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Absolute Convergence of Orthogonal Expansion in Eigen-Functions of Odd Order Differential Operator

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Abstract. We consider an ordinary differential operator of odd order. Absolute and uniform convergence of orthogonal expansion of the function from the class $W_1^1(G)$, $G = (0, 1)$ in eigenfunctions of the given operator are studied, rate of uniform convergence in the interval $\overline{G} = [0, 1]$ is estimated.

Key Words and Phrases: absolute convergence, uniform convergence, eigenfunction, orthogonal expansion.

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1. Statement of results

Consider the odd order differential operator

$$Lu = u^{(n)} + P_2(x)\, u^{(n-2)} + ... + P_n(x)\, u,$$

on the interval $G = (0, 1)$, where $n = 2m + 1$, $m = 1, 2, ...$, $P_l(x) \in L_1(G)$, $l = \overline{2, n}$.

Denote by $D_n(G)$ a class of functions absolutely continuous on $\overline{G} = [0, 1]$ together with their derivatives up to $(n-1)$-th order $\left(D_n(G) \equiv W_1^{(n)}(G)\right)$.

By the eigenfunction of the operator L corresponding to the eigenvalue λ we understand any indentically nonzero function $u(x) \in D_n(G)$ satisfying the equation $Lu + \lambda u = 0$ almost everywhere in G (see [1]).

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Assume that the system $\{u_k(x)\}_{k=1}^{\infty}$ is a complete system of orthonormal eigenfunctions in the space $L_2(G)$, and $\{\lambda_k\}_{k=1}^{\infty}$ is an appropriate system of eigenvalues, and $\mathrm{Re}\lambda_k = 0$, $k = 1, 2, \ldots.$ Denoting

$$\mu_k = \begin{cases} (-i\lambda_k)^{1/n}, & \mathrm{Im}\lambda_k \geq 0, \\ (i\lambda_k)^{1/n}, & \mathrm{Im}\lambda_k < 0, \end{cases}$$

we define partial sums

$$\sigma_\nu(x, f) = \sum_{\mu_k \leq \nu} f_k u_k(x), \quad \nu > 0,$$

of orthogonal expansion of the function $f(x) \in W_1^1(G)$ in the system $\{u_k(x)\}_{k=1}^{\infty}$, where the Fourier coefficients f_k are defined by the formula $f_k = (f, u_k) = \int_G f(x)\overline{u_k(x)}dx$.

Denote

$$R_\nu(x, f) = f(x) - \sigma_\nu(x, f).$$

In this paper we prove the following theorem.

Theorem 1. *Let the system $\{u_k(x)\}_{k=1}^{\infty}$ be uniformly bounded and the conditions*

$$\left| f(1)\overline{u_k^{(2m)}(1)} - f(0)\overline{u_k^{(2m)}(0)} \right| \leq C_1(f)\mu_k^\alpha, \quad 0 < \alpha < 2m, \quad \mu_k \geq 1; \tag{1}$$

$$\sum_{k=2}^{\infty} \omega_1\left(f', k^{-1}\right) k^{-1} < \infty, \tag{2}$$

be satisfied for the function $f(x) \in W_1^1(G)$ and the system $\{u_k(x)\}_{k=1}^{\infty}$.

Then the orthogonal expansion of the function $f(x)$ in the system $\{u_k(x)\}_{k=1}^{\infty}$ absolutely and uniformly converges on $\overline{G} = [0, 1]$ and the estimate

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \leq const \left\{ C_1(f)\nu^{\alpha-2m} + \sum_{n=[\nu]}^{\infty} n^{-1}\omega_1\left(f', n^{-1}\right) + \right.$$

$$\left. +\nu^{-1}\left(\sum_{l=2}^{2m+1} \nu^{2-l}\|P_l\|_1 + 1\right)\left(\|f\|_\infty + \|f'\|_1\right) \right\}, \quad \nu \geq \nu_0, \tag{3}$$

is valid. Here $\omega_1(g, \delta)$ is the continuity modulus of the function $g(x)$ on $L_1(G)$, $\|P_l\|_p = \|P_l\|_{L_p(G)}$, const is independent of $f(x)$, $\nu_0 = 4\pi/(\min_j |\mathrm{Re}\omega_j|)$, ω_j, $j = \overline{1, 2m+1}$ are the roots of the number $(-1)^{2m+1}$ of degree $(2m+1)$.

Corollary 1. *If* $f'(x) \in H_1^\alpha(G)$, $0 < \alpha \le 1$ *and* $f(0) = f(1) = 0$, *then the conditions (1), (2) of Theorem 1 are satisfied and the estimate (3) takes the following form*

$$\|R_\nu(\cdot, f)\|_{C[0,1]} \le const\nu^{-\alpha} \|f'\|_1^\alpha, \qquad \nu \ge \nu_0.$$

Here $H_1^\alpha(G)$ *is a Nikolski class,* $\|f'\|_1^\alpha = \|f'\|_1 + \sup\limits_{\delta>0} \delta^{-\alpha}\omega_1(f', \delta)$, *const is independent of* $f(x)$.

Note that similar results were obtained in [2-4], for second order differential operators, in [5] for a third order differential operator, and in [6] for an arbitrary differential operator of even order.

2. Proof of the results

To prove the theorem, we must estimate the Fourier coefficients of the function $f(x) \in W_1^1(G)$ in the system $\{u_k(x)\}_{k=1}^\infty$. To this end, we use representation of the eigenfunction $u_k(x)$. Let us introduce the following function

$$R(z) \equiv R_k(z) = \begin{cases} \sum\limits_{j=1}^n \omega_j e^{i\omega_j \mu_k (sign \mathrm{Im}\lambda_k) z}, & n = 4q+1, \\ \sum\limits_{j=1}^n \omega_j e^{-i\omega_j \mu_k (sign \mathrm{Im}\lambda_k) z}, & n = 4q-1, \end{cases}$$

where the numbers $\omega_j, j = \overline{1,n}$, are different roots of the number $(-1)^n$ of n-th degree,

$$X_j^\pm \equiv X_{jk}^\pm(0) = \frac{(i)^{n+1}}{n\mu^{n-1}} \sum_{r=0}^{n-1} (\pm i\mu_k)^r \omega_j^{r-1} u_k^{(n-1-r)}(0);$$

$$M(\xi, u_k) = \frac{(i)^{n-1}}{n\mu_k^{n-1}} \sum_{r=2}^n P_r(\xi) u_k^{(n-r)}(\xi), \quad i = \sqrt{-1}, \quad n = 2m+1.$$

Lemma 1. *(see [7]). If* $\lambda_k \ne 0$, *then the following representation is valid for the eigenfunction* $u_k(x)$:

$$u_k^{(l)}(t) = \sum_{j=1}^n (-i\omega_j\mu_k)^l X_j^- e^{-i\omega_j\mu_k t} + \int_0^1 M(\xi, u_k) \times \tag{4}$$

$$\times R_t^{(l)}(\xi - t)\, d\xi, \quad if \quad n = 4q-1, \quad \mathrm{Im}\lambda_k > 0 \ or$$

$$n = 4q+1, \quad \mathrm{Im}\lambda_k < 0; \quad \ell = \overline{0, n-1};$$

$$u_k^{(l)}(t) = \sum_{j=1}^{n} (i\omega_j \mu_k)^l X_j^+ e^{i\omega_j \mu_k t} + \int_0^t M(\xi, u_k), \tag{5}$$

$$R_t^{(l)}(\xi - t)\, d\xi, \quad \textit{if} \quad n = 4q-1, \quad \mathrm{Im}\lambda_\mathrm{k} < 0 \ \textit{or}$$
$$n = 4q+1, \quad \mathrm{Im}\lambda_\mathrm{k} > 0, \quad \ell = \overline{0, \mathrm{n}-1}.$$

Let us rewrite the formulas (4) and (5) in more convenient form

$$\mu_k^{-l} u_k^{(l)}(t) = \sum_{\mathrm{Im}\omega_\mathrm{j} \le 0} (-i\omega_j)^l X_{jk}^-(0)\, e^{-i\omega_j \mu_k t} +$$
$$+ \sum_{\mathrm{Im}\omega_\mathrm{j} > 0} (-i\omega_j)^l B_{jk}^-(0)\, e^{i\omega_j \mu_k (1-t)} +$$
$$+ \sum_{\mathrm{Im}\omega_\mathrm{j} \le 0} (-i)^l \omega_j^{l+1} \int_0^t M(\xi, u_k)\, e^{i\omega_j \mu_k (\xi - t)} d\xi -$$
$$- \sum_{\mathrm{Im}\omega_\mathrm{j} > 0} (-i)^l \omega_j^{l+1} \int_t^1 M(\xi, u_k)\, e^{i\omega_j \mu_k (\xi - t)} d\xi, \tag{4'}$$

if $n = 4q-1, \ \mathrm{Im}\lambda_\mathrm{k} > 0$ or $n = 4q+1, \ \mathrm{Im}\lambda_\mathrm{k} < 0$;

$$\mu_k^{-l} u_k^{(l)}(t) = \sum_{\mathrm{Im}\omega_\mathrm{j} \le 0} (i\omega_j)^l X_{jk}^+(0)\, e^{i\omega_j \mu_k t} +$$
$$+ \sum_{\mathrm{Im}\omega_\mathrm{j} < 0} (i\omega_j)^l B_{jk}^+(0)\, e^{-i\omega_j \mu_k (1-t)} +$$
$$+ \sum_{\mathrm{Im}\omega_\mathrm{j} \ge 0} (i)^l \omega_j^{l+1} \int_0^t M(\xi, u_k)\, e^{-i\omega_j \mu_k (\xi - t)} d\xi -$$
$$- \sum_{\mathrm{Im}\omega_\mathrm{j} < 0} (i)^l \omega_j^{l+1} \int_t^1 M(\xi, u_k)\, e^{-i\omega_j \mu_k (\xi - t)} d\xi, \tag{5'}$$

if $n = 4q-1$, $\mathrm{Im}\lambda_\mathrm{k} < 0$ or $n = 4q+1$, $\mathrm{Im}\lambda_\mathrm{k} > 0$. In these relations

$$B_{jk}^+(0) = X_{jk}^+(0)\, e^{i\omega_j \mu_k} + \omega_j \int_0^1 M(\xi, u_k)\, e^{-i\omega_j \mu_k (\xi - t)} d\xi,$$

$$B_{jk}^{-}(0) = X_{jk}^{-}(0)\, e^{-i\omega_j\mu_k} + \omega_j \int_0^1 M(\xi, u_k)\, e^{i\omega_j\mu_k(\xi - t)} d\xi.$$

The following estimates are true for the coefficients $X_{jk}^{\pm}(0)$ and $B_{jk}^{\pm}(0)$ (see [8,9], p. 443):

$$\left| X_{jk}^{\pm}(0) \right| \le const \left\| u_k \right\|_2 \le const, \quad \text{if} \quad \mathrm{Im}\omega_{\mathrm{j}} = 0; \tag{6}$$

$$\left| X_{jk}^{-}(0) \right| \le const \left\| u_k \right\|_\infty, \quad \text{if} \quad \mathrm{Im}\omega_{\mathrm{j}} < 0; \tag{7}$$

$$\left| X_{jk}^{+}(0) \right| \le const \left\| u_k \right\|_\infty, \quad \text{if} \quad \mathrm{Im}\omega_{\mathrm{j}} > 0; \tag{8}$$

$$\left| B_{jk}^{-}(0) \right| \le const \left\| u_k \right\|_\infty, \quad \text{if} \quad \mathrm{Im}\omega_{\mathrm{j}} > 0; \tag{9}$$

$$\left| B_{jk}^{+}(0) \right| \le const \left\| u_k \right\|_\infty, \quad \text{if} \quad \mathrm{Im}\omega_{\mathrm{j}} < 0. \tag{10}$$

Lemma 2. *Let $f(x) \in W_1^1(G)$, $\{u_k(x)\}_{k=1}^{\infty}$ be uniformly bounded, and the condition (1) be satisfied. Then for the Fourier coefficients f_k the estimate*

$$|f_k| \le const \left\{ C_1(f)\, \mu_k^{\alpha - 2m - 1} + \mu_k^{-1}\omega_1(f', \mu_k^{-1}) + \right.$$

$$\left. + \mu_k^{-2} \left(1 + \sum_{l=2}^{2m+1} \mu_k^{2-l} \left\| P_l \right\|_1 \right) \left(\left\| f' \right\|_1 + \left\| f \right\|_\infty \right) \right\}, \tag{11}$$

is valid. Here const is independent of $f(x)$ and k; $\mu_k \ge 4\pi \left(\min\limits_j |\mathrm{Re}\omega_{\mathrm{j}}| \right)^{-1}$.

Proof. By the definition of the eigenfunction $u_k(x)$, for Fourier coefficients $f_k, \mu_k \ge 1$ we have

$$f_k = (f, u_k) = \left(f, -\lambda_k^{-1} L u_k \right) =$$

$$= -\left(\overline{\lambda}_k \right)^{-1} \left(f, u_k^{(2m+1)} \right) - \left(\overline{\lambda}_k \right)^{-1} \sum_{l=2}^{2m+1} \left(f, P_l u_k^{(2m-l+1)} \right). \tag{12}$$

To estimate the second term on the right-hand side of this equality, we apply the known estimate (see [8,9])

$$\left\| u_k^{(s)} \right\|_\infty \le const \left(1 + |\mu_k| \right)^s \left\| u_k \right\|_\infty, \quad s = \overline{0, 2m},$$

and take into account uniform boundedness of the system $\{u_k(x)\}_{k=1}^{\infty}$:

$$\left| -\left(\overline{\lambda}_k\right)^{-1} \sum_{l=2}^{2m+1} \left(f, P_l u_k^{(2m-l+1)}\right) \right| \leq$$

$$\leq \frac{\|f\|_\infty}{\mu_k^{2m+1}} \sum_{l=2}^{2m+1} \|P_l\|_1 \left\|u_k^{(2m-l+1)}\right\|_\infty \leq const \mu_k^{-2} \|f\|_\infty \left(\sum_{l=2}^{2m+1} \mu_k^{2-l} \|P_l\|_1\right) \|u_k\|_\infty \leq$$

$$\leq const \mu_k^{-2} \|f\|_\infty \sum_{l=2}^{2m+1} \mu_k^{2-l} \|P_l\|_1 . \qquad (13)$$

We integrate by parts the first term on the right-hand side of (12) and get

$$-\left(\overline{\lambda}_k\right)^{-1} \left(f, P_l u_k^{(2m-l+1)}\right) = -\left(\overline{\lambda}_k\right)^{-1} \left\{ f(1) \overline{u_k^{(2m)}(1)} - f(0) \overline{u_k^{(2m)}(0)} \right\} +$$

$$+ \left(\overline{\lambda}_k\right)^{-1} \int_0^1 f'(x) \overline{u_k^{(2m)}(x)} dx.$$

Taking into account condition (1), we have

$$\left| -\left(\overline{\lambda}_k\right)^{-1} \right| \left| f, u_k^{(2m+1)} \right| \leq C_1(f) \mu_k^{\alpha-2m-1} + \mu_k^{-2m-1} \left| \left(f', u_k^{(2m)}\right) \right| . \qquad (14)$$

To estimate the term $\mu_k^{-2m-1} \left| \left(f', u_k^{(2m)}\right) \right|$, we consider the case $m = 2q$ (i.e. $n = 4q+1$) and apply formulas (4′) and (5′). For simplicity, we consider the case $\mathrm{Im}\lambda_{\mathrm{k}} > 0$. For $l = 2m$, by formula (5′)

$$\mu_k^{-2m-1} \left(f', u_k^{(2m)}\right) = \left(f', \mu_k^{-2m} u_k^{(2m)}\right) \mu_k^{-1} =$$

$$= \mu_k^{-1} \sum_{\mathrm{Im}\omega_{\mathrm{j}} \geq 0} \left(f', (\omega_j \mu_k)^{2m} X_{jk}^{+}(0) e^{i\omega_j \mu_k t}\right) +$$

$$+ \mu_k^{-1} \sum_{\mathrm{Im}\omega_{\mathrm{j}} < 0} \left(f', (i\omega_j)^{2m} B_{jk}^{+}(0) e^{i\omega_j \mu_k (1-t)}\right) +$$

$$+ \mu_k^{-1} \sum_{\mathrm{Im}\omega_{\mathrm{j}} \geq 0} \left(f', i^{2m} \omega_j^{2m+1} \int_0^t M(\xi, u_k) e^{-i\omega_j \mu_k (\xi - t)} d\xi\right) -$$

$$- \mu_k^{-1} \sum_{\mathrm{Im}\omega_{\mathrm{j}} < 0} \left(f', i^{2m} \omega_j^{2m+1} \int_t^1 M(\xi, u_k) e^{-i\omega_j \mu_k (\xi - t)} d\xi\right) . \qquad (15)$$

Let us estimate the terms on the right-hand side of this equality. It is clear that

$$\overline{\left(f',(i\omega_j)^{2m}X^{+}_{jk}(0)\,e^{i\omega_j\mu_k t}\right)}=(i\omega_j)^{2m}X^{+}_{jk}(0)\int\limits_0^1\overline{f}(t)\,e^{i\omega_j\mu_k t}dt,\quad \mathrm{Im}\omega_{\mathrm{j}}\geq 0.$$

Hence, taking into account estimates (6), (8) and uniform boundedness of the system $\{u_k(x)\}_{k=1}^{\infty}$, and applying the inequality

$$\left|\int\limits_0^1\overline{f'(t)}e^{i\omega_j\mu_k t}dt\right|\leq const\left\{\omega_1\left(f',\mu_k^{-1}\right)+\mu_k^{-1}\left|f'\right|_1\right\},\ \ \mu_k\geq 4\pi/\left(\min_j|\mathrm{Re}\omega_{\mathrm{j}}|\right),$$

(see [10,11], Lemma 6) we have

$$\left|\left(f',(i\omega_j)^{2m}X^{+}_{jk}(0)\,e^{i\omega_j\mu_k t}\right)\right|\leq const\left\{\omega_1\left(f',\mu_k^{-1}\right)+\mu_k^{-1}\left\|f'\right\|_1\right\}. \tag{16}$$

For $\mathrm{Im}\omega_{\mathrm{j}}<0$, taking into account estimate (10), uniform boundedness of the system $\{u_k(x)\}_{k=1}^{\infty}$ and inequality

$$\int\limits_0^1\overline{f'(t)}e^{-i\omega_j\mu_k(1-t)}dt\leq const\left\{\omega_1\left(f',\mu_k^{-1}\right)+\mu_k^{-1}\left\|f'\right\|_1\right\}$$

(see [10,11]), we have

$$\left|\left(f',(i\omega_j)^{2m}B^{+}_{jk}(0)\,e^{-i\omega_j\mu_k(1-t)}\right)\right|\leq const\left\{\omega_1\left(f',\mu_k^{-1}\right)+\mu_k^{-1}\left\|f'\right\|_1\right\}. \tag{17}$$

From the uniform boundedness of the system $\{u_k(x)\}_{k=1}^{\infty}$ and estimate (13) we get

$$|M(\xi,u_k)|\leq\frac{const}{\mu_k}\left[\sum_{l=2}^{-2m+1}|P_l(\xi)|\,\mu_k^{2-l}\right]. \tag{18}$$

By inequality (18), we estimate the third and the fourth summands in equality (15) as follows

$$\left|\mu_k^{-1}\sum_{\mathrm{Im}\omega_{\mathrm{j}}\geq 0}\left(f',i^{2m}\omega_j^{2m+1}\int\limits_0^t M(\xi,u_k)\,e^{-i\omega_j\mu_k(\xi-t)}d\xi\right)\right|\leq$$

$$\leq const\ \mu_k^{-2}\left(\sum_{r=2}^{2m+1}\|P_r\|_1\,\mu_k^{2-r}\right)\left\|f'\right\|_1, \tag{19}$$

$$\left|\mu_k^{-1}\sum_{\mathrm{Im}\omega_{\mathrm{j}}<0}\left(f',i^{2m}\omega_j^{2m+1}\int\limits_t^1 M\left(\xi,u_k\right)e^{-i\omega_j\mu_k(\xi-t)}d\xi\right)\right|\le$$

$$\le const\ \mu_k^{-2}\left(\sum_{r=2}^{2m+1}\|P_r\|_1\,\mu_k^{2-r}\right)\|f'\|_1\,. \tag{20}$$

Thus, by inequalities (16), (17), (19) and (20), from (15) we get

$$\mu_k^{-2m-1}\left|\left(f',u_k^{(2m)}\right)\right|\le\frac{const}{\mu_k}\left\{\omega_1\left(f',\mu_k^{-1}\right)+\mu_k^{-1}\|f'\|_1\left(1+\sum_{r=2}^{2m+1}\mu_k^{2-r}\|P_r\|_1\right)\right\}. \tag{21}$$

Considering the inequalities (13), (14) and (21) in (12), for the case $n=4q+1$, $\mathrm{Im}\lambda_{\mathrm{k}}>0$ we get the validity of the estimate (11). Lemma 2 is proved. ◄

Proof of Theorem 1. To prove the theorem, we must show that the series $\sum\limits_{k=1}^{\infty}|f_k|\,|u_k(x)|$ uniformly converges on $\overline{G}=[0,1]$. To this end, we rewrite it in the form

$$\sum_{k=1}^{\infty}|f_k|\,|u_k(x)|=\sum_{0\le\mu_k<\gamma}|f_k|\,|u_k(x)|+\sum_{\mu_k\ge\gamma}|f_k|\,|u_k(x)|\,,$$

$$\gamma=4\pi/\left(\min_j|\mathrm{Re}\omega_{\mathrm{j}}|\right).$$

By orthogonality of the system $\{u_k(x)\}_{k=1}^{\infty}$ in $L_2(G)$ (see [8,9]),

$$\sum_{\tau\le\mu_k\le\tau+1}1\le const,\quad\forall\tau\ge0. \tag{22}$$

From inequality (22) and by uniform boundedness of the system $\{u_k(x)\}_{k=1}^{\infty}$, we have

$$\sum_{0\le\mu_k<\gamma}|f_k|\,|u_k(x)|\le const\,\|f\|_1\sum_{0\le\mu_k<\gamma}1\le const\,\|f\|_1\,.$$

Denote $I(\mu,x)=\sum\limits_{\mu_k\ge\mu}|f_k|\,|u_k(x)|$, where $\mu=\gamma$. Taking into account relations (1), (11) and (22), we get

$$I(\mu,x)=\sum_{\mu_k\ge\mu}|f_k|\,|u_k(x)|\le const\sum_{\mu_k\ge\mu}|f_k|\le$$

$$\leq const\left\{C_1(f)\sum_{\mu_k\geq\mu}\mu_k^{\alpha-2m-1}+\sum_{\mu_k\geq\mu}\mu_k^{-1}\omega_1\left(f',\mu_k^{-1}\right)+\right.$$

$$\left.+\left(\left\|f'\right\|_1+\|f\|_\infty\right)\sum_{\mu_k\geq\mu}\mu_k^{-2}\left(1+\sum_{l=2}^{2m+1}\mu_k^{2-l}\|P_l\|_1\right)\right\}\leq$$

$$\leq const\left\{C_1(f)\sum_{r=[\mu]}^{\infty}\sum_{r\leq\mu_k\leq r+1}\mu_k^{\alpha-2m-1}+\sum_{r=[\mu]}^{\infty}\sum_{r\leq\mu_k\leq r+1}\mu_k^{-1}\omega_1\left(f',\mu_k^{-1}\right)+\right.$$

$$\left.+\left(\left\|f'\right\|_1+\|f\|_\infty\right)\sum_{r=[\mu]}^{\infty}\sum_{r\leq\mu_k\leq r+1}\mu_k^{-2}\left(1+\sum_{l=2}^{2m+1}\mu_k^{2-l}\|P_l\|_1\right)\right\}\leq$$

$$\leq const\left\{C_1(f)\mu^{\alpha-2m}+\sum_{r=[\mu]}^{\infty}r^{-1}\omega_1\left(f',r^{-1}\right)+\left(\left\|f'\right\|_1+\|f\|_\infty\right)\times\right.$$

$$\left.\times\left(\sum_{r=[\mu]}^{\infty}r^{-2}+\sum_{l=2}^{2m+1}\|P_l\|_1\sum_{r=[\mu]}^{\infty}r^{-l}\right)\right\}\leq$$

$$\leq const\left\{C_1(f)\mu^{\alpha-2m}+\sum_{r=[\mu]}^{\infty}r^{-1}\omega_1\left(f',r^{-1}\right)+\right.$$

$$\left.+[\mu]^{-1}\left(\left\|f'\right\|_1+\|f\|_\infty\right)\left(1+\sum_{l=2}^{2m+1}[\mu]^{2-l}\|P_l\|_1\right)\right\}<\infty.$$

Thus, the series $\sum_{k=1}^{\infty}|f_k|\,|u_k(x)|$ uniformly converges on $\overline{G}=[0,1]$, i.e. the series $\sum_{k=1}^{\infty}f_k u_k(x)$ absolutely and uniformly converges on $\overline{G}=[0,1]$. By the completeness of the system $\{u_k(x)\}_{k=1}^{\infty}$ in $L_2(G)$ and absolute continuity of the function $f(x)$, the equality

$$f(x)=\sum_{k=1}^{\infty}f_k u_k(x),\qquad x\in\overline{G}, \tag{23}$$

is valid.

Now estimate the difference $R_\nu(x,f)$. Assume that $\nu\geq\gamma$. Then by equality (25)

$$\|R_\nu(\cdot,f)\|_{C[0,1]}=\|f-\sigma_\nu(\cdot,f)\|_{C[0,1]}=$$

$$= \left\| \sum_{\mu_k > \nu} f_k u_k (\cdot) \right\|_{C[0,1]} \leq \max_{x \in \overline{G}} I(\nu, x) \leq$$

$$\leq const \left\{ C_1(f) \nu^{\alpha - 2m} + \sum_{r=[\nu]}^{\infty} r^{-1} \omega_1 \left(f', r^{-1}\right) + \right.$$

$$\left. + \nu^{-1} \left(\left\| f' \right\|_1 + \| f \|_{\infty} \right) \left(1 + \sum_{l=2}^{2m+1} \nu^{2-l} \| P_l \|_1 \right) \right\}.$$

Thus, the validity of the estimate (3) is proved. Theorem 1 is proved. ◄

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