the positive periodic solution does not exist.

Key Words and Phrases: semilinear parabolic equation, global solution, periodic solution, Harnack inequality.

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1. Introduction

We will use the following notations: $x=(x_1,...,x_n)\in R^n$, $n\geq 3$, $r=|x|=\sqrt{x_1^2+...+x_n^2}$, $B_R=\{x;|x|<R\}$, $B_{R_1,R_2}=\{x;R_1<|x|<R_2\}$, $B_R^{'}=\{x;|x|>R\}$,$Q_T^R=B_R\times(0,T)$, $Q_T^{R_1,R_2}=B_{R_1,R_2}\times(0,T)$, $Q_T^{R,\infty}=B_R^{'}\times(0,T)$, $Q_R^{'}=B_R^{'}\times(-\infty,+\infty)$, $S_R=\{x;|x|=R\}\times(-\infty,+\infty)$, Ω is a bounded domain in R^n, $Q_T=\Omega\times(0,T)$, $Q=\Omega\times(-\infty,+\infty)$.

Consider the equation

$$\frac{\partial u}{\partial t}=div\left(|x|^{\alpha}A\nabla u\right)+a_0\left(x,t\right)|u|^{q-1}u, \tag{1}$$

in the domain $Q_R^{'}$, where $q>1$, $\alpha<2$, $A=A\left(x,t\right)=\left(a_{ij}\left(x,t\right)\right)_{i,j=1}^{n}$, $a_{ij}\left(x,t\right)$, $a_0\left(x,t\right)$ are bounded, measurable, T- periodic in t functions, and there exist the constants ν_1,ν_2 such that

$$\nu_1|\xi|^2\leq(A\xi,\xi)\leq\nu_2|\xi|^2 \tag{2}$$

for every $(x,t)\in Q_R^{'}$, $\xi=(\xi_1,...,\xi_n)\in R^n$.

Here $A\nabla u$ denotes the action of the matrix A on the vector $\nabla u = \left(\frac{\partial u}{\partial x_1}, ..., \frac{\partial u}{\partial x_n}\right)$, i.e., $A\nabla u = \left(\sum\limits_{j=1}^{n} a_{ij}\frac{\partial u}{\partial x_j}\right)_{i=1}^{n}$ and $(A\xi, \eta) = \sum\limits_{i,j=1}^{n} a_{ij}\xi_i\eta_j$, $\xi = (\xi_1, ..., \xi_n), \eta = (\eta_1, ..., \eta_n)$.

We will study the existence of a global positive solution. Before giving a definition for solution, we consider the following function space:

$$W_2^{1,\frac{1}{2}}(Q_T) = \Big\{u(x,t); u(x,t+T) = u(x,t), u(x,t) \in W_2^{1,0}(Q_T),$$

$$\sum_{k=-\infty}^{+\infty} |k| \int\limits_{\Omega} |u_k(x)|^2 dx < \infty \Big\},$$

where

$$u_k(x) = \frac{1}{T}\int\limits_0^T u(x,t)\, exp\left\{ik\frac{2\pi}{T}t\right\} dt.$$

The norm in this space is defined as follows.

$$\|u\|^2_{W_2^{1,\frac{1}{2}}(Q_T)} = \|u\|^2_{L_2(Q_T)} + \|\nabla u\|^2_{L_2(Q_T)} + \sum_{k=-\infty}^{+\infty} |k| \int\limits_{\Omega} |u_k(x)|^2 dx.$$

By $\overset{\circ}{W}{}_2^{1,\frac{1}{2}}(Q_T)$ we mean a completion of $C^{0,\infty}(Q_T)$ with respect to the norm $\|\cdot\|_{W_2^{1,\frac{1}{2}}(Q_T)}$, where $C^{0,\infty}(Q_T)$ is a set of infinitely differentiable functions on Q_T, which are T periodic in t and vanish in the vicinity of $\partial\Omega$.

The function $u(x,t)$ is called a solution of the equation (1) in $Q_T^{R,\infty}$, if $u(x,t) \in W_{2,loc}^{1,\frac{1}{2}}\left(Q_T^{R,\infty}\right) \cap L_{\infty,loc}\left(Q_T^{R,\infty}\right)$ and the integral identity

$$2\pi \sum_{k=-\infty}^{+\infty} (ik) \int\limits_{B'_R} u_k(x)\,\varphi_{-k}(x)\, dx + \iint\limits_{Q_T^{R,\infty}} |x|^{\alpha} (A\nabla u, \nabla\varphi)\, dxdt =$$

$$= \iint\limits_{Q_T^{R,\infty}} a_0(x,t)\, |u|^{q-1} u\varphi dxdt,$$

holds for every $\varphi(x,t) \in \overset{\circ}{W}{}_2^{1,\frac{1}{2}}\left(Q_T^{R,\infty}\right)$.

Throughout this paper we will assume that $a_{ij}\left(x,t\right), i,j=\overline{1,n}$ are bounded, measurable, T periodic in t functions which satisfy the condition (2). All the constants appearing in different estimates will be denoted by C, although they are different in different estimates.

The matters of existence and non-existence of global solutions for different classes of differential equations and inequalities play an important role both in theory and applications, that's why they have always been the cause for constant interest from mathematicians. A lot of works have been dedicated to these matters (see [1-11]). For useful reviews of such works, we refer the readers to the article [12], the monograph [13], and the book [14].

In particular, the existence of solutions to the periodic parabolic equations has also been a study object for many researchers (see [15-22]). One of the earliest works dedicated to periodic parabolic equations was Seidman's [15], which treated the existence of non-trivial periodic solution for the following problem:

$$\frac{\partial u}{\partial t}=\Delta u+a(x,t)u^{q},(x,t)\in\Omega\times(0,+\infty),u/_{\partial\Omega}=0, \tag{3}$$

with $q=0$, where $a(x,t)$ is a periodic in t function and $\Omega\subset R^{n}$ is a bounded domain. Since then, many authors have considered the problem (3) for $q>0$. Beltramo and Hess [16] studied the problem (3) for $q=1$ and showed that for specially chosen $a(x,t)$ it may have non-trivial periodic solutions. Esteban [17, 18] proved that for every $q>1$ when $n\leq 2$, and for $1<q<\frac{n}{n-2}$ when $n>2$ the problem (3) has positive periodic solutions for any kind of $a(x,t)>0$. He also proved that for $n>2$, $q\geq\frac{n+2}{n-2}$, this problem has no positive periodic solution. In 2004, Quittner [21] proved, with some restrictions on $a(x,t)$, that this problem has positive solutions for $1<q<\frac{n+2}{n-2}$.

In [23], the equation (1) has been considered for $\alpha=0$ in $Q_{R}^{'}$, and it was proved that if $a_{0}(x,t)\geq c|x|^{\sigma}$, then there is no positive solution for $2+\sigma+(2-n)(q-1)\geq 0$. In [24], the equation (1) has been again considered for $\alpha=0$ in $Q_{R}^{'}$ and it was proved that if $a_{0}(x,t)\geq c|x|^{\sigma}\ln^{s}|x|$, then there is no positive solution for $2+\sigma+(2-n)(q-1)>0$, $s\in(-\infty,+\infty)$ and for $2+\sigma+(2-n)(q-1)=0$, $s\geq -1$.

In this work, we consider the equation (1) for $\alpha<2$ and obtain an exact criterion for non-existence of positive time-periodic solutions.

2. Auxiliary facts

Denote

$$Lu\equiv div\left(|x|^{\alpha}A\nabla u\right)-\frac{\partial u}{\partial t}.$$

Lemma 1. *Let* $W(x,t) \in L_{\infty,loc}(Q_T^{R,\infty})$*,* $W(x,t+T) = W(x,t)$*. If* $0 < u(x,t)$ *is a solution of the inequality* $Lu + Wu \le 0$*, then for every* $f(x) \in C_0^\infty\left(B_R'\right)$ *there holds*

$$\iint\limits_{Q_T^{R,\infty}} W(x,t)\, f^2(x)\, dxdt \le C \int\limits_{B_R'} |x|^\alpha |\nabla f|^2 dx.$$

Proof. In the definition of solution, we take $\varphi_{\bar h}(x,t) = \frac{1}{h}\int\limits_{t-h}^{t} \varphi(x,\tau)\, d\tau$ as a test function, where $\varphi(x,t) = \frac{f^2(x)}{u_h(x,t)}$,

$$f(x) \in C_0^\infty\left(B_R'\right), \quad u_h(x,t) = \frac{1}{h}\int\limits_t^{t+h} u(x,\tau)\, d\tau.$$

Taking into account that

$$\int\limits_0^T u\varphi_{\bar h} dt = \int\limits_0^T u_h \varphi dt,$$

we obtain

$$2\pi \sum_{k=-\infty}^{+\infty} (ik) \int\limits_{B_R'} u_k(x)\, \varphi_{\bar h(-k)}(x)\, dx + \iint\limits_{Q_T^{R,\infty}} |x|^\alpha \left(A\nabla u, \nabla \varphi_{\bar h}\right) dxdt \ge$$

$$\ge \iint\limits_{Q_T^{R,\infty}} W(x,t)\, u\varphi_{\bar h} dxdt, \tag{4}$$

$$2\pi \sum_{k=-\infty}^{+\infty} (ik) \int\limits_{B_R'} u_k(x)\, \varphi_{\bar h(-k)}(x)\, dx = -\iint\limits_{Q_T^{R,\infty}} u \frac{\partial \varphi_{\bar h}}{\partial t} dxdt =$$

$$= -\iint\limits_{Q_T^{R,\infty}} u \left(\frac{\partial \varphi}{\partial t}\right)_{\bar h} dxdt = -\iint\limits_{Q_T^{R,\infty}} u_h \frac{\partial \varphi}{\partial t} dxdt =$$

$$= \iint\limits_{Q_T^{R,\infty}} \frac{\partial u_h}{\partial t} \varphi dxdt = \iint\limits_{Q_T^{R,\infty}} \frac{\partial u_h}{\partial t} \frac{f^2}{u_h} dxdt = \iint\limits_{Q_T^{R,\infty}} \frac{\partial \ln u_h}{\partial t} f^2 dxdt = 0,$$

$$\iint\limits_{Q_T^{R,\infty}} |x|^\alpha \left(A\nabla u, \nabla \varphi_{\bar h}\right) dxdt = \iint\limits_{Q_T^{R,\infty}} |x|^\alpha \left((A\nabla u)_h, \nabla \varphi\right) dxdt =$$

$$= -\iint\limits_{Q_T^{R,\infty}} |x|^\alpha \sum_{i,j=1}^n \left(a_{ij}\frac{\partial u}{\partial x_j}\right)_h \frac{\partial u_h}{\partial x_i}\frac{f^2}{u_h^2}dxdt+$$

$$+\iint\limits_{Q_T^{R,\infty}} 2\,|x|^\alpha \sum_{i,j=1}^n \left(a_{ij}\frac{\partial u}{\partial x_j}\right)_h \frac{\partial f}{\partial x_i}\frac{f}{u_h}dxdt.$$

Considering the last relations in (4) and passing to the limit as $h \to 0$, we get

$$\iint\limits_{Q_T^{R,\infty}} Wf^2dxdt \le -\iint\limits_{Q_T^{R,\infty}} |x|^\alpha \left(A\nabla u, \nabla u\right)\frac{f^2}{u^2}dxdt+$$

$$+\iint\limits_{Q_T^{R,\infty}} 2\,|x|^\alpha \left(A\nabla u, \nabla f\right)\frac{f}{u}dxdt \le -\iint\limits_{Q_T^{R,\infty}} |x|^\alpha \left(A\nabla u, \nabla u\right)\frac{f^2}{u^2}dxdt+$$

$$+\iint\limits_{Q_T^{R,\infty}} 2\,|x|^\alpha \frac{f}{u}\left(A\nabla u, \nabla u\right)^{\frac{1}{2}}\left(A\nabla f, \nabla f\right)^{\frac{1}{2}}dxdt \le$$

$$\le \iint\limits_{Q_T^{R,\infty}} |x|^\alpha \left(A\nabla f, \nabla f\right)dxdt \le \nu_2 T\int\limits_{B_R'} |x|^\alpha \left|\nabla f\right|^2 dx.$$

Lemma 1 is proved. ◄

Lemma 2. *Let $n \ge 3$, $2-n \le \alpha < 2$, the non-negative, continuous function $u(x,t) \in W_{2,loc}^{1,\frac{1}{2}}\left(Q_T^{R,\infty}\right)$ on $\overline{Q}_T^{R,\infty}$ satisfy the inequality $Lu \le 0$ and $u(x,t) > 0$ on S_R. Then there exists $\beta_0 = const > 0$ such that $u(x,t) \ge \beta_0 |x|^{2-n-\alpha}$ for $(x,t) \in Q_T^{R,\infty}$.*

Proof. We first consider the case $2-n < \alpha < 2$. Let's continue the coefficients $a_{ij}(x,t)$ to Q_T^R, assuming $a_{ij} = \delta_{ij}$ in Q_T^R. Let $\Gamma(x,t)$ be a fundamental solution of the equation $Lu = 0$ with a singularity at zero. From [25, 26] it follows that if $\alpha < 2$, then the following estimates are true for $\Gamma(x,t)$, $t > 0$:

$$\mu_1 t^{-\frac{n}{2-\alpha}} e^{-\alpha_1\frac{|x|^{2-\alpha}}{t}} \le \Gamma(x,t) \le \mu_2 t^{-\frac{n}{2-\alpha}} e^{-\alpha_2\frac{|x|^{2-\alpha}}{t}},$$

with $\Gamma(x,t) = 0$ for $t \le 0$, where $\mu_1, \mu_2, \alpha_1, \alpha_2$ are positive constants.

Consider the function

$$\Gamma'(x,t) = \sum_q \Gamma(x, t+Tq). \tag{5}$$

Obviously, the series (5) is convergent and determines the solution of the equation $Lu = 0$, According to the estimate from below for $\Gamma(x, t + Tq)$, we obtain

$$\Gamma'(x,t) = \sum_q \Gamma(x, t+Tq) \geq \sum_q \mu_1 |t+Tq|^{-\frac{n}{2-\alpha}} \exp\left\{-\alpha_1 \frac{|x|^{2-\alpha}}{t+Tq}\right\} \geq$$

$$\geq C \int_{-\frac{t}{T}}^{\infty} |t+Ts|^{-\frac{n}{2-\alpha}} \exp\left\{-\alpha_1 \frac{|x|^{2-\alpha}}{t+Ts}\right\} ds - C|x|^{-n} \geq$$

$$\geq C|x|^{2-n-\alpha} - C|x|^{-n} \geq C|x|^{2-n-\alpha}.$$

As $u(x,t) > 0$ on S_R, there exists a constant $\beta_0 = const > 0$, such that $u(x,t) \geq \beta_0 \Gamma'(x,t)$. Assume $W(x,t) = u(x,t) - \beta_0 \Gamma'(x,t)$. Then we have

$$LW(x,t) \leq 0,\ W(x,t) \geq 0 \ \text{ on } \ S_R \ \text{ and} \, W(x,t+T) = W(x,t).$$

Because of $u(x,t) \geq 0$ in $Q_T^{R,\infty}$ and $\Gamma'(x,t) \to 0$ as $|x| \to \infty$, we have $lim_{|x|\to\infty} W(x,t) \geq 0$.

The maximum principle implies $W(x,t) \geq 0$ in $Q_T^{R,\infty}$.

Therefore, $u(x,t) \geq \beta_0 |x|^{2-n-\alpha}$.

Now let $\alpha = 2 - n$.

Consider the following auxiliary problem

$$Lv_\rho = 0 \ \text{ in } \ Q_T^{R,\rho}, \tag{6}$$

$$v_\rho|_{|x|=R} = 1, \quad v_\rho|_{|x|=\rho} = 0, \quad v_\rho(x, t+T) = v_\rho(x,t). \tag{7}$$

Let's prove that $v_\rho \uparrow 1$ as $\rho \to \infty$.

Consider the following function

$$\varphi_\rho(x) = \begin{cases} 1, & |x| \leq \rho^{\frac{1}{e}}, \\ 1 + \ln\ln \rho^{\frac{1}{e}} - \ln\ln|x|, & \rho^{\frac{1}{e}} < |x| < \rho. \end{cases}$$

Let's take $v_\rho - \varphi_\rho$ as a test function in the definition of solution of the problem (6),(7).

Then we obtain

$$2\pi \sum_{k=-\infty}^{+\infty} (ik) \int_{B_{R,\rho}} v_{\rho_k} (v_\rho - \varphi_\rho)_{-k} \, dx +$$

$$+\iint\limits_{Q_T^{R,\rho}} |x|^{2-n} \left(A\nabla v_\rho, \nabla(v_\rho - \varphi_\rho)\right) dxdt = 0. \tag{8}$$

Just like in Lemma 1, it is easy to show that the first term is equal to zero.

Then from (8) we obtain

$$\iint\limits_{Q_T^{R,\rho}} |x|^{2-n} \left(A\nabla v_\rho, \nabla v_\rho\right) dxdt = \iint\limits_{Q_T^{R,\rho}} |x|^{2-n} \left(A\nabla v_\rho, \nabla \varphi_\rho\right) \le$$

$$\le \frac{1}{2}\iint\limits_{Q_T^{R,\rho}} |x|^{2-n} \left(A\nabla v_\rho, \nabla v_\rho\right) dxdt + \frac{1}{2}\nu_2 \iint\limits_{Q_T^{R,\rho}} |x|^{2-n} \left|\nabla \varphi_\rho\right|^2 dxdt.$$

Hence

$$\iint\limits_{Q_T^{R,\rho}} |x|^{2-n} \left(A\nabla v_\rho, \nabla v_\rho\right) dxdt \le \nu_2 \iint\limits_{Q_T^{R,\rho}} |x|^{2-n} \left|\nabla \varphi_\rho\right|^2 dxdt \le$$

$$\le C\int\limits_{\rho^{\frac{1}{e}}}^{\rho} \frac{dr}{r\ln^2 r} = C\left(-\frac{1}{\ln r}\right) /_{\rho^{\frac{1}{e}}}^{\rho} =$$

$$= C\left(e-1\right)\frac{1}{\ln \rho} \to 0 \quad \text{as} \quad \rho \to \infty.$$

It follows from the maximum principle that $v_{\rho_1} \le v_{\rho_2}$ for $\rho_1 \le \rho_2$. Therefore, $v_\rho \uparrow 1$ as $\rho \to \infty$.

Obviously, for every $\rho > R^e$ we have $u(x,t) \ge C_2 v_\rho$, where $C_2 = \frac{1}{2}\min_{|x|=R} u(x,t)$. Then, passing here to the limit as $\rho \to \infty$, we get the statement of the lemma.

So Lemma 2 is proved. ◄

Let's prove the following analog of the Caccioppoli inequality.

Lemma 3. *Let $\alpha < 2$, and $v\,(x,t)$ be a non-negative solution of the equation $Lv + \beta^2 |x|^{\alpha-2} v = 0$ in $Q_T^{R,\infty}$. Then the following inequality holds for $\rho > 2R$:*

$$\iint\limits_{Q_T^{\rho,2\rho}} |x|^{\alpha} |\nabla v|^2 dxdt \le C \int\limits_{Q_T^{\rho/2,5\rho 2}} |x|^{\alpha-2} v^2 dxdt.$$

Proof.

Consider the following function

$$\eta(x)=\begin{cases} 0, & \text{as} \quad |x|\le \frac{\rho}{2},\\ \frac{2}{\rho}\left(|x|-\frac{\rho}{2}\right), & \text{as} \quad \frac{\rho}{2}\le\ |x|\le\rho,\\ 1, & \text{as} \quad \rho\le\ |x|\le 2\rho,\\ \frac{2}{\rho}\left(\frac{5\rho}{2}-|x|\right), & \text{as} \quad 2\rho\le\ |x|\le\frac{5\rho}{2},\\ 0, & \text{as} \quad |x|\le\frac{5\rho}{2}. \end{cases}$$

Take $\eta^2(x)v(x,t)$ as a test function in the definition of solution. Then we obtain

$$2\pi\sum_{k=-\infty}^{+\infty}(ik)\int\limits_{B_{\rho/2,5\rho/2}} v_k\left(\eta^2 v\right)_{-k}dx+\iint\limits_{Q_T^{\rho/2,5\rho/2}}\eta^2|x|^{\alpha}\left(A\nabla v,\nabla v\right)dxdt+$$

$$+\iint\limits_{Q_T^{\rho/2,5\rho/2}}2\eta v|x|^{\alpha}\left(A\nabla v,\nabla\eta\right)dxdt-\beta^2\iint\limits_{Q_T^{\rho/2,5\rho/2}}|x|^{\alpha-2}\eta^2v^2dx=0.$$

As in Lemma 1, it is easy to show that the first term is equal to zero. Using the inequality $2ab\le\varepsilon a^2+\frac{1}{\varepsilon}b^2$ in the third integral, we get the statement of the lemma. ◄

Lemma 4. *Let $\alpha<2$, $0\le W(x,t)\in L_{\infty,loc}(Q_T^{R,\infty})$, $W(x,t+T)=W(x,t)$ and $|x|^{2-\alpha}W(x,t)\to\infty$ as $x\to\infty$. Then there exists no positive solution in $Q_T^{R,\infty}$ to the inequality*

$$Lu+W(x,t)u\le 0.$$

Proof. Assume the contrary, i.e. assume there is a positive solution $u(x,t)$. Then all the conditions of Lemma 1 hold. Let the function $f(x)$ satisfy the following conditions: $f\in C_0^{\infty}(B_{\rho,2\rho})$, $0\le f\le 1$, $f=1$ for $\frac{5\rho}{4}\le|x|\le\frac{7\rho}{4}$ and $|\nabla f|<\frac{5}{\rho}$. Then, by Lemma 1,

$$\inf_{Q_T^{\rho,2\rho}}W(x,t)\int\limits_{B_{\rho,2\rho}}f^2dx\le\frac{1}{T}\iint\limits_{Q_T^{\rho,2\rho}}W(x,t)f^2dx\le$$

$$\le C\int\limits_{B_{\rho,2\rho}}|x|^{\alpha}|\nabla f|^2dx.$$

Hence

$$\rho^n\inf_{Q_T^{\rho,2\rho}}W(x,t)\le C\rho^{\alpha-2+n},$$

and

$$|x|^{2-\alpha}W(x,t) \le C,$$

which contradicts the conditions of the lemma.

Lemma 4 is proved. ◄

Lemma 5. *Let $\alpha = 2-n$, $0 \le W(x,t) \in L_{\infty,loc}(Q_T^{R,\infty})$, $W(x,t+T) = W(x,t)$ and $|x|^n \ln|x| W(x,t) \to \infty$ as $|x| \to \infty$. Then there exists no positive solution in $Q_T^{R,\infty}$ to the inequality*

$$Lu + W(x,t)u \le 0.$$

Proof. Assume the contrary, i.e. assume there is a positive solution $u(x,t)$. Consider the following function

$$f(x) = \begin{cases} 0, & \text{as} \quad |x| \le \rho^{\frac{1}{e}}, \\ 1 + \ln\ln|x| - \ln\ln\rho, & \text{as} \quad \rho^{\frac{1}{e}} \le |x| \le \rho, \\ 1, & \text{as} \quad \rho \le |x| \le 2\rho, \\ 1 + \ln\ln(2\rho) - \ln\ln|x|, & \text{as} \quad 2\rho \le |x| \le (2\rho)^e, \\ 0, & \text{as} \quad |x| \ge (2\rho)^e, \end{cases}$$

where $\rho > R^e$.

Again, applying Lemma 1 to such $f(x)$, we obtain

$$\rho^n \inf_{Q_T^{\rho,2\rho}} W(x,t) \le C \iint\limits_{Q_T^{\rho^{\frac{1}{e}},(2\rho)^e}} W(x,t) f^2 dxdt \le$$

$$\le C \int\limits_{B_{\rho^{\frac{1}{e}},(2\rho)^e}} |x|^{2-n} |\nabla f|^2 dx =$$

$$= C \left(\int\limits_{B_{\rho^{\frac{1}{e}},\rho}} |x|^{2-n} |\nabla f|^2 dx + \int\limits_{B_{2\rho,(2\rho)^e}} |x|^{2-n} |\nabla f|^2 dx \right) \le$$

$$\le C \left(\int\limits_{\rho^{\frac{1}{e}}}^{\rho} \frac{1}{r\ln^2 r} dr + \int\limits_{2\rho}^{(2\rho)^e} \frac{1}{r\ln^2 r} dr \right) =$$

$$= C \left(\frac{1}{\ln\rho}(e-1) + \frac{1}{\ln(2\rho)}\left(1 - \frac{1}{e}\right) \right) \le C \frac{1}{\ln\rho}.$$

Hence

$$\rho^n \ln \rho \inf_{Q_T^{\rho,2\rho}} W(x,t) \le C,$$

$$|x|^n \ln |x| W(x,t) \le C,$$

which contradicts the conditions of the lemma. So Lemma 5 is proved. ◄

3. Main results and their proofs

The main results of this work are the following theorems.

Theorem 1. *Let $n \ge 3$, $q > 1$, $2 - n \le \alpha < 2$, $a_0(x,t) \ge C|x|^\sigma$, $\sigma \in R$, $C = const > 0$. If $\sigma + 2 - \alpha + (2 - n - \alpha)(q - 1) \ge 0$, then the equation (1) has no positive solution in $Q_T^{R,\infty}$.*

Proof. For simplicity, we assume $R = 1$.

I. Let first $2+\sigma-\alpha+(2-n-\alpha)(q-1) > 0$. Denote $W(x,t) = a_0(x,t)|u|^{q-1}$. If $u(x,t)$ is a positive solution of the equation (1), then all the conditions of Lemma 2 are satisfied. So by Lemma 2 we have $W(x,t) \ge C|x|^\sigma |x|^{(2-n-\alpha)(q-1)} = C|x|^{\alpha-2}|x|^{\sigma+2-\alpha+(2-n-\alpha)(q-1)}$. Hence

$$\lim_{|x|\to\infty} |x|^{2-\alpha} W(x,t) = \infty.$$

Then, by Lemma 4, the equation (1) has no positive solution in $Q_T^{1,\infty}$.

II. Now let $\sigma + 2 - \alpha + (2 - n - \alpha)(q - 1) = 0$ and $\alpha \ne 2 - n$. If the equation (1) has a positive solution $u(x,t)$, then, by Lemma 2

$$Lu + \beta^2 |x|^{\alpha-2} u \le 0.$$

Consider the following linear equation in $Q_T^{1,\infty}$:

$$Lv + \beta^2 |x|^{\alpha-2} v = 0. \tag{9}$$

Assume that the equation (9) has a non-negative solution $v(x,t)$.

In the definition of solution of the equation (9), we take the test function $\varphi(x,t)$ as follows:

$$0 \le \varphi(x,t) = \varphi(x) \in C_0^\infty, \ \ |\nabla\varphi| \le C/|x|^2,$$

$$\varphi(x) = \begin{cases} 0, & \text{as} \quad |x| \le 1, \\ 1, & \text{as} \quad 2 \le |x| \le \rho, \\ 0, & \text{as} \quad |x| \ge 2\rho. \end{cases}$$

Then

$$\beta^2 \iint\limits_{Q_T^{1,\infty}} |x|^{\alpha-2} v\varphi dxdt = \iint\limits_{Q_T^{1,\infty}} |x|^{\alpha}(A\nabla u, \nabla\varphi)dxdt. \tag{10}$$

Let's estimate the left-hand side of (10) from below, and the right-hand side from above. By Lemma 2, we obtain

$$\beta^2 \iint\limits_{Q_T^{1,\infty}} |x|^{\alpha-2} v\varphi dxdt \geq \beta^2 \iint\limits_{Q_T^{2,\rho}} |x|^{\alpha-2} v dxdt \geq \beta^2 \iint\limits_{Q_T^{2,\rho}} |x|^{\alpha-2+2-n-\alpha} dxdt \geq$$

$$\geq \beta_2^2 \int\limits_2^\rho \frac{dr}{r} = \beta_2^2 \ln\frac{\rho}{2}, \tag{11}$$

$$\iint\limits_{Q_T^{1,\infty}} |x|^{\alpha}(A\nabla v, \nabla\varphi)dxdt = \iint\limits_{Q_T^{1,2}} |x|^{\alpha}(A\nabla v, \nabla\varphi)dxdt +$$

$$+ \iint\limits_{Q_T^{\rho,2\rho}} |x|^{\alpha}(A\nabla v, \nabla\varphi)dxdt \leq C + \iint\limits_{Q_T^{\rho,2\rho}} |x|^{\alpha}(A\nabla v, \nabla v)^{\frac{1}{2}}(A\nabla\varphi, \nabla\varphi)^{\frac{1}{2}} dxdt \leq$$

$$\leq C + \left(\iint\limits_{Q_T^{\rho,2\rho}} |x|^{\alpha}|\nabla v|^2 dxdt\right)^{\frac{1}{2}} \left(\int\limits_{\beta_{\rho,2\rho}} |x|^{\alpha}|\nabla\varphi|^2 dx\right)^{\frac{1}{2}} \leq$$

$$\leq C + C\rho^{\frac{n-2+\alpha}{2}} \left(\iint\limits_{Q_T^{\rho,2\rho}} |x|^{\alpha}|\nabla v|^2 dxdt\right)^{\frac{1}{2}}. \tag{12}$$

Using Harnack inequality and Lemma 3, we estimate the latter integral as follows

$$\left(\iint\limits_{Q_T^{\rho,2\rho}} |x|^{\alpha}|\nabla v|^2 dxdt\right)^{\frac{1}{2}} \leq C \left(\iint\limits_{Q_T^{\rho/2,5\rho/2}} |x|^{\alpha-2} v^2 dxdt\right)^{\frac{1}{2}} \leq$$

$$\leq C\rho^{\frac{n-2+\alpha}{2}} \min_{Q_T^{\rho/2,5\rho/2}} v. \tag{13}$$

Then, by virtue of (11), (12) and (13), from (10) we obtain

$$\beta_2^2 \ln \frac{\rho}{2} \le C + C\rho^{n-2+\alpha} \min_{Q_T^{\rho/2,5\rho/2}} v.$$

Hence

$$\min_{Q_T^{\rho/2,5\rho/2}} v \ge \frac{1}{C}\rho^{2-n-\alpha}(\beta^2 \ln \frac{\rho}{2} - C).$$

As a result

$$v(x,t) \ge C|x|^{2-n-\alpha} \ln \frac{|x|}{C}. \tag{14}$$

Now let's show that the equation (9) has in fact positive solutions. For this aim, consider the following auxiliary problem:

$$LV_R + \beta^2 |x|^{\alpha-2} V_R = 0, \tag{15}$$

$$V_R\big|_{|x|=1} = 1, \quad V_R\big|_{|x|=R} = 0. \tag{16}$$

Obviously, the problem (15), (16) has a solution V_R. Let's show that $V_R \ge 0$ and $V_R \le 1$. We first prove that $V_R \le 1$. Assume the contrary, i.e. assume there exists $Q' \in Q_T^{1,\infty}$ such that $V_R > 1$ in Q'.

Let $\psi(x,t) = V_R - 1$ for $(x,t) \in Q'$ and $\psi(x,t) = 0$ for $(x,t) \notin Q'$. It is clear that $\psi(x,t) \in \overset{\circ}{W}{}_2^{1,\frac{1}{2}}\left(Q_T^{1,R}\right)$. Then, if we take $V_R\ \psi(x,t)$ as a test function in the definition of solution, we get

$$2\pi \sum_{k=-\infty}^{+\infty} \int\limits_{1<|x|<R} V_{R_k}(x)\psi_{-k}(x)dx + \iint\limits_{Q'} |x|^\alpha \sum_{i,j=1}^{n} a_{ij}(x,t) \frac{\partial V_R}{\partial x_j} \frac{\partial (V_R - 1)}{\partial x_i} dxdt -$$

$$-\beta^2 \iint\limits_{Q'} |x|^{\alpha-2} V_R (V_R - 1) dxdt = 0.$$

Using the averaging $\psi_{\bar{h}}(x,t) = h^{-1} \int\limits_{t-h}^{t} \psi(x,\tau)d\tau$ and then passing to the limit as $h \to 0$, we obtain as in the proof of Lemma 1 that the first term in the latter equality is equal to zero.

As $\alpha \ne 2 - n$, using the Poincare inequality and taking into account the condition (2) we obtain

$$\nu_1 \iint\limits_{Q'} |x|^\alpha |\nabla V_R|^2 dxdt + \beta^2 \iint\limits_{Q'} |x|^{\alpha-2} V_R =$$

$$= \beta^2 \iint\limits_{Q'} |x|^{\alpha-2} V_R^2 dxdt \le \beta^2 C \int\limits_{Q'} |x|^{\alpha} |\nabla V_R|^2 dxdt.$$

Hence,

$$(\nu_1 - \beta^2 C) \iint\limits_{Q'} |x|^{\alpha} |\nabla V_R|^2 dxdt + \beta^2 \iint\limits_{Q'} |x|^{\alpha-2} V_R dxdt \le 0.$$

As β^2 can be chosen sufficiently small, all the terms on the left-hand side of this inequality are positive. Then we obtain a contradiction. Therefore, $V_R \le 1$ in $Q_T^{1,R}$. Similarly we can show that $V_R \ge 0$.

As $V_R \ge 0$ and $V_R \le 1$ for every R and V_R is a solution of the problem (15), (16), the function $V_R(x,t)$ converges uniformly, in every compact subset, to some function $V(x,t)$, which is the sought non-negative solution of the equation (9). Consider the function $W_R(x,t) = u(x,t) - CV_R(x,t)$, where $C = \frac{1}{2} \min_{|x|=1} u(x,t)$. Then

$$LW_R + \beta^2 |x|^{\alpha-2} W_R \le 0,$$

$$W_R > 0 \text{ for } |x| = 1, \; W_R \ge 0 \text{ for } |x| = R, \; W_R(x, t+T) = W_R(x,t).$$

From here, similar to the case $W_R \le 1$, it is easy to derive that $W_R(x,t) \ge 0$ in $Q_T^{1,R}$.

As a result, we have $u(x,t) \ge CV_R(x,t)$ for every R. Then, passing to the limit as $R \to \infty$ and taking into account (14), we obtain

$$u(x,t) \ge C|x|^{2-n-\alpha} \ln \frac{|x|}{C}.$$

Using this inequality, as in case I, we arrive at the conclusion that the equation (1) has no positive solution for $\alpha \ne 2-n$, $\sigma + 2 - \alpha + (2-n-\alpha)(q-1) = 0$, too.

III. Now let $\alpha = 2-n$.

Denote $W(x,t) = a_0(x,t)|u|^{q-1}$. By Lemma 2, for $\alpha = 2-n$ we have $u(x,t) \ge C$. Then

$$W(x,t) \ge C|x|^{\sigma} = C|x|^{\alpha-2}|x|^{\sigma+2-\alpha} = C|x|^{-n}|x|^{\sigma+n}.$$

Consequently, for $\sigma \ge -n$ we have

$$\lim_{|x|\to\infty} |x|^n \ln|x| W(x,t) \ge \lim_{|x|\to\infty} \ln|x| |x|^{\sigma+n} = \infty.$$

Hence, by Lemma 5, the equation (1) has no positive solution.

So Theorem 1 is completely proved. ◄

Theorem 2. *Let $n \geq 3$, $q > 1$, $\alpha < 2-n$, $a_0(x,t) \geq C|x|^{\sigma}$, $\sigma \in R$, $C = const > 0$. Then, for every $\sigma \in (-\infty, +\infty), q > 1$ the equation* (1) *has no positive solution in $Q_T^{R,\infty}$.*

Proof. Again, for simplicity we assume $R = 1$. Let $u(x,t) > 0$ be a solution of the equation (1). Take $u^{-\theta}\varphi^s$ as a test function in the definition of solution, where $0 < \theta < 1$, $s = 2\frac{q-\theta}{q-1}$, $\varphi(x) = \xi(x)\psi(x)$,

$$\xi(x) = \begin{cases} \sin(\frac{|x|-1}{\rho-1}\frac{\pi}{2}), & 1 \leq |x| \leq \rho, \\ 1, & |x| \geq \rho, \end{cases}$$

$$\psi(x) = \begin{cases} 1, & 1 \leq |x| \leq \rho, \\ (2 - \frac{|x|}{\rho}), & \rho \leq |x| \leq \rho, \\ 0, & |x| \geq \rho. \end{cases}$$

Then we obtain

$$\iint\limits_{Q_T^{1,\infty}} |x|^{\sigma} |u|^{q-\theta} \varphi^s dxdt \leq$$

$$\leq 2\pi \sum_{k=-\infty}^{+\infty} (ik) \int\limits_{B_1'} u_k(x)\, \varphi_{-k}(x)\, dx + (-\theta) \iint\limits_{Q_T^{1,\infty}} |x|^{\alpha} (A\nabla u, \nabla u)\, u^{-\theta-1}\varphi^s dxdt +$$

$$+ s \iint\limits_{Q_T^{1,\infty}} |x|^{\alpha} (A\nabla u, \nabla\varphi)\, u^{-\theta}\varphi^{s-1} dxdt.$$

Obviously, the first term on the right-hand side is equal to zero. Then, using the inequalities

$$(A\nabla u, \nabla\varphi) \leq (A\nabla u, \nabla u)^{\frac{1}{2}} (A\nabla\varphi, \nabla\varphi)^{\frac{1}{2}},$$

$$ab \leq \frac{\varepsilon}{2}a^2 + \frac{1}{2\varepsilon}b^2,$$

we obtain

$$\iint\limits_{Q_T^{1,\infty}} |x|^{\sigma} |u|^{q-\theta} \varphi^s dxdt \leq (-\theta) \iint\limits_{Q_T^{1,\infty}} |x|^{\alpha} (A\nabla u, \nabla u)\, u^{-\theta-1}\varphi^s dxdt +$$

$$+ \frac{|\theta|}{2} \iint\limits_{Q_T^{1,\infty}} |x|^{\alpha} (A\nabla u, \nabla u)\, u^{-\theta-1}\varphi^s dxdt + \frac{s^2}{2|\theta|} \iint\limits_{Q_T^{1,\infty}} |x|^{\alpha} (A\nabla\varphi, \nabla\varphi)\, u^{-\theta+1}\varphi^{s-2} dxdt \leq$$

$$\leq \frac{s^2}{2|\theta|} \iint\limits_{Q_T^{1,\infty}} |x|^{\alpha} |\nabla\varphi|^2 u^{-\theta+1} \varphi^{s-2} dxdt \leq \left(\iint\limits_{Q_T^{1,\infty}} |x|^{\sigma} |u|^{q-\theta} \varphi^s dxdt \right)^{\frac{1}{p}} \times$$

$$\times \left(\iint\limits_{Q_T^{1,\infty}} \frac{|x|^{\alpha p'} |\nabla\varphi|^{2p'} \varphi^{s-2p'}}{|x|^{\sigma(p'-1)}} dxdt \right)^{\frac{1}{p'}},$$

where $p = \frac{q-\theta}{1-\theta}$, $p' = \frac{q-\theta}{q-1}$.

Hence

$$\iint\limits_{Q_T^{1,\infty}} |x|^{\sigma} |u|^{q-\theta} \varphi^s dxdt \leq \iint\limits_{Q_T^{1,\infty}} \frac{|x|^{\alpha p'} |\nabla\varphi|^{2p'} \varphi^{s-2p'}}{|x|^{\sigma(p'-1)}} dxdt =$$

$$= C \int\limits_{B_{1,\rho}} \frac{|x|^{\alpha p'} |\nabla\xi|^{2p'} \psi^{2p'}}{|x|^{\sigma(p'-1)}} dx + C \int\limits_{B_{\rho,2\rho}} \frac{|x|^{\alpha p'} |\nabla\psi|^{2p'} \xi^{2p'}}{|x|^{\sigma(p'-1)}} dx \leq$$

$$\leq C \int\limits_1^{\rho} \left| \frac{\partial\xi}{\partial r} \right|^{2p'} r^{\alpha p' + n - 1 - \sigma(p'-1)} dr + C \int\limits_{\rho}^{2\rho} \left| \frac{\partial\psi}{\partial r} \right|^{2p'} r^{\alpha p' + n - 1 - \sigma(p'-1)} dr =$$

$$= C \frac{1}{(\rho-1)^{2p'}} \rho^{\alpha p' + n - \sigma(p'-1)} + C \frac{1}{\rho^{2p'}} \rho^{\alpha p' + n - \sigma(p'-1)} \leq$$

$$\leq C\rho^{\alpha p' - 2p' + n - \sigma(p'-1)} = C\rho^{(\alpha-2-\sigma)(p'-1)+\alpha-2+n} =$$

$$= C\rho^{-\frac{\sigma+2-\alpha+(2-n-\alpha)(p-1)}{p-1}} = C\rho^{-\frac{(\alpha-2-\sigma)(1-\beta)+(2-n-\alpha)(q-1)}{q-1}}. \tag{17}$$

If now $\alpha < 2 - n$, then we choose $\theta \in (0,1)$ in such a way that $(\alpha - 2 - \sigma)(1 - \theta) + (2 - n - \alpha)(q - 1) > 0$. Then, passing to the limit as $\rho \to +\infty$, from (17) we obtain

$$\iint\limits_{Q_T^{1,\infty}} |x|^{\sigma} |u|^{q-\theta} \varphi^s dxdt \leq 0.$$

It follows that $u \equiv 0$.

Theorem is proved. ◀

Now let's show that the estimate we obtained for the non-existence of positive solution is exact. To do so, we have to show that for $\sigma+2-\alpha+(2-n-\alpha)(q-1) < 0$ there exists an equation which has a global positive solution.

Consider the equation

$$\frac{\partial u}{\partial t} = div(|x|^{\alpha}\nabla u) + |x|^{\sigma}|u|^{q-1}u. \tag{18}$$

We seek the solution of this equation in the form $u(x,t) = A|x|^{-\mu}$. Substituting it into the equation, we obtain

$$-\mu(\alpha-\mu-2)|x|^{\alpha-\mu-2} - n\mu|x|^{\alpha-\mu-2} + A^{q-1}|x|^{\sigma-q\mu} = 0.$$

Take $\alpha - \mu - 2 = \sigma - q\mu$.
Hence, $\mu = \frac{\sigma+2-\alpha}{q-1}$.
Let's find out when $A > 0$ holds. We have

$$A^{q-1} = \mu(\alpha-\mu-2) + n\mu = \mu(\alpha-2+n-\mu) =$$

$$= \frac{\sigma+2-\alpha}{q-1}\left(\alpha-2+n-\frac{\sigma+2-\alpha}{q-1}\right) =$$

$$= -\frac{\sigma+2-\alpha}{q-1}\frac{\sigma+2-\alpha+(2-n-\alpha)(q-1)}{q-1}.$$

So, if $\sigma+2-\alpha+(2-n-\alpha)(q-1) < 0$, then $A^{q-1} > 0$ and $A > 0$.
Therefore, for $\sigma+2-\alpha+(2-n-\alpha)(q-1) < 0$ the equation (18) has a positive solution

$$u(x,t) = \left[-\frac{\sigma+1-\alpha}{q-1}\frac{\sigma+1-\alpha+(2-n2)(q-1)}{q-1}\right]^{\frac{1}{q-1}}|x|^{\frac{\sigma+2-\alpha}{q-1}}.$$

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