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A New Type of p-Ideals in BCI-Algebras Based on Hesitant Fuzzy Sets

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Abstract. Using the concept of hesitant union ($\Cup$), the notion of $\Cup$-hesitant fuzzy p-ideals is introduced, and their properties are investigated. Relations between $\Cup$-hesitant fuzzy ideals and $\Cup$-hesitant fuzzy p-ideals are considered. Conditions for the $\Cup$-hesitant fuzzy ideal to be a $\Cup$-hesitant fuzzy p-ideal are provided. The reduction property for $\Cup$-hesitant fuzzy p-ideals is established.

Key Words and Phrases: hesitant union, $\Cup$-hesitant fuzzy ideal, $\Cup$-hesitant fuzzy p-ideal.

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1. Introduction

As a generalization of fuzzy sets, Torra introduced the notion of hesitant fuzzy sets (see [9, 10]), and it is a very useful tool to deal with uncertainty, which can be accurately and perfectly described in terms of the opinions of decision makers. Xu and Xia [14] proposed a variety of distance measures for hesitant fuzzy sets, based on which the corresponding similarity measures can be obtained. They investigated the connections of the aforementioned distance measures and further developed a number of hesitant ordered weighted distance measures and hesitant ordered weighted similarity measures. Xu and Xia [15] defined the distance and correlation measures for hesitant fuzzy information and then discussed their properties in detail. Also, hesitant fuzzy set theory is used in decision making problem etc. (see [7, 11, 12, 13, 15]), and is applied to BCK/BCI-algebras, EQ-algebras, residuated lattices and MTL-algebras (see [2, 3, 4, 6]). In [8], Song et al. introduced the notion of $\Cup$-hesitant fuzzy subalgebras and $\Cup$-hesitant fuzzy ideals, and investigated several properties.

In this paper, we introduce the notion of $\Cup$-hesitant fuzzy p-ideals in BCI-algebras and investigate several properties. We consider relations between $\Cup$-hesitant fuzzy ideals and $\Cup$-hesitant fuzzy p-ideals, and provide conditions for the $\Cup$-hesitant fuzzy ideal to be a $\Cup$-hesitant fuzzy p-ideal. We establish the reduction property for $\Cup$-hesitant fuzzy p-ideals.

2. Preliminaries

A BCK/BCI-algebra is an important class of logical algebras introduced by K. Iséki and was extensively investigated by several researchers.

An algebra $(X; *, 0)$ of type $(2, 0)$ is called a *BCI-algebra* if it satisfies the following conditions:

(I) $(\forall x, y, z \in X)$ $(((x * y) * (x * z)) * (z * y) = 0)$,

(II) $(\forall x, y \in X)$ $((x * (x * y)) * y = 0)$,

(III) $(\forall x \in X)$ $(x * x = 0)$,

(IV) $(\forall x, y \in X)$ $(x * y = 0, y * x = 0 \Rightarrow x = y)$.

If a BCI-algebra X satisfies the following identity:

(V) $(\forall x \in X)$ $(0 * x = 0)$,

then X is called a *BCK-algebra*.

Any BCK/BCI-algebra X satisfies the following conditions:

$$(\forall x \in X)\,(x * 0 = x)\,, \tag{1}$$
$$(\forall x, y, z \in X)\,(x \leq y \;\Rightarrow\; x * z \leq y * z,\; z * y \leq z * x)\,, \tag{2}$$
$$(\forall x, y, z \in X)\,((x * y) * z = (x * z) * y)\,, \tag{3}$$
$$(\forall x, y, z \in X)\,((x * z) * (y * z) \leq x * y) \tag{4}$$

where $x \leq y$ if and only if $x * y = 0$.

Any BCI-algebra X satisfies the following conditions:

$$(\forall x, y, z \in X)\,(0 * (0 * ((x * z) * (y * z))) = (0 * y) * (0 * x))\,, \tag{5}$$
$$(\forall x, y \in X)\,(0 * (0 * (x * y)) = (0 * y) * (0 * x))\,, \tag{6}$$
$$(\forall x \in X)\,(0 * (0 * (0 * x)) = 0 * x)\,. \tag{7}$$

A BCI-algebra X is said to be p-semisimple (see [1]) if $0 * (0 * x) = x$ for all $x \in X$.

Every p-semisimple BCI-algebra X satisfies:

$$(\forall x, y, z \in X)\,((x * z) * (y * z) = x * y)\,. \tag{8}$$

A subset A of a BCK/BCI-algebra X is called an *ideal* of X if it satisfies:

$$0 \in A, \tag{9}$$

$$(\forall x \in X)\,(x * y \in A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{10}$$

A subset A of a BCI-algebra X is called a p-*ideal* of X (see [16]) if it satisfies (9) and

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \in A,\ y \in A \ \Rightarrow\ x \in A)\,. \tag{11}$$

Note that an ideal A of a BCI-algebra X is a p-ideal of X if and only if the following assertion is valid (see [16]):

$$(\forall x, y, z \in X)\,((x * z) * (y * z) \in A \ \Rightarrow\ x * y \in A)\,. \tag{12}$$

We refer the reader to the books [1, 5] for further information regarding BCK/BCI-algebras.

A hesitant fuzzy set on a reference set X (see [9]) is defined in terms of a function $\mathcal{G}$ that when applied to X returns a subset of $[0, 1]$, that is, $\mathcal{G} : X \to \mathscr{P}([0, 1])$.

Given a hesitant fuzzy set $\mathcal{G}$ on X, we define $\mathrm{Inf}\mathcal{G}$ as follows:

$$\mathrm{Inf}\mathcal{G}(x) = \begin{cases} \text{minimum of } \mathcal{G}(x) & \text{if } \mathcal{G}(x) \text{ is finite,} \\ \text{infimum of } \mathcal{G}(x) & \text{otherwise.} \end{cases} \tag{13}$$

for all $x \in X$. It is obvious that $\mathrm{Inf}\mathcal{G}$ is fuzzy set in X.

For a hesitant fuzzy set $\mathcal{G}$ on X and $x, y \in X$, we define

$$\mathcal{G}(x) \Cup \mathcal{G}(y) := \{t \in \mathcal{G}(x) \cup \mathcal{G}(y) \mid t \geq \max\{\mathrm{Inf}\mathcal{G}(x), \mathrm{Inf}\mathcal{G}(y)\}\}. \tag{14}$$

We say that $\mathcal{G}(x) \Cup \mathcal{G}(y)$ is the *hesitant union* of $\mathcal{G}(x)$ and $\mathcal{G}(y)$. Note that the following assertions are always true:

$$\mathcal{G}(x) \Cup \mathcal{G}(x) = \mathcal{G}(x), \tag{15}$$

$$\mathcal{G}(a) \subseteq \mathcal{G}(x),\ \mathcal{G}(b) \subseteq \mathcal{G}(y) \ \Rightarrow\ \mathcal{G}(a) \Cup \mathcal{G}(b) \subseteq \mathcal{G}(x) \Cup \mathcal{G}(y) \tag{16}$$

for all $a, b, x, y \in X$. For any hesitant fuzzy set $\mathcal{G}$ on X and $\tau \in \mathscr{P}([0, 1])$, we consider the set

$$L(\mathcal{G}, \tau) := \{x \in X \mid \mathcal{G}(x) \subseteq \tau\},$$

which is called the *lower hesitant τ-level set* of $\mathcal{G}$ on X.

A hesitant fuzzy set $\mathcal{G}$ on a BCK/BCI-algebra X is called a *hesitant fuzzy ideal* of X based on the hesitant union ($\Cup$) (briefly, $\Cup$-*hesitant fuzzy ideal* of X) (see [8]) if it satisfies:

$$(\forall x \in X)\,(\mathcal{G}(0) \subseteq \mathcal{G}(x))\,, \tag{17}$$

$$(\forall x, y \in X)\,(\mathcal{G}(x) \subseteq \mathcal{G}(x * y) \Cup \mathcal{G}(y))\,. \tag{18}$$

3. p-ideals of BCI-algebras based on the hesitant union

In what follows, let X denote a BCI-algebra unless otherwise specified.

Definition 1. *A hesitant fuzzy set $\mathcal{G}$ on X is called a hesitant fuzzy p-ideal of a BCI-algebra X based on the hesitant union ($\Cup$) (briefly, $\Cup$-hesitant fuzzy p-ideal of X) if it satisfies* (17) *and*

$$(\forall x, y, z \in X)\,(\mathcal{G}(x) \subseteq \mathcal{G}((x * z) * (y * z)) \Cup \mathcal{G}(y))\,. \tag{19}$$

Example 1. *Let $X = \{0, a, b, c\}$ be a BCI-algebra with the following Cayley table (see [1]):*

$*$	0	a	b	c
0	0	a	b	c
a	a	0	c	b
b	b	c	0	a
c	c	b	a	$0.$

Define a hesitant fuzzy set $\mathcal{G}$ on X as follows:

$$\mathcal{G} : X \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} \{0.4, 0.45\} \cup (0.5, 0.7) & \text{if } x \in \{0, c\} \\ [0.4, 0.7] & \text{otherwise,} \end{cases}$$

It is routine to verify that $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X.

Theorem 1. *Every $\Cup$-hesitant fuzzy p-ideal of X is a $\Cup$-hesitant fuzzy ideal of X.*

Proof. Let $\mathcal{G}$ be a $\Cup$-hesitant fuzzy p-ideal of X. Since $x * 0 = x$ for all $x \in X$, taking $z := 0$ in (19) yields

$$\mathcal{G}(x) \subseteq \mathcal{G}((x * 0) * (y * 0)) \Cup \mathcal{G}(y) = \mathcal{G}(x * y) \Cup \mathcal{G}(y)$$

for all $x, y \in X$. Therefore $\mathcal{G}$ is a $\Cup$-hesitant fuzzy ideal of X.

The converse of Theorem 1 is not true in general as seen in the following example.

Example 2. *Consider a BCI-algebra* $X = \{0, 1, 2, a, b\}$ *with the following Cayley table (see [1]):*

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	1	b	a
2	2	2	0	a	a
a	a	a	a	0	0
b	b	a	b	1	0.

Define a hesitant fuzzy set $\mathcal{G}$ *on* X *as follows:*

$$\mathcal{G} : X \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} \{0.4\} \cup [0.5, 0.6) \cup \{0.7\} & \text{if } x \in \{0, 2, a\}, \\ [0.4, 0.6] \cup \{0.7\} & \text{otherwise,} \end{cases}$$

Then $\mathcal{G}$ *is a* $\Cup$*-hesitant fuzzy ideal of* X*. But it is not a* $\Cup$*-hesitant fuzzy* p*-ideal of* X *since*

$$\begin{aligned} &\mathcal{G}((b * b) * (a * b)) \Cup \mathcal{G}(a) = \mathcal{G}(0) \Cup \mathcal{G}(a) \\ &= \{t \in \mathcal{G}(0) \cup \mathcal{G}(a) \mid t \geq \max\{\mathrm{Inf}\mathcal{G}(0), \mathrm{Inf}\mathcal{G}(a)\}\} \\ &= \{t \in \{0.4\} \cup [0.5, 0.6) \cup \{0.7\} \mid t \geq 0.4\} \\ &= \{0.4\} \cup [0.5, 0.6) \cup \{0.7\} \not\supseteq [0.4, 0.6] \cup \{0.7\} = \mathcal{G}(b). \end{aligned}$$

Lemma 1 ([8])**.** *Every* $\Cup$*-hesitant fuzzy ideal* $\mathcal{G}$ *of* X *satisfies:*

$$(\forall x, y \in X)\,(x \leq y \;\Rightarrow\; \mathcal{G}(x) \subseteq \mathcal{G}(y))\,, \tag{20}$$

$$(\forall x, y, z \in X)\,(x * y \leq z \;\Rightarrow\; \mathcal{G}(x) \subseteq \mathcal{G}(y) \Cup \mathcal{G}(z))\,. \tag{21}$$

Proposition 1. *Every* $\Cup$*-hesitant fuzzy* p*-ideal* $\mathcal{G}$ *of* X *satisfies the following assertion:*

$$(\forall x \in X)\,(\mathcal{G}(x) \subseteq \mathcal{G}(0 * (0 * x)))\,. \tag{22}$$

$$(\forall x, y, z \in X)\,(\mathcal{G}((x * z) * (y * z)) \subseteq \mathcal{G}(x * y))\,. \tag{23}$$

Proof. Let $\mathcal{G}$ be a $\Cup$-hesitant fuzzy p-ideal of X. If we put $z := x$ and $y := 0$ in (19), then

$$\mathcal{G}(x) \subseteq \mathcal{G}((x * x) * (0 * x)) \Cup \mathcal{G}(0) = \mathcal{G}(0 * (0 * x)) \Cup \mathcal{G}(0) \subseteq \mathcal{G}(0 * (0 * x))$$

for all $x \in X$ by (III), (17), (15) and (16). Thus (22) holds.

Note that $\mathcal{G}$ is a $\Cup$-hesitant fuzzy ideal of X by Theorem 1. Since $(x * z) * (y * z) \leq x * y$ for all $x, y, z \in X$, it follows from (20) that

$$\mathcal{G}((x * z) * (y * z)) \subseteq \mathcal{G}(x * y),$$

which proves (23).

Given a hesitant fuzzy set $\mathcal{G}$ on X, we consider the following inclusion:

$$(\forall x, y, z \in X)\,(\mathcal{G}(x * y) \subseteq \mathcal{G}((x * z) * (y * z)))\,. \tag{24}$$

The following example shows that there is a $\Cup$-hesitant fuzzy ideal $\mathcal{G}$ of X which does not satisfy the condition (24).

Example 3. *Let $X = \{0, 1, 2, a, b\}$ be a BCI-algebra with the following Cayley table (see [5]):*

$*$	0	1	2	a	b
0	0	0	0	a	a
1	1	0	0	a	a
2	2	2	0	b	a
a	a	a	a	0	0
b	b	b	a	2	0.

Define a hesitant fuzzy set $\mathcal{G}$ on X as follows:

$$\mathcal{G} : X \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} \{0.4, 0.45, 0.5\} & \text{if } x = 0, \\ \{0.4\} \cup [0.45, 0.5] & \text{if } x = 1, \\ \{0.4\} \cup [0.45, 0.6) & \text{if } x = 2, \\ [0.4, 0.5] & \text{if } x = a, \\ [0.4, 0.6] & \text{if } x = b. \end{cases}$$

It is routine to verify that $\mathcal{G}$ is a $\Cup$-hesitant fuzzy ideal of X. But $\mathcal{G}$ does not satisfy the condition (24) because

$$\mathcal{G}(2 * a) = \mathcal{G}(b) = [0.4, 0.6] \supset [0.4, 0.5] = \mathcal{G}((2 * 2) * (a * 2)).$$

We provide conditions for a $\Cup$-hesitant fuzzy ideal to be a $\Cup$-hesitant fuzzy p-ideal.

Theorem 2. *If a $\Cup$-hesitant fuzzy ideal $\mathcal{G}$ of X satisfies the condition (24), then it is a $\Cup$-hesitant fuzzy p-ideal of X.*

Proof. Let $\mathcal{G}$ be a $\Cup$-hesitant fuzzy ideal of X satisfying the condition (24). Then

$$\mathcal{G}(x) \subseteq \mathcal{G}(x * y) \Cup \mathcal{G}(y) \subseteq \mathcal{G}((x * z) * (y * z)) \Cup \mathcal{G}(y)$$

for all $x, y, z \in X$ by (18) and (16). Therefore $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X.

Lemma 2 ([8]). *Every $\Cup$-hesitant fuzzy ideal $\mathcal{G}$ of X satisfies the following assertion:*

$$(\forall x \in X)\,(\mathcal{G}(0*(0*x)) \subseteq \mathcal{G}(x))\,. \tag{25}$$

Theorem 3. *If a $\Cup$-hesitant fuzzy ideal $\mathcal{G}$ of X satisfies the condition* (22), *then it is a $\Cup$-hesitant fuzzy p-ideal of X.*

Proof. Let $x, y, z \in X$. Using (22), (6), (5) and Lemma 2, we have

$$\begin{aligned}\mathcal{G}(x*y) &\subseteq \mathcal{G}(0*(0*(x*y))) = \mathcal{G}((0*y)*(0*x))\\ &= \mathcal{G}(0*(0*((x*z)*(y*z))))\\ &\subseteq \mathcal{G}((x*z)*(y*z)).\end{aligned}$$

It follows from Theorem 2 that $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X.

Theorem 4. *In a p-semisimple BCI-algebra, every $\Cup$-hesitant fuzzy ideal is a $\Cup$-hesitant fuzzy p-ideal.*

Proof. Let $\mathcal{G}$ be a $\Cup$-hesitant fuzzy ideal of a p-semisimple BCI-algebra X. Using (18) and (8), we have

$$\mathcal{G}(x) \subseteq \mathcal{G}(x*y) \Cup \mathcal{G}(y) = \mathcal{G}((x*z)*(y*z)) \Cup \mathcal{G}(y)$$

for all $x, y, z \in X$. Therefore $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X.

Corollary 1. *In a BCI-algebra X in which its BCK-part is $\{0\}$, every $\Cup$-hesitant fuzzy ideal is a $\Cup$-hesitant fuzzy p-ideal.*

Corollary 2. *In a BCI-algebra X in which every element is minimal, every $\Cup$-hesitant fuzzy ideal is a $\Cup$-hesitant fuzzy p-ideal.*

Corollary 3. *Let X be a BCI-algebra in which any one of the following conditions is true:*

(1) $(x*y)*(z*u) = (x*z)(*y*u)$,

(2) $0*(y*x) = x*y$,

(3) $(x*y)*(x*z) = z*y$,

(4) $z*x = z*y \;\Rightarrow\; x = y$,

(5) $x*y = 0 \;\Rightarrow\; x = y$

for all $x, y, z, u \in X$. Then every $\Cup$-hesitant fuzzy ideal is a $\Cup$-hesitant fuzzy p-ideal.

Lemma 3 ([8])**.** *If $\mathcal{G}$ is a $\Cup$-hesitant fuzzy ideal of X, then the lower hesitant τ-level set $L(\mathcal{G},\tau)$ of $\mathcal{G}$ on X is an ideal of X for all $\tau \in \mathscr{P}([0,1])$ with $L(\mathcal{G},\tau) \neq \emptyset$.*

Lemma 4 ([16])**.** *An ideal A of X is a p-ideal of X if and only if it satisfies*

$$(\forall x \in X)\,(0 * (0 * x) \in A \;\Rightarrow\; x \in A)\,. \tag{26}$$

Theorem 5. *If $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X, then the lower hesitant τ-level set $L(\mathcal{G},\tau)$ of $\mathcal{G}$ on X is a p-ideal of X for all $\tau \in \mathscr{P}([0,1])$ with $L(\mathcal{G},\tau) \neq \emptyset$.*

Proof. Let $\tau \in \mathscr{P}([0,1])$ be such that $L(\mathcal{G},\tau) \neq \emptyset$. If $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X, then it is a $\Cup$-hesitant fuzzy ideal of X, and so $L(\mathcal{G},\tau)$ is an ideal of X by Lemma 3. Assume that $0 * (0 * x) \in L(\mathcal{G},\tau)$ for all $x \in X$. Then $\mathcal{G}(0 * (0 * x)) \subseteq \tau$, which implies by (22) that $\mathcal{G}(x) \subseteq \mathcal{G}(0 * (0 * x)) \subseteq \tau$. Hence $x \in L(\mathcal{G},\tau)$, and thus $L(\mathcal{G},\tau)$ is a p-ideal of X by Lemma 4.

Corollary 4. *If $\mathcal{G}$ is a $\Cup$-hesitant fuzzy ideal of a p-semisimple BCI-algebra X, then the lower hesitant τ-level set $L(\mathcal{G},\tau)$ of $\mathcal{G}$ on X is a p-ideal of X for all $\tau \in \mathscr{P}([0,1])$ with $L(\mathcal{G},\tau) \neq \emptyset$.*

Corollary 5. *Let X be a BCI-algebra in which at least one of the five conditions in Corollary 3 is true. If $\mathcal{G}$ is a $\Cup$-hesitant fuzzy ideal of X, then the lower hesitant τ-level set $L(\mathcal{G},\tau)$ of $\mathcal{G}$ on X is a p-ideal of X for all $\tau \in \mathscr{P}([0,1])$ with $L(\mathcal{G},\tau) \neq \emptyset$.*

The following example shows that the converse of Theorem 5 is not true in general.

Example 4. *Consider a BCI-algebra $X = \{0, 1, a, b, c\}$ with the following Cayley table (see [1]):*

$*$	0	1	a	b	c
0	0	0	a	b	c
1	1	0	a	b	c
a	a	a	0	c	b
b	b	b	c	0	a
c	c	c	b	a	0.

Let $\mathcal{G}$ be a hesitant fuzzy set on X given as follows:

$$\mathcal{G} : X \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} (0.5, 0.6) & \text{if } x \in \{0,1\}, \\ [0.5, 0.6) \cup \{0.7\} & \text{if } x = a, \\ \{0.4\} \cup [0.5, 0.7] & \text{otherwise.} \end{cases}$$

Then we have

$$L(\mathcal{G},\tau) = \begin{cases} \{0,1\} & \textit{if } (0.5,0.6) \subseteq \tau \textit{ and } [0.5,0.6) \cup \{0.7\} \nsubseteq \tau, \\ \{0,1,a\} & \textit{if } [0.5,0.6) \cup \{0.7\} \subseteq \tau \textit{ and } \{0.4\} \cup [0.5,0.7] \nsubseteq \tau, \\ X & \textit{if } \{0.4\} \cup [0.5,0.7] \subseteq \tau, \\ \emptyset & \textit{otherwise,} \end{cases}$$

and so $L(\mathcal{G},\tau)$ is a p-ideal of X for all $\tau \in \mathscr{P}([0,1])$ with $L(\mathcal{G},\tau) \neq \emptyset$. Note that $\mathcal{G}(0) \subseteq \mathcal{G}(x)$ for all $x \in X$. But $\mathcal{G}(b) = \{0.4\} \cup [0.5,0.7]$ and

$$\begin{aligned} &\mathcal{G}((b*a)*(1*a)) \Cup \mathcal{G}(1) = \mathcal{G}(b) \Cup \mathcal{G}(1) \\ &= \{t \in \mathcal{G}(b) \cup \mathcal{G}(1) \mid t \geq \max\{\mathrm{Inf}\mathcal{G}(b), \mathrm{Inf}\mathcal{G}(1)\}\} \\ &= \{t \in \{0.4\} \cup [0.5,0.7] \mid t \geq \max\{0.4, 0.5\}\} \\ &= [0.5, 0.7], \end{aligned}$$

and thus $\mathcal{G}(b) \nsubseteq \mathcal{G}((b*a)*(1*a)) \Cup \mathcal{G}(1)$. Therefore $\mathcal{G}$ is not a $\Cup$-hesitant fuzzy p-ideal of X.

We provide a condition for the converse of Theorem 5 to be true.

Theorem 6. *Let $\mathcal{G}$ be a hesitant fuzzy set on X satisfying the condition*

$$(\forall x, y \in X)\,(\mathcal{G}(x) \Cup \mathcal{G}(y) = \mathcal{G}(x) \cup \mathcal{G}(y))\,. \tag{27}$$

If the lower hesitant τ-level set $L(\mathcal{G},\tau)$ of $\mathcal{G}$ on X is a p-ideal of X for all $\tau \in \mathscr{P}([0,1])$ with $L(\mathcal{G},\tau) \neq \emptyset$, then $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X.

Proof. For any $x \in X$, let $\mathcal{G}(x) = \tau_x$. Then $x \in L(\mathcal{G},\tau_x)$, and so $L(\mathcal{G},\tau_x)$ is a p-ideal of X by assumption. Thus $0 \in L(\mathcal{G},\tau_x)$, and hence $\mathcal{G}(0) \subseteq \tau_x = \mathcal{G}(x)$. For any $x, y, z \in X$, taking $\tau = \mathcal{G}((x*z)*(y*z)) \cup \mathcal{G}(y)$ yields

$$(x*z)*(y*z) \in L(\mathcal{G},\tau) \text{ and } y \in L(\mathcal{G},\tau).$$

Hence $x \in L(\mathcal{G},\tau)$, and so

$$\mathcal{G}(x) \subseteq \tau = \mathcal{G}((x*z)*(y*z)) \cup \mathcal{G}(y) = \mathcal{G}((x*z)*(y*z)) \Cup \mathcal{G}(y)$$

by using the condition (27). Therefore $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X.

Theorem 7. *Given a nonempty proper subset A of X, define a hesitant fuzzy set $\mathcal{G}$ on X as follows:*

$$\mathcal{G} : X \to \mathscr{P}([0,1]), \quad x \mapsto \begin{cases} \tau_1 & \text{if } x \in A, \\ \tau_2 & \text{otherwise,} \end{cases} \tag{28}$$

where $\tau_1, \tau_2 \in \mathscr{P}([0,1])$ with $\tau_1 \subsetneq \tau_2$. If $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X, then A is a p-ideal of X.

Proof. Note that

$$L(\mathcal{G},\tau)=\begin{cases} A & \text{if } \tau_1\subseteq\tau \text{ and } \tau_2\nsubseteq\tau,\\ X & \text{if } \tau_2\subseteq\tau,\\ \emptyset & \text{otherwise.}\end{cases}$$

If $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X, then $L(\mathcal{G},\tau)$ is a p-ideal of X for all $\tau\in\mathscr{P}([0,1])$ with $L(\mathcal{G},\tau)\neq\emptyset$ by Theorem 5. Hence A should be a p-ideal of X.

Lemma 5. *If* $\mathrm{Inf}\tau_1=\mathrm{Inf}\tau_2\in\tau_1\cap\tau_2$, *then the hesitant fuzzy set* $\mathcal{G}$ *in Theorem 7 satisfies the condition (27).*

Proof. Let $x,y\in X$. If $x,y\in A$, then

$$\begin{aligned}\mathcal{G}(x)\Cup\mathcal{G}(y)&=\{t\in\mathcal{G}(x)\cup\mathcal{G}(y)\mid t\geq\max\{\mathrm{Inf}\mathcal{G}(x),\mathrm{Inf}\mathcal{G}(y)\}\}\\&=\{t\in\tau_1\mid t\geq\mathrm{Inf}\tau_1\}\}\\&=\tau_1=\mathcal{G}(x)\cup\mathcal{G}(y).\end{aligned}$$

If $x,y\in X\setminus A$, then

$$\begin{aligned}\mathcal{G}(x)\Cup\mathcal{G}(y)&=\{t\in\mathcal{G}(x)\cup\mathcal{G}(y)\mid t\geq\max\{\mathrm{Inf}\mathcal{G}(x),\mathrm{Inf}\mathcal{G}(y)\}\}\\&=\{t\in\tau_2\mid t\geq\mathrm{Inf}\tau_2\}\}\\&=\tau_2=\mathcal{G}(x)\cup\mathcal{G}(y).\end{aligned}$$

If $x\in A$ and $y\in X\setminus A$, then $\mathcal{G}(x)=\tau_1\subsetneq\tau_2=\mathcal{G}(y)$, and so

$$\begin{aligned}\mathcal{G}(x)\Cup\mathcal{G}(y)&=\{t\in\mathcal{G}(x)\cup\mathcal{G}(y)\mid t\geq\max\{\mathrm{Inf}\mathcal{G}(x),\mathrm{Inf}\mathcal{G}(y)\}\}\\&=\{t\in\tau_2\mid t\geq\max\{\mathrm{Inf}\tau_1,\mathrm{Inf}\tau_2\}\}\\&=\{t\in\tau_2\mid t\geq\mathrm{Inf}\tau_2\}\\&=\tau_2=\mathcal{G}(x)\cup\mathcal{G}(y).\end{aligned}$$

Similarly, if $x\in X\setminus A$ and $y\in A$, then $\mathcal{G}(x)\Cup\mathcal{G}(y)=\mathcal{G}(x)\cup\mathcal{G}(y)$. Thus $\mathcal{G}$ satisfies the condition (27).

Theorem 8. *If* $\mathrm{Inf}\tau_1=\mathrm{Inf}\tau_2\in\tau_1\cap\tau_2$ *and* A *is a* p*-ideal of* X, *then the hesitant fuzzy set* $\mathcal{G}$ *in Theorem 7 is a* $\Cup$*-hesitant fuzzy* p*-ideal of* X.

Proof. It is by Theorem 6 and Lemma 5.

Theorem 9. (Reduction property for $\Cup$-hesitant fuzzy p-ideals) *Let* $\mathcal{G}$ *and* $\mathcal{H}$ *be* $\Cup$*-hesitant fuzzy ideals of* X *such that* $\mathcal{G}(0)=\mathcal{H}(0)$ *and* $\mathcal{G}(x)\supseteq\mathcal{H}(x)$ *for all* $x\in X$. *If* $\mathcal{G}$ *is a* $\Cup$*-hesitant fuzzy* p*-ideal of* X, *then so is* $\mathcal{H}$.

Proof. Assume that $\mathcal{G}$ is a $\Cup$-hesitant fuzzy p-ideal of X. Note that

$$0 * (0 * (x * (0 * (0 * x)))) = (0 * (0 * (0 * x))) * (0 * x) = (0 * x) * (0 * x) = 0$$

for all $x \in X$ by (6), (7) and (III). Using the hypothesis and (22) yields

$$\begin{aligned}\mathcal{H}(x * (0 * (0 * x))) &\subseteq \mathcal{G}(x * (0 * (0 * x)))\\ &\subseteq \mathcal{G}(0 * (0 * (x * (0 * (0 * x)))))\\ &= \mathcal{G}(0) = \mathcal{H}(0),\end{aligned}$$

and so

$$\begin{aligned}\mathcal{H}(x) &\subseteq \mathcal{H}(x * (0 * (0 * x))) \Cup \mathcal{H}(0 * (0 * x))\\ &\subseteq \mathcal{H}(0) \Cup \mathcal{H}(0 * (0 * x))\\ &\subseteq \mathcal{H}(0 * (0 * x)) \Cup \mathcal{H}(0 * (0 * x))\\ &= \mathcal{H}(0 * (0 * x))\end{aligned}$$

for all $x \in X$ by (18), (16), (17) and (15). Therefore $\mathcal{H}$ is a $\Cup$-hesitant fuzzy p-ideal of X by Theorem 3.

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