*Azerbaijan Journal of Mathematics*
*V. 8, No 2, 2018, July*
*ISSN 2218-6816*

# Reconstruction of the Sturm–Liouville Operators with a Finite Number of Tranmission and Parameter Dependent Boundary Conditions

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**Abstract.** In this paper, we study discontinuous Sturm–Liouville problems with the eigenvalue parameter linearly contained in boundary conditions, and the coefficients are recovered by $(i)$ Weyl function and $(ii)$ spectral data. Further eigenparameter appears in both of the boundary conditions.

**Key Words and Phrases**: inverse Sturm–Liouville problem, Weyl M–function, discontinuous and parameter dependent boundary conditions.

**2010 Mathematics Subject Classifications**: 34B20, 34L05, 34B24, 47A10

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## 1. Introduction

We consider the Sturm–Liouville problem

$$\ell y := -y'' + qy = \lambda y, \tag{1}$$

with the eigenparameter dependent boundary conditions

$$U(y) := (\lambda - h_1)y'(0) + (\lambda h - h_2)y(0) = 0, \tag{2}$$
$$V(y) := (\lambda - H_1)y'(\pi) + (\lambda H - H_2)y(\pi) = 0, \tag{3}$$

and discontinuous conditions for $i = 1, \dots, m-1$

$$y(d_i + 0) = a_i y(d_i - 0), \quad y'(d_i + 0) = b_i y'(d_i - 0) + c_i y(d_i - 0), \tag{4}$$

where $q(x) \in L_2(0, \pi)$ is a real–valued function, $h$, $h_1$, $h_2$, $H$, $H_1$, $H_2$, $d_i$, $a_i$, $b_i$, $c_i \in \mathbb{R}$ and $r_1 := h_2 - hh_1 > 0$, $r_2 := HH_1 - H_2 > 0$, $d_0 = 0 < d_1 < d_2 <$

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$\cdots < d_{m-1} < d_m = \pi$ and $a_i b_i > 0$. where $\{d_i\}_{i=1}^{m-1}$ are discontinuous points. For simplicity we use the notation $L := L\,(q(x), d_i, a_i, b_i, c_i, h, h_1, h_2, H, H_1, H_2)$ for the problem (1)–(4).

Spectral problems for Sturm–Liouville operators with eigenvalue dependent boundary conditions have been studied extensively. Boundary value problems with discontinuities inside the interval often appear in mathematics, mechanics, physics, geophysics and other branches of natural sciences. Further, it is known that inverse spectral problems play an important role in the study of some nonlinear evolution equations of mathematical physics.

The inverse problem of recovering higher–order differential operators from the Weyl functions has been studied in [19]. In [1], the Sturm–Liouville problem with discontinuities in the case where an eigenparameter linearly appears not only in the differential equation, but also in both of the boundary conditions is investigated. Paper [2] is dedicated to the study of inverse problems by (i) one spectrum and a sequence of norming constants; (ii) two spectra. Recently these results were extended to finite number of transmission conditions in [15, 24]. We obtain necessary conditions for eigenvalues and norming constants in Section 2. In Section 3 we prove that the operator $L$ in (1)–(4) is unique, also in this section we construct it using Weyl M-function. In Section 4 by using spectral data we obtain inverse problem solution. Furthermore, we present an example in the end of this section. We refer to [1, 2, 3, 4, 7, 8, 10, 11, 13, 14, 16, 17, 18, 20] and [22] for further aspects of this field. For general information on inverse Sturm–Liouville problems we refer (e.g.) to the monographs [6, 9, 12, 19, 21] and [23].

## 2. Preliminaries

Let $\varphi(x,\lambda)$ and $\psi(x,\lambda)$ be the solutions of (1) satisfying the initial conditions

$$\varphi(0,\lambda) = \lambda - h_1, \quad \varphi'(0,\lambda) = -\lambda h + h_2,$$

and

$$\psi(\pi,\lambda) = \lambda - H_1, \quad \psi'(\pi,\lambda) = -\lambda H + H_2,$$

and the jump conditions (4). Without loss of generality, in [15, Lemma 3.2] we can suppose that $a_i b_i = 1$. From the linear differential equations it follows that the Wronskian

$$W(u,v) := u(x)v'(x) - u'(x)v(x),$$

is constant on $[0, d_1) \cup (d_1, d_2) \cup \cdots \cup (d_{m-1}, \pi]$ for two solutions $\ell u = \lambda u$ and $\ell v = \lambda v$ satisfying the transmission conditions (4). Set

$$\chi(\lambda) := W(\varphi(\lambda), \psi(\lambda)) = V(\varphi) = -U(\psi).$$

Then $\chi(\lambda)$ is an entire function whose roots $\lambda_n$ coincide with the eigenvalues of $L$. Define the inner product in the Hilbert space $\mathbb{H} = L_2(0,\pi) \oplus \mathbb{C}^2$ by

$$\langle F, G\rangle_{\mathbb{H}} := \int_0^\pi F_1(x)\overline{G_1}(x)dx + \frac{1}{r_1}F_2\overline{G_2} + \frac{1}{r_2}F_3\overline{G_3},$$

where

$$F := \begin{pmatrix} F_1(x) \\ F_2 \\ F_3 \end{pmatrix}, G := \begin{pmatrix} G_1(x) \\ G_2 \\ G_3 \end{pmatrix} \in \mathbb{H}.$$

Define the operator $T$ in $\mathbb{H}$ by

$$T(F) := \begin{pmatrix} -F_1''(x) + q(x)F_1(x) \\ h_1F_1'(0) + h_2F_1(0) \\ H_1F_1'(\pi) + H_2F_1(\pi) \end{pmatrix},$$

with

$$D(T) = \left\{ F \in \mathbb{H} \;\middle|\; \begin{array}{c} F_1(x) \text{ and } F_1'(x) \in AC[0,d_1) \cup (d_1,d_2) \cdots \cup (d_{m-1},\pi], \\ \ell F_1 \in L_2(0,\pi),\ F_1(d_i+0) = a_iF_1(d_i-0), \\ F_1'(d_i+0) = b_iF_1'(d_i-0) + c_iF_1(d_i-0), \\ F_2 := F_1'(0) + hF_1(0),\ F_3 := F_1'(\pi) + HF_1(\pi). \end{array} \right\}.$$

Denote

$$\Phi_n(x) := \begin{pmatrix} \varphi(x,\lambda_n) \\ \varphi'(0,\lambda_n) + h\varphi(0,\lambda_n) \\ \varphi'(\pi,\lambda_n) + H\varphi(\pi,\lambda_n) \end{pmatrix}.$$

The eigenfunctions of Eqs. (1)–(4) are $\{\Phi_n\}_{n=0}^\infty$. It is easy to see that the set of all eigenfunctions are orthogonal, i.e.

$$\langle \Phi_{n_1}, \Phi_{n_2}\rangle = 0, \quad \text{for } n_1 \neq n_2.$$

Also we note that

$$\varphi(x,\lambda_n) = \begin{cases} \varphi_1(x,\lambda_n), & 0 \le x < d_1, \\ \varphi_2(x,\lambda_n), & d_1 < x < d_2, \\ \vdots & \\ \varphi_m(x,\lambda_n), & d_{m-1} < x \le \pi. \end{cases}$$

We define the norming constants by

$$\Gamma_n = \|\Phi_n\|^2 = \\ = \sum_{i=0}^{m-1} \int_{d_i}^{d_{i+1}} \varphi_{i+1}^2(x,\lambda_n)dx + \frac{(\varphi_1'(0,\lambda_n) + h\varphi_1(0,\lambda_n))^2}{r_1} + \tag{5}$$

$$+\frac{(\varphi_m'(\pi,\lambda_n)+H\varphi_m(\pi,\lambda_n))^2}{r_2},$$

where the functions $\varphi_i(x,\lambda_n)$ for $(i=1,\dots,m)$ are defined in Theorem 1.

**Remark 1.** *The set of eigenvalues and norming constants ($\{\lambda_n,\Gamma_n\}_{n\geq 0}$) is called the spectral data of the problem* (1)–(4).

**Theorem 1.** *The following asymptotic forms hold:*

$$\varphi(x,\lambda)=\begin{cases}\rho^2\cos\rho x+O(\rho\exp(|\tau|x)), & 0\leq x<d_1,\\ \rho^2[\alpha_1\cos\rho x+\alpha_1'\cos\rho(2d_1-x)]+O(\rho\exp(|\tau|x)), & d_1<x<d_2,\\ \rho^2[\alpha_1\alpha_2\cos\rho x+\alpha_1'\alpha_2\cos\rho(2d_1-x)+\alpha_1\alpha_2'\cos\rho(2d_2-x) & \\ \quad+\alpha_1'\alpha_2'\cos\rho(2d_2-2d_1-x)]+O(\rho\exp(|\tau|x)), & d_2<x<d_3,\\ \quad\vdots & \\ \rho^2[\alpha_1\alpha_2\dots\alpha_{m-1}\cos\rho x+ & \\ \quad+\alpha_1'\alpha_2\dots\alpha_{m-1}\cos\rho(2d_1-x)+\cdots & \\ \quad+\alpha_1\alpha_2\dots\alpha_{m-1}'\cos\rho(2d_{m-1}-x)+ & \\ \quad+\alpha_1'\alpha_2'\alpha_3\dots\alpha_{m-1}\cos\rho(2d_2-2d_1-x)+\cdots & \\ \quad+\alpha_1\dots\alpha_i'\dots\alpha_j'\dots\alpha_{m-1}\cos\rho(2d_j-2d_i-x) & \\ \quad+\alpha_1\dots\alpha_i'\dots\alpha_j'\dots\alpha_k'\dots\alpha_{m-1}\cos\rho(2d_k-2d_j-2d_i-x)+\cdots & \\ \quad+\alpha_1'\alpha_2'\dots\alpha_{m-1}'\cos\rho(2d_{m-1}-2d_{m-2}\dots-2(-1)^{m-2}d_2-2(-1)^{m-1}d_1-x)] & \\ \quad+O(\rho\exp(|\tau|x)), & d_{m-1}<x\leq\pi,\end{cases}\tag{6}$$

*where* $\alpha_i=\frac{1}{2}(a_i+b_i)$, $\alpha_i'=\frac{1}{2}(a_i-b_i)$, $\tau=\mathrm{Im}\rho$ *and* $\rho\to\infty$, *and the asymptotic form of eigenvalues is*

$$\rho_n=n+o(n),\tag{7}$$

*where* $\lambda_n=\rho_n^2$ *and* $n\to\infty$. *Also,*

$$\gamma_n=\sum_{i=0}^{m-1}\int_{d_i}^{d_{i+1}}\varphi_{i+1}^2(x,\lambda_n)dx=\rho_n^4\mu(\rho_n;d_i;a_i;b_i)\left[1+O\left(\frac{1}{n}\right)\right]\tag{8}$$

*where* $\mu(\rho_n; d_i; a_i; b_i) =$

$$\begin{cases} \frac{\pi}{2}, \quad \textit{for } m=1, \\ \frac{d_1}{2} + \frac{\pi-d_1}{2}\big(\alpha_1^2 + {\alpha'}_1^2 + \alpha_1\alpha'_1 \cos\rho_n(2d_1)\big), \quad \textit{for } m=2, \\ \frac{d_1}{2} + \frac{d_2-d_1}{2}(\alpha_1^2 + {\alpha'}_1^2 + \alpha_1{\alpha'}_1 \cos\rho_n(2d_1)) \\ \quad +\frac{\pi-d_2}{2}\big(\alpha_1^2\alpha_2^2 + {\alpha'}_1^2\alpha_1^2 + {\alpha'}_1^2{\alpha'}_2^2 + 2\alpha_1\alpha_2{\alpha'}_1\alpha_2\cos\rho_n(2d_1) \\ \quad +\alpha_1^2\alpha_2{\alpha'}_2\cos\rho_n(2d_2) + 2\alpha_1\alpha_2\alpha'_1\alpha'_2\cos\rho_n(2d_2-2d_1) \\ \quad +{\alpha'}_1^2\alpha_2\alpha'_2\cos\rho_n(2d_2-4d_1)\big), \quad \textit{for } m=3, \\ \quad \vdots \\ \frac{d_1}{2} + \frac{d_2-d_1}{2}(\alpha_1^2 + {\alpha'}_1^2 + \alpha_1{\alpha'}_1 \cos\rho_n(2d_1)) \\ \quad +\frac{d_3-d_2}{2}\big(\alpha_1^2\alpha_2^2 + {\alpha'}_1^2\alpha_1^2 + {\alpha'}_1^2{\alpha'}_2^2 + 2\alpha_1\alpha_2{\alpha'}_1\alpha_2\cos\rho_n(2d_1) \\ \quad +2\alpha_1\alpha_2\alpha'_1\alpha'_2\cos\rho_n(2d_2-2d_1) + {\alpha'}_1^2\alpha_2\alpha'_2\cos\rho_n(2d_2-4d_1) \\ \quad +\alpha_1^2\alpha_2{\alpha'}_2\cos\rho_n(2d_2)\big) + \cdots \\ \quad +\frac{1}{2}\big(\alpha_1^2\alpha_2^2\ldots\alpha_{m-1}^2 + {\alpha'}_1^2\alpha_2^2\ldots\alpha_{m-1} + \cdots\alpha_1^2\alpha_2^2\ldots{\alpha'}_{m-1}^2 \\ \quad +\cdots + {\alpha'}_1^2{\alpha'}_2^2\ldots{\alpha'}_{m-1}^2 + \alpha_1{\alpha'}_1\alpha_2^2\ldots\alpha_{m-1}^2\cos\rho_n(2d_1) \\ \quad +\alpha_1^2\alpha_2\alpha'_2\alpha_3^2\ldots\alpha_{m-1}^2\cos\rho_n(2d_2) \\ \quad +\alpha_1^2\alpha_2^2\ldots\alpha_{m-1}\alpha'_{m-1}\cos\rho_n(2d_{m-1}) + \cdots \\ \quad +{\alpha'}_1^2\alpha_2{\alpha'}_2\ldots\alpha_{m-1}\alpha'_{m-1}\cos\rho_n(2[(-1)^{m-1}+1]d_1 + 2(-1)^{m-2}d_2 + \cdots - 2d_{m-1}) \\ \quad +\alpha_1\alpha'_1\alpha'_2\ldots\alpha_{m-1}\alpha'_{m-1}\cos\rho_n(2(-1)^{m-1}d_1 + 2[(-1)^{m-2}+1]d_2 + \cdots - 2d_{m-1}) \\ \quad +\cdots + \alpha_1{\alpha'}_1{\alpha'_2}^2\ldots{\alpha'_{m-1}}^2\cos\rho_n(2d_1) \\ \quad +\cdots + {\alpha'}_1^2{\alpha'_2}^2\ldots\alpha_{m-1}\alpha'_{m-1}\cos\rho_n(2d_{m-1})\big), \quad \textit{for } m\geq 2. \end{cases}$$

*Proof.* Eqs. (6) and (7) have been obtained in [15, Theorems 5.3 and 5.4], and by using (5) and (6) we get (8). ◄

## 3. Reconstruction by Weyl M–function

Using the properties of the spectrum of the Weyl function [15, Section 5], we can define the Weyl M–function by

$$M(\lambda) := -\frac{\psi'(0,\lambda) + h_1\psi(0,\lambda)}{r_1\chi(\lambda)}, \tag{9}$$

$$\theta(x,\lambda) := \frac{\psi(x,\lambda)}{\chi(\lambda)} = S(x,\lambda) - M(\lambda)\varphi(x,\lambda), \tag{10}$$

where $S(x,\lambda)$ is a solution of (1) under the initial conditions $S(0,\lambda) = 0$, $S'(0,\lambda) = 1$, and the jump conditions (4). The functions $\theta(x,\lambda)$ and $M(\lambda)$ are called the Weyl solution and the Weyl function of the operator $L$, respectively. From [15] we have

$$M(\lambda) = -\frac{1}{r_1\lambda} + O\left(\lambda^{-\frac{3}{2}}\right).$$

Now we are ready to prove our uniqueness theorem for the solutions of the problems (1)–(4). For this purpose, together with $L$, we consider a boundary value problem $\tilde{L}$ of the same form but with different coefficients $\tilde{q}(x)$, $\tilde{h}$, $\tilde{h}_1$, $\tilde{h}_2$, $\tilde{H}$, $\tilde{H}_1$, $\tilde{H}_2$, $\tilde{a}_i$, $\tilde{b}_i$, $\tilde{c}_i$, $\tilde{d}_i$. If a certain symbol $\eta$ denotes an object related to $L$, then $\tilde{\eta}$ will denote the analogous object related to $\tilde{L}$.

**Theorem 2** (see [15])**.** *If $M(\lambda) = \tilde{M}(\lambda)$, then $L = \tilde{L}$, i.e., $q(x) = \tilde{q}(x)$, a.e. and $d_i = \tilde{d}_i$, $a_i = \tilde{a}_i$, $h = \tilde{h}$, $h_1 = \tilde{h}_1$, $h_2 = \tilde{h}_2$, $H = \tilde{H}$, $H_1 = \tilde{H}_1$, $H_2 = \tilde{H}_2$.*

*Proof.* Let us define the matrix $P(x,\lambda) = [P_{jk}(x,\lambda)]_{j,k=1,2}$ by the formula

$$P(x,\lambda)\begin{pmatrix}\tilde{\varphi}(x,\lambda) & \tilde{\theta}(x,\lambda)\\ \tilde{\varphi}'(x,\lambda) & \tilde{\theta}'(x,\lambda)\end{pmatrix} = \begin{pmatrix}\varphi(x,\lambda) & \theta(x,\lambda)\\ \varphi'(x,\lambda) & \theta'(x,\lambda)\end{pmatrix}. \tag{11}$$

Using (10) and (11) we have

$$\begin{cases} P_{j1}(x,\lambda) = \varphi^{(j-1)}(x,\lambda)\tilde{\theta}'(x,\lambda) - \theta^{(j-1)}(x,\lambda)\tilde{\varphi}'(x,\lambda),\\ P_{j2}(x,\lambda) = \theta^{(j-1)}(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi^{(j-1)}(x,\lambda)\tilde{\theta}(x,\lambda),\end{cases} \tag{12}$$

$$\begin{cases} \varphi(x,\lambda) = P_{11}(x,\lambda)\tilde{\varphi}(x,\lambda) + P_{12}(x,\lambda)\tilde{\varphi}'(x,\lambda),\\ \theta(x,\lambda) = P_{11}(x,\lambda)\tilde{\theta}(x,\lambda) + P_{12}(x,\lambda)\tilde{\theta}'(x,\lambda).\end{cases} \tag{13}$$

According to (9) and (12), for fixed $x$, the functions $P_{jk}(x,\lambda)$ are meromorphic functions in $\lambda$ with simple poles in the points $\lambda_n$ and $\tilde{\lambda}_n$. Denote $G_\delta^0 = G_\delta \bigcap \tilde{G}_\delta$, where

$$G_\delta = \{\lambda : |\lambda - \lambda_n| > \delta,\ n = 1, 2, ...\}$$

and

$$\tilde{G}_\delta = \{\lambda : |\lambda - \tilde{\lambda}_n| > \delta,\ n = 1, 2, ...\}.$$

From $\rho \in G_\delta$ and using the asymptotic form of $\theta^v(x,\lambda)$ we get $|\theta^v(x,\lambda)| \le C_\delta|\rho|^{v-3}\exp(-|\tau|x)$. Thus it follows that

$$|P_{11}(x,\lambda)| \le C_\delta, \quad |P_{12}(x,\lambda)| \le C_\delta|\rho|^{-1}, \quad \rho \in G_\delta^0. \tag{14}$$

Using (9) and (12) we get

$$P_{11}(x,\lambda) = \varphi(x,\lambda)\tilde{S}'(x,\lambda) - S(x,\lambda)\tilde{\varphi}'(x,\lambda) + (\tilde{M}(\lambda) - M(\lambda))\varphi(x,\lambda)\tilde{\varphi}'(x,\lambda),$$
$$P_{12}(x,\lambda) = S(x,\lambda)\tilde{\varphi}(x,\lambda) - \varphi(x,\lambda)\tilde{S}(x,\lambda) + (M(\lambda) - \tilde{M}(\lambda))\varphi(x,\lambda)\tilde{\varphi}(x,\lambda).$$

Thus, if $M(\lambda) \equiv \tilde{M}(\lambda)$, then for each fixed $x$ the function $P_{1k}(x,\lambda)$ is entire in $\lambda$. Together with (14) this yields $P_{12}(x,\lambda) \equiv 0$, $P_{11}(x,\lambda) \equiv A(x)$. Using (13) we derive

$$\varphi(x,\lambda) \equiv A(x)\tilde{\varphi}(x,\lambda), \quad \theta(x,\lambda) \equiv A(x)\tilde{\theta}(x,\lambda). \tag{15}$$

From $W(\theta(x,\lambda),\varphi(x,\lambda)) \equiv 1$ and $W(\tilde{\theta}(x,\lambda),\tilde{\varphi}(x,\lambda)) \equiv 1$, we have $A(x) = 1$. So from (15) we obtain $\varphi(x,\lambda) \equiv \tilde{\varphi}(x,\lambda)$ and $\theta(x,\lambda) \equiv \tilde{\theta}(x,\lambda)$ for all $x$ and $\lambda$. Consequently, $L = \tilde{L}$. ◀

Now we construct the solution of the inverse problem. For this purpose, we let

$$d_i = \tilde{d}_i, \quad a_i = \tilde{a}_i,$$

and denote

$$\begin{cases} D(x,\lambda,\mu) := \frac{W(\varphi(x,\lambda),\varphi(x,\mu))}{\lambda-\mu} = \int_0^x \varphi(t,\lambda)\varphi(t,\mu)dt, & r(x,\lambda,\mu) = D(x,\lambda,\mu)\hat{M}(\mu), \\ \tilde{D}(x,\lambda,\mu) := \frac{W(\tilde{\varphi}(x,\lambda),\tilde{\varphi}(x,\mu))}{\lambda-\mu} = \int_0^x \tilde{\varphi}(t,\lambda)\tilde{\varphi}(t,\mu)dt, & \tilde{r}(x,\lambda,\mu) = \tilde{D}(x,\lambda,\mu)\hat{M}(\mu). \end{cases} \tag{16}$$

From Theorem 1 we have

$$D(x,\lambda_n,\mu_n) = \begin{cases} \int_0^x \varphi_1(t,\rho_n)\varphi_1(t,\eta_n)dt, & 0 \le x < d_1, \\ \int_0^{d_1} \varphi_1(t,\rho_n)\varphi_1(t,\eta_n)dt + \int_{d_1}^x \varphi_2(t,\rho_n)\varphi_2(t,\eta_n)dt, & d_1 < x < d_2, \\ \int_0^{d_1} \varphi_1(t,\rho_n)\varphi_1(t,\eta_n)dt + \int_{d_1}^{d_2} \varphi_2(t,\rho_n)\varphi_2(t,\eta_n)dt + & \\ \int_{d_2}^x \varphi_3(t,\rho_n)\varphi_3(t,\eta_n)dt, & d_2 < x < d_3, \\ \vdots & \\ \int_0^{d_1} \varphi_1(t,\rho_n)\varphi_1(t,\eta_n)dt + \sum_{i=1}^{m-2} \int_{d_i}^{d_{i+1}} \varphi_{i+1}(t,\rho_n)\varphi_{i+1}(t,\eta_n)dt + & \\ \int_{d_{m-1}}^x \varphi_m(t,\rho_n)\varphi_m(t,\eta_n)dt, & d_{m-1} < x \le \pi, \end{cases}$$

where $\rho_n^2 = \lambda_n$ and $\eta_n^2 = \mu_n$.

Now let us consider the contour $\gamma = \gamma' \cup \gamma''$ where $\gamma'$ is a bounded closed contour encircling the set $\{\lambda = \rho^2 : \text{Im}\rho \ge 0 : \chi(\rho) = 0\}$ and $\gamma''$ is the two-sided cut along the arc $\{\lambda : \lambda > 0,\ \lambda \notin \text{int}\ \gamma'\}$.

**Theorem 3.** *The following relations hold:*

$$\tilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde{r}(x,\lambda,\mu)\varphi(x,\mu)d\mu \tag{17}$$

*and*

$$r(x,\lambda,\mu) - \tilde{r}(x,\lambda,\mu) + \frac{1}{2\pi i}\int_\gamma \tilde{r}(x,\lambda,\xi)r(x,\xi,\mu)d\xi = 0. \tag{18}$$

*The relation (17) is called the main equation of the inverse problem.*

*Proof.* For $\lambda,\ \mu \in \gamma,\ \pm\mathrm{Re}\rho,\ \mathrm{Re}\eta \geq 0$, using Lemma 2.2.1 in [5], we have

$$|r(x,\lambda,\mu)|,\ |\tilde{r}(x,\lambda,\mu)| \leq \frac{C_x}{|\mu|(|\rho \pm \eta|+1)}, \quad |\varphi(x,\lambda)| \leq C.$$

Denote $J_\gamma = \{\lambda:\ \lambda \notin \gamma \cup \gamma'\}$. Consider the contour $\gamma_R = \gamma \cap \{\lambda:\ |\lambda| \leq R\}$ with counterclockwise circuit, and also consider the contour $\gamma_R^0 = \gamma_R \cup \{\lambda:\ |\lambda| = R\}$ with clockwise circuit. By Cauchy's integral formula

$$P_{1k}(x,\lambda) - \delta_{1k} = \frac{1}{2\pi i}\int_{\gamma_R^0} \frac{P_{1k}(x,\mu)-\delta_{1k}}{\lambda-\mu} d\mu, \quad \lambda \in \mathrm{int}\ \gamma_R^0,$$

$$\frac{P_{jk}(x,\lambda) - P_{jk}(x,\mu)}{\lambda-\mu} = \frac{1}{2\pi i}\int_{\gamma_R^0} \frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)} d\xi, \quad \lambda,\mu \in \mathrm{int}\ \gamma_R^0.$$

Using (14) we get

$$\lim_{R\to\infty}\int_{|\mu|=R} \frac{P_{1k}(x,\mu)-\delta_{1k}}{\lambda-\mu} d\mu = 0, \quad \lim_{R\to\infty}\int_{|\xi|=R} \frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)} d\xi = 0.$$

and consequently

$$P_{1k}(x,\lambda) = \delta_{1k} + \frac{1}{2\pi i}\int_{\gamma} \frac{P_{1k}(x,\mu)-\delta_{1k}}{\lambda-\mu} d\mu, \quad \lambda \in J_\gamma, \tag{19}$$

$$\frac{P_{jk}(x,\lambda) - P_{jk}(x,\mu)}{\lambda-\mu} = \frac{1}{2\pi i}\int_{\gamma} \frac{P_{jk}(x,\xi)}{(\lambda-\xi)(\xi-\mu)} d\xi, \quad \lambda,\ \mu \in J_\gamma. \tag{20}$$

By virtue of (13) and (19),

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_{\gamma} \frac{\tilde{\varphi}(x,\lambda)P_{11}(x,\mu) + \tilde{\varphi}'(x,\lambda)P_{12}(x,\mu)}{\lambda-\mu} d\mu, \quad \lambda \in J_\gamma.$$

Taking (11) into account we get

$$\varphi(x,\lambda) = \tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_{\gamma} \Big[\tilde{\varphi}(x,\lambda)(\varphi(x,\mu)\tilde{\theta}'(x,\mu) - \theta(x,\mu)\tilde{\varphi}'(x,\mu)) +$$

$$+\tilde{\varphi}'(x,\lambda)(\theta(x,\mu)\tilde{\varphi}(x,\mu) - \varphi(x,\mu)\tilde{\theta}(x,\mu))\Big]\frac{d\mu}{\lambda-\mu}.$$

In view of (9), this yields (17). According to (20) and the proof of Lemma 1.6.3 in [5], we arrive at

$$D(x,\lambda,\mu) - \tilde{D}(x,\lambda,\mu) =$$

$$= \frac{1}{2\pi i}\int_{\gamma}\left[\frac{W(\tilde{\varphi}(x,\lambda),\tilde{\theta}(x,\xi))W(\varphi(x,\xi),\varphi(x,\mu))}{(\lambda-\xi)(\xi-\mu)} - \frac{W(\tilde{\varphi}(x,\lambda),\tilde{\varphi}(x,\xi))W(\theta(x,\xi),\varphi(x,\mu))}{(\lambda-\xi)(\xi-\mu)}\right]d\xi.$$

In view of (9) and (16) this yields (18). ◄

**Theorem 4.** *The following relations hold:*

$$q(x) = \tilde{q}(x) + \varepsilon(x), \tag{21}$$

$$H = \tilde{H} + \varepsilon_0(\pi), \quad H_2 = \tilde{H}_2 + \varepsilon_0(\pi)\tilde{H}_1, \quad H_1 = \tilde{H}_1, \tag{22}$$

$$h = \tilde{h} + \varepsilon_0(0), \quad h_2 = \tilde{h}_2 + \varepsilon_0(0)\tilde{h}_1, \quad h_1 = \tilde{h}_1 \tag{23}$$

$$c_i = \tilde{c}_i - (a_i^3 - b_i)\varepsilon_0(d_i - 0), \tag{24}$$

*where*

$$\varepsilon_0(x) = \frac{1}{2\pi i}\int_\gamma \tilde{\varphi}(x,\mu)\varphi(x,\mu)M(\mu)d\mu, \ \varepsilon(x) = -2\varepsilon_0'(x). \tag{25}$$

*Proof.* By (16), (17) and (25) we get

$$\tilde{\varphi}'(x,\lambda) - \varepsilon_0(x)\tilde{\varphi}(x,\lambda) = \varphi'(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde{r}(x,\lambda,\mu)\varphi'(x,\mu)d\mu, \tag{26}$$

$$\begin{aligned}\tilde{\varphi}''(x,\lambda) = \varphi(x,\lambda) + \frac{1}{2\pi i}\int_\gamma \tilde{r}(x,\lambda,\mu)\varphi''(x,\mu)d\mu + \frac{1}{2\pi i}\int_\gamma 2\tilde{\varphi}(x,\lambda)\tilde{\varphi}(x,\mu)\\ + M(\mu)\varphi'(x,\mu)d\mu + \frac{1}{2\pi i}\int_\gamma 2(\tilde{\varphi}(x,\lambda)\tilde{\varphi}(x,\mu))'M(\mu)\varphi(x,\mu)d\mu.\end{aligned} \tag{27}$$

In (27) we replace the second derivatives by using equation (1), and we replace $\varphi(x,\lambda)$ using (17). This yields

$$\begin{aligned}\tilde{q}(x)\tilde{\varphi}(x,\lambda) =& q(x)\tilde{\varphi}(x,\lambda) + \frac{1}{2\pi i}\int_\gamma W(\varphi(x,\lambda),\varphi(x,\mu))M(\mu)\varphi(x,\mu)d\mu\\ &+ \frac{1}{2\pi i}\int_\gamma 2\tilde{\varphi}(x,\lambda),\tilde{\varphi}(x,\mu)M(\mu)\varphi'(x,\mu)d\mu\\ &+ \frac{1}{2\pi i}\int_\gamma (\tilde{\varphi}(x,\lambda),\tilde{\varphi}(x,\mu))'M(\mu)\varphi(x,\mu)d\mu.\end{aligned}$$

After canceling terms with $\varphi'(x,\lambda)$ we arrive at (21). Taking $x = 0, \pi$ in (26) and (17) and using Cauchy's theorem, we get (22)–(23). Applying (4) in (25) in the points $d_i$, we get

$$\varepsilon_0(d_i + 0) = a_i^2\varepsilon_0(d_i - 0). \tag{28}$$

From (4), (26) and (28) we obtain (24). ◄

## 4. Reconstruction by spectral data

Let two sequences of real numbers $\{\lambda_n\}$ and $\{\Gamma_n\}, (n \in \mathbb{Z}_+)$ be given with the properties stated in Theorem 1. Now we consider the inverse problem of recovering $L$ from the spectral data $\{\lambda_n, \Gamma_n\}_{n\geq 0}$. Let us choose $\tilde{r}_1 := \tilde{h}_2 - \tilde{h}\tilde{h}_1 > 0$, $\tilde{r}_2 := \tilde{H}\tilde{H}_1 - \tilde{H}_2 > 0$ such that $\omega = \tilde{\omega}$ with $\omega = h + H + \frac{1}{2}\int_0^\pi q(t)dt$ for the operator $\tilde{L}$. Let

$$a_i = \tilde{a}_i,\ d_i = \tilde{d}_i,\ \text{and} \sum_{n=0}^{\infty} \xi_n |\rho_n| < \infty, \tag{29}$$

where $\xi_n := |\rho_n - \tilde{\rho}_n| + |\Gamma_n - \tilde{\Gamma}_n|$. Denote

$$\begin{aligned}
&\lambda_{n0} = \lambda_n, \quad \lambda_{n1} = \tilde{\lambda}_n,\\
&\Gamma_{n0} = \Gamma_n, \quad \Gamma_{n1} = \tilde{\Gamma}_n\\
&\varphi_{ni}(x) = \varphi(x, \lambda_{ni}), \quad \tilde{\varphi}_{ni}(x) = \tilde{\varphi}(x, \lambda_{ni})\\
&\tilde{Q}_{kj}(x, \lambda) = \frac{W(\tilde{\varphi}(x, \lambda), \quad \tilde{\varphi}_{kj}(x))}{\Gamma_{kj}(\lambda - \lambda_{kj})} = \frac{1}{\Gamma_{kj}} \int_0^x \tilde{\varphi}(t, \lambda)\tilde{\varphi}_{kj}(t)dt, \qquad (30)\\
&\tilde{Q}_{ni,kj}(x, \lambda) = \tilde{Q}_{kj}(x, \lambda_{ni}).
\end{aligned}$$

Also we can rewrite (30) as follows:

$$\tilde{Q}_{kj}(x, \lambda) = \frac{1}{\Gamma_{kj}} \begin{cases} \int_0^x \tilde{\varphi}_1(t, \lambda)\tilde{\varphi}_1(t, \lambda_{kj})dt, & 0 \leq x < d_1,\\ \int_0^{d_1} \tilde{\varphi}_1(t, \lambda)\tilde{\varphi}_1(t, \lambda_{kj})dt + \int_{d_1}^x \tilde{\varphi}_2(t, \lambda)\tilde{\varphi}_2(t, \lambda_{kj})dt, & d_1 < x < d_2,\\ \int_0^{d_1} \tilde{\varphi}_1(t, \lambda)\tilde{\varphi}_1(t, \lambda_{kj})dt + \int_{d_1}^{d_2} \tilde{\varphi}_2(t, \lambda)\tilde{\varphi}_2(t, \lambda_{kj})dt + & \\ \int_{d_2}^x \tilde{\varphi}_3(t, \lambda)\tilde{\varphi}_3(t, \lambda_{kj})dt, & d_2 < x < d_3,\\ \vdots & \\ \int_0^{d_1} \tilde{\varphi}_1(t, \lambda)\tilde{\varphi}_1(t, \lambda_{kj})dt + \sum_{r=1}^{m-2} \int_{d_r}^{d_{r+1}} \tilde{\varphi}_{r+1}(t, \lambda)\tilde{\varphi}_{r+1}(t, \lambda_{kj})dt + & \\ \int_{d_{m-1}}^x \tilde{\varphi}_m(t, \lambda)\tilde{\varphi}_m(t, \lambda_{kj})dt, & d_{m-1} < x \leq \pi, \end{cases}$$

where $i, j = 0, 1$ and $n, k \geq 0$.

**Lemma 1** (see [24] Lemma 3.1)**.** *Let $\varphi_{ni}(x)$, $Q_{ni,kj}(x)$ be defined as above. Then the following estimates are valid for $x \in (d_s, d_{s+1})$, $1 \leq s \leq m-1$:*

$$|\varphi_{ni}(x)| \leq C(n+1)^s, \quad |\varphi_{n0}(x) - \varphi_{n1}(x)| \leq C(n+1)^{s-\frac{1}{m}}$$

$$\begin{cases} |Q_{ni,kj}(x)| \leq \frac{C(n+1)^s}{(|n-k|+1)(k+1)^{2(m-1)-s}},\\ |Q_{ni,k0}(x) - Q_{ni,k1}(x)| \leq \frac{C(n+1)^s}{(|n-k|+1)(k+1)^{2(m-1)-s+\frac{1}{m}}},\\ |Q_{n0,kj}(x) - Q_{n1,kj}(x)| \leq \frac{C(n+1)^{s-\frac{1}{m}}}{(|n-k|+1)(k+1)^{2(m-1)-s}}, \end{cases}$$

*where $n, k \geq 0$, $i, j = 0, 1$ and $C$ is a positive constant. The analogous estimates are also valid for $\tilde{\varphi}_{ni}(x)$, $\tilde{Q}_{ni,kj}(x)$.*

**Lemma 2.** *The following relation holds:*

$$\tilde{\varphi}(x,\lambda) = \varphi(x,\lambda) + \sum_{k=0}^{\infty} (\tilde{Q}_{k0}(x)\varphi_{k0}(x) - \tilde{Q}_{k1}(x)\varphi_{k1}(x)). \tag{31}$$

*Proof.* By virtue of (29) we have

$$a = \tilde{a}, \quad \alpha_1 = \tilde{\alpha}_1.$$

It follows from Theorem 1 that

$$|\varphi^{(v)}(x,\lambda)| = O(|\rho|^{v+2} \exp(|\tau|x)), \quad 0 \le x \le \pi, \ v = 0, 1.$$

So,

$$|\varphi^{v}(x,\lambda) - \tilde{\varphi}^{v}(x,\lambda)| \le C|\rho|^{v+1} \exp(|\tau|x).$$

Similarly, by substituting $x$ with $\pi - x$ we get the asymptotic form of $\psi(x,\lambda)$ and $\psi'(x,\lambda)$. In particular,

$$|\psi^{(v)}(x,\lambda)| = O(|\rho|^{v+2} \exp(|\tau|(\pi - x))), \quad 0 \le x \le \pi, \ v = 0, 1,$$
$$|\psi^{v}(x,\lambda) - \tilde{\psi}^{v}(x,\lambda)| \le C|\rho|^{v+1} \exp(|\tau|(\pi - x)). \tag{32}$$

Denote $G_\delta^0 = G_\delta \cap \tilde{G}_\delta$, where $G_\delta$ and $\tilde{G}_\delta$ are defined in Theorem 9. From (10) and (32) we have

$$|\theta^{v}(x,\lambda) - \tilde{\theta}^{v}(x,\lambda)| \le C_\delta |\rho|^{v} \exp(-|\tau|x), \quad \rho \in G_\delta^0 \ v = 0, \ 1.$$

Let $P(x,\lambda)$ be the matrix defined in Theorem 2 and $\Omega = \{\lambda = u + iv : u = (2h^2)^{-2}v^2 - h^2\}$ be the image of the set $\mathrm{Im}\rho = \pm h$ under the mapping $\lambda = \rho^2$. Denote $\Omega_n = \Gamma \cap \{\lambda : |\lambda| \le r_n\}$, and $\Omega_{n0} = \Omega_n \cup \{\lambda : |\lambda| = r_n, \lambda \notin \text{ int } \Omega\}, \Omega_{n1} = \Omega_n \cup \{\lambda : |\lambda| = r_n, \lambda \in \text{ int } \Omega\}$. Since for each fixed $x$, the functions $P_{1k}$ are meromorphic in $\lambda$ with simple poles $\lambda_n$ and $\tilde{\lambda}_n$, we get by Cauchy's theorem

$$P_{1k}(x,\lambda) - \delta_{1k} = \frac{1}{2\pi i} \int_{\Omega_{n0}} \frac{P_{1k}(x,\xi) - \delta_{1k}}{\xi - \lambda} d\xi, \quad k = 1, 2, \tag{33}$$

where $\lambda \in \Omega_{n0}$, and $\delta_{jk}$ is the Kronecker delta. Further, (10) and (12) imply

$$P_{11}(x,\lambda) = 1 + (\varphi(x,\lambda) - \tilde{\varphi}(x,\lambda))\tilde{\Phi}'(x,\lambda) - (\Phi(x,\lambda) - \tilde{\Phi}(x,\lambda))\tilde{\varphi}'(x,\lambda).$$

Also we obtain

$$|P_{1k}(x,\lambda) - \delta_{1k}| \le C_\delta |\rho|^{-1}, \quad \rho \in G_\delta^0. \tag{34}$$

By virtue of (34)

$$\lim_{n\to\infty}\frac{1}{2\pi i}\int_{|\xi|=r_n}\frac{P_{1k}(x,\xi)-\delta_{1k}}{\xi-\lambda}d\xi=0,$$

and consequently, (33) yields

$$P_{1k}(x,\lambda)-\delta_{1k}=\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Omega_{n1}}\frac{P_{1k}(x,\xi)-\delta_{1k}}{\xi-\lambda}d\xi.$$

Substituting into (13) we obtain

$$\varphi(x,\lambda)=\tilde\varphi(x,\lambda)+\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Omega_{n1}}\frac{\tilde\varphi(x,\lambda)P_{11}(x,\xi)+\tilde\varphi'(x,\lambda)P_{12}(x,\xi)}{\lambda-\xi}d\xi.$$

Taking (12) into account we calculate

$$\varphi(x,\lambda)=\tilde\varphi(x,\lambda)+\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Omega_{n1}}\Big[\tilde\varphi(x,\lambda)(\varphi(x,\xi)\tilde\theta'(x,\xi)-\theta(x,\xi)\tilde\varphi'(x,\xi))$$
$$+\tilde\varphi'(x,\lambda)(\theta(x,\xi)\tilde\varphi(x,\xi)-\varphi(x,\xi)\tilde\theta(x,\xi))\Big]\frac{d\xi}{\lambda-\xi},$$

or, in view of (9),

$$\tilde\varphi(x,\lambda)=\varphi(x,\lambda)+\lim_{n\to\infty}\frac{1}{2\pi i}\int_{\Omega_{n1}}\frac{W(\tilde\varphi(x,\lambda),\tilde\varphi(x,\xi))}{\lambda-\xi}\hat M(\xi)\varphi(x,\xi)d\xi. \tag{35}$$

Then we have

$$\mathrm{Res}_{\xi=\lambda_{kj}}\frac{W(\tilde\varphi(x,\lambda),\tilde\varphi(x,\xi))}{\lambda-\xi}\hat M(\xi)\varphi(x,\xi)=\tilde Q_{kj}(x,\lambda)\varphi_{kj}(x).$$

Now with calculation in the integral in (35), by residue theorem we arrive at (31). ◀

It follows from (31) that

$$\tilde\varphi_{ni}(x)=\varphi_{ni}(x)+\sum_{k=0}^{\infty}(\tilde Q_{ni,k0}(x)\varphi_{k0}(x)-\tilde Q_{ni,k1}(x)\varphi_{k1}(x)).$$

Let $K$ be a set of indices $u=(n,i)$, $n\ge 0$, $i=0,1$. For each fixed $x\in[0,d_1)\cup(d_2,d_3)\cup\cdots\cup(d_{m-1},\pi]$, we define the vector

$$\phi(x)=[\phi_u(x)]_{u\in K}=\begin{bmatrix}\phi_{n0}(x)\\ \phi_{n1}(x)\end{bmatrix}_{n\ge 0}=[\phi_{00},\phi_{01},\phi_{10},\phi_{11},\dots]^T$$

by the formulae

$$\begin{bmatrix}\phi_{n0}(x)\\ \phi_{n1}(x)\end{bmatrix} = \begin{bmatrix}\frac{n+1}{\rho_n^{m-1}} & -\frac{n+1}{\rho_n^{m-1}}\\ 0 & \frac{1}{\rho_n^{m-1}}\end{bmatrix}\begin{bmatrix}\varphi_{n0}(x)\\ \varphi_{n1}(x)\end{bmatrix} = \begin{bmatrix}\frac{(n+1)(\varphi_{n0}(x)-\varphi_{n1}(x))}{\rho_n^{m-1}}\\ \frac{\varphi_{n1}(x)}{\rho_n^{m-1}}\end{bmatrix},$$

therefore $\varphi_{n0}, \varphi_{n1}$ can be found by the formula

$$\begin{bmatrix}\varphi_{n0}(x)\\ \varphi_{n1}(x)\end{bmatrix} = \begin{bmatrix}\frac{\rho_n^{m-1}}{n+1} & \rho_n^{m-1}\\ 0 & \rho_n^{m-1}\end{bmatrix}\begin{bmatrix}\phi_{n0}(x)\\ \phi_{n1}(x)\end{bmatrix}.$$

We also define the block matrix

$$H(x) = [H_{u,v}(x)]_{u,v\in K} = \begin{bmatrix}H_{n0,k0}(x) & H_{n0,k1}(x)\\ H_{n1,k0}(x) & H_{n1,k1}(x)\end{bmatrix},$$

$$n, k \geq 0,\ u = (n, i),\ v = (k, j)$$

by the formulae

$$\begin{bmatrix}H_{n0,k0}(x) & H_{n0,k1}(x)\\ H_{n1,k1}(x) & H_{n1,k1}(x)\end{bmatrix} = \begin{bmatrix}\frac{n+1}{\rho_n^{m-1}} & -\frac{n+1}{\rho_n^{m-1}}\\ 0 & \frac{1}{\rho_n^{m-1}}\end{bmatrix}\begin{bmatrix}Q_{n0,k0}(x) & Q_{n0,k1}(x)\\ Q_{n1,k0}(x) & Q_{n1,k1}(x)\end{bmatrix}\begin{bmatrix}\frac{\rho_k^{m-1}}{k+1} & \rho_k^{m-1}\\ 0 & -\rho_k^{m-1}\end{bmatrix} =$$

$$\begin{bmatrix}\frac{n+1}{k+1}(\frac{\rho_k}{\rho_n})^{m-1}(Q_{n0,k0}(x) - Q_{n1,k0}(x)) & \frac{n+1}{\rho_n^{m-1}}(Q_{n0,k0}(x) - Q_{n0,k1}(x) - Q_{n1,k0}(x) + Q_{n1,k1}(x))\\ \frac{\rho_k^{m-1}}{k+1}Q_{n1,k0}(x) & Q_{n1,k0}(x) - Q_{n1,k1}(x)\end{bmatrix}.$$

Analogously we define $\tilde{\phi}(x)$, $\tilde{H}(x)$ by replacing in the previous definitions $\varphi_{ni}(x)$ by $\tilde{\varphi}_{ni}(x)$ and $Q_{ni,kj}(x)$ by $\tilde{Q}_{ni,kj}(x)$. Also for $x \in (d_s, d_{s+1})$ we have

$$|\phi_{ni}(x)|, |\tilde{\phi}_{ni}(x)| \leq \frac{C}{(n+1)^{m-s-1}}. \tag{36}$$

Similarly

$$|H_{ni,kj}(x)|, |\tilde{H}_{ni,kj}(x)| \leq \frac{C}{(|n-k|+1)(n+1)^{m-s-1}(k+1)^{m-s}}. \tag{37}$$

Let us consider the Banach space $m$ of bounded sequences $\alpha = [\alpha_u]_{u\in K}$ with the norm $\|\alpha\|_m = \sup_{u\in K} |\alpha_u|$. It follows from (36) and (37) that for each fixed $x \in [0, \pi]$, the operators $E + \tilde{H}(x)$ and $E - \tilde{H}(x)$ (here $E$ is the identity operator), acting from $m$ to $m$, are linear bounded operators, and

$$\|H(x)\|, \|\tilde{H}(x)\| \leq C \sup_n \sum_{k=0}^{\infty} \frac{1}{(|n-k|+1)(n+1)^{m-s-1}(k+1)^{m-s}} < \infty.$$

Taking into account our notation, we can rewrite (31) in the form

$$\tilde{\phi}_{ni}(x) = \phi_{ni}(x) + \sum_{k=0}^{\infty}(\tilde{H}_{ni,k0}(x)\phi_{k0}(x) + \tilde{H}_{ni,k1}(x)\phi_{k1}(x))$$

or

$$\tilde{\phi}(x) = (E + \tilde{H}(x))\phi(x). \tag{38}$$

Thus, for each fixed $x$, the vector $\phi(x) \in m$ is a solution of equation (38) in the Banach space m. Equation (38) is called the main equation of the inverse problem. Solving (38), we find the vector $\phi(x)$ and consequently, the functions $\varphi_{ni}(x), n \geq 0, i = 0, 1$. Since $\varphi_{ni}(x) = \varphi(x, \lambda_{ni})$ are the solutions of (1), we can construct the function $q(x)$ by the formula

$$q(x) = \lambda_n + \frac{\varphi''{}_{n0}(x)}{\varphi_{n0}(x)}. \tag{39}$$

Also we obtain the coefficients $h$, $h_1$, $h_2$, $H$, $H_1$, and $H_2$ from the linear system of equations

$$\begin{cases} (\lambda_n - H_1)\varphi'_{n0}(\pi) + (\lambda_n H - H_2)\varphi_{n0}(\pi) = 0, & n \geq 0, \\ (\lambda_n - h_1)\varphi'_{n0}(0) + (\lambda_n h - h_2)\varphi_{n0}(0) = 0, & n \geq 0. \end{cases}$$

So, finally we obtain

$$\begin{cases} c_1 = \frac{\varphi'_{n0}(d_1+0)}{\varphi_{n0}(d_1-0)} - \frac{\varphi'_{n0}(d_1-0)}{\varphi_{n0}(d_1+0)}, \\ c_2 = \frac{\varphi'_{n0}(d_2+0)}{\varphi_{n0}(d_2-0)} - \frac{\varphi'_{n0}(d_2-0)}{\varphi_{n0}(d_2+0)}, \\ \quad\vdots \\ c_{m-1} = \frac{\varphi'_{n0}(d_{m-1}+0)}{\varphi_{n0}(d_{m-1}-0)} - \frac{\varphi'_{n0}(d_{m-1}-0)}{\varphi_{n0}(d_{m-1}+0)}. \end{cases} \tag{40}$$

**Example 1.** *Take* $\tilde{L} = L\left(\tilde{q}(x) = 0,\, d_i,\, a_i,\, a_i^{-1},\, 0,\, 0,\, 0,\, 0,\, \tilde{h}_2 < 0,\, \tilde{H}_2 < 0\right)$*. Let* $\{\tilde{\lambda}_n, \tilde{\Gamma}_n\}_{n\geq 0}$ *be the spectral data of* $\tilde{L}$ *and* $h$*,* $H$ *be given. Clearly,* $\tilde{\lambda}_0 = 0$*, and* $\tilde{\Gamma}_0$ *is obtained using (5)*

$$\tilde{\varphi}_{00}(x) = \begin{cases} 1, & 0 \leq x < d_1, \\ a_1, & d_1 < x < d_2, \\ a_1 a_2, & d_2 < x < d_3, \\ \quad\vdots & \quad\vdots \\ a_1 a_2 ... a_{m-1}, & d_{m-1} < x \leq \pi. \end{cases}$$

*Let* $\lambda_n = \tilde{\lambda}_n (n \geq 0)$*,* $\Gamma_n = \tilde{\Gamma_n}(n \geq 1)$*, and* $\Gamma_0 > 0$ *be an arbitrary positive number. Denote* $A := \frac{1}{\Gamma_0} - \frac{1}{\tilde{\Gamma}_0}$*. Then (31) yields*

$$\varphi_{00}(x) = \tilde{\varphi}_{00}(x)\left(1 + A\int_0^x \tilde{\varphi}_{00}^2(t)dt\right)^{-1}.$$

*So, we have*

$$\varphi_{00}(x) = \begin{cases} (1+Ax)^{-1}, & 0 \leq x < d_1, \\ a_1(B_1 + Aa_1^2 x)^{-1}, & d_1 < x < d_2, \\ a_1 a_2 (B_2 + A(a_1a_2)^2 x)^{-1}, & d_2 < x < d_3, \\ \quad\vdots & \quad\vdots \\ a_1a_2...a_{m-1}(B_{m-1} + A(a_1a_2...a_{m-1})^2 x)^{-1}, & d_{m-1} < x \leq \pi. \end{cases}$$

*where for* $m \geq 1$

$$B_m = 1 + A\sum_{i=1}^{m} d_i(1 - a_i^2)\prod_{j=0}^{i-1} {a_j}^2, \quad a_0 = 1.$$

*Using (39) and the value* $\lambda_{00} = 0$*, it is easy to see that*

$$q(x) = \begin{cases} 2A^2(1+Ax)^{-2}, & 0 \leq x < d_1, \\ 2A^2a_1^4(B_1 + Aa_1^2x)^{-2}, & d_1 < x < d_2. \\ 2A^2(a_1a_2)^4(B_2 + A(a_1a_2)^2x)^{-2}, & d_2 < x < d_3, \\ \quad\vdots & \quad\vdots \\ 2A^2(a_1a_2\cdots a_{m-1})^4(B_{m-1} + A(a_1\cdots a_{m-1})^2x)^{-2}, & d_{m-1} < x \leq \pi. \end{cases}$$

*Also, we can obtain the following relations*

$$H_1 = \frac{a_1\cdots a_{m-1}}{B_{m-1} + A(a_1\cdots a_{m-1})^2\pi}, \; H_2 = \frac{-A(a_1\cdots a_{m-1})^3}{(B_{m-1} + A(a_1\cdots a_{m-1})^2\pi)^2}.$$

*And*

$$h_1 = -1, \quad h_2 = -A.$$

*From relation (40) we have*

$$\begin{cases} c_1 = \frac{A(a_1^{-1} - a_1^3)}{1 + Aa_1}, & x \in [0, d_1) \cup (d_1, d_2), \\ c_2 = \frac{Aa_1^2(a_2^{-1} - a_2^3)}{B_1 + Aa_1^2 a_2}, & x \in (d_1, d_2) \cup (d_2, d_3), \\ \quad\vdots & \\ c_{m-1} = \frac{A(a_1\cdots a_{m-2})^2(a_{m-1}^{-1} - a_{m-1}^3)}{B_{m-2} + A(a_1\cdots a_{m-2})^2 d_{m-1}}, & x \in (d_{m-2}, d_{m-1}) \cup (d_{m-1}, \pi]. \end{cases}$$

## Acknowledgements

The authors are thankful to the referee(s) for their valuable comments.

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Received 20 January 2015
Accepted 12 April 2017