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On One Method for Solving Matrix Riccati Equation

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Abstract. In this paper we find a symmetric solution of the matrix Riccati equation:

$$XCX + XB^T + BX + A = 0,$$

where A, B, C, X are real square matrices of order n, A and C are symmetric. This equation appears when one seeks for the normal form of a quadratic Hamiltonian with the help of a canonical transformation. We consider the case where zero eigenvalues of the corresponding Hamiltonian matrix

$$V = \begin{pmatrix} B^T & C \\ -A & -B \end{pmatrix},$$

form blocks of even order in the matrix of Jordan normal form (with the exception, perhaps, of one pair, forming two blocks of the first order). Matrix Riccati equation arises in optimal control problems. Solution method relies on finding appropriate eigenvectors and generalized eigenvectors of the Hamiltonian matrix V. We prove that the vectors satisfy the symplectic conditions. We obtain sufficient conditions for existence of a solution.

Key Words and Phrases: Hamiltonian matrix, Jordan normal form, Hamiltonian, eigenvalue, eigenvector.

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The matrix Riccati equation plays an important role in optimal control problems [2, 5]. This equation arises frequently in mechanics in case where one seeks for the normalization of a quadratic Hamiltonian with the help of a canonical transformation [9]. The matrix Riccati equation is an equation of the following form:

$$XCX + XB^T + BX + A = 0, \tag{1}$$

where A, B, C, X are real square matrices of order n, A and C are symmetric. The aim is to find a symmetric solution X of this equation.

Let us consider a Jordan basis of the Hamiltonian matrix

$$V = \begin{pmatrix} B^T & C \\ -A & -B \end{pmatrix}, \tag{2}$$

(consisting of $2n$ eigenvectors and generalized eigenvectors). Let $\{e_1, \ldots, e_n\}$ be a subset of this basis consisting of n vectors and satisfying the following condition: if a vector of this basis corresponding to an eigenvalue λ_i is in this subset, then all the basis vectors obtained from it by applying $V - \lambda_i E$ and its iterations also belong to this subset. Let

$$\begin{pmatrix} e_1 & e_2 & \ldots & e_n \end{pmatrix} := \begin{pmatrix} F \\ G \end{pmatrix}, \tag{3}$$

where F and G are square matrices of order n. If one has $\det F \neq 0$, then the matrix $X = GF^{-1}$ is a solution of the equation (1): [8, 6].

The set of eigenvalues of the matrix V consists of pairs (λ_i, λ_i') of opposite numbers: $\lambda_i' = -\lambda_i$. In [4, 6, 8], the case of a non-degenerate matrix V (i.e. $\lambda_i \neq 0$) has been considered. Assume that the subset $\{e_1, \ldots, e_n\}$ of the Jordan basis is chosen so that if a vector corresponding to an eigenvalue λ belongs to this subset, then no vector corresponding to the eigenvalue $-\lambda = \lambda_i'$ is in it. Then the matrix is symmetric: $X = X^T$ and thus it is a symmetric solution of (1): [4, 6, 8].

Suppose that zero eigenvalues of the Hamiltonian matrix V form blocks of even order in the Jordan normal form with the exception, perhaps, of one pair forming two blocks of the first order. In this case there exists a symplectic transformation of the matrix V to the Jordan normal form [4]:

$$\Phi = \begin{pmatrix} U & I \\ O & -U^T \end{pmatrix}, \tag{4}$$

where

$$U = U_1 \oplus U_2 \oplus \ldots \oplus U_k; \; I = I_1 \oplus I_2 \oplus \ldots \oplus I_k;$$

$$U_i = \begin{pmatrix} \lambda_i & 1 & 0 & \ldots & 0 \\ 0 & \lambda_i & 1 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \\ 0 & 0 & 0 & \ldots & \lambda_i \end{pmatrix}, I_i = \begin{pmatrix} 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ldots & 0 \\ 0 & 0 & 0 & \ldots & \varepsilon_i \end{pmatrix}, \tag{5}$$

$$\text{where } \varepsilon_i = \begin{cases} 0 & \text{if } \lambda_i \neq 0, \\ \pm 1 & \text{if } \lambda_i = 0, \; i < k, \\ 0, \pm 1 & \text{if } \lambda_k = 0. \end{cases}$$

Submatrices U_i, I_i $(i = \overline{1,k})$ have the same order and nonzero eigenvalues of the submatrix U satisfy the condition

$$\lambda_i + \lambda_j \neq 0, \text{ if } \lambda_i \neq 0,\ \lambda_j \neq 0,\ i,j = \overline{1,n}. \tag{6}$$

In the case of zero eigenvalue with block of order $2l$, the corresponding submatrices have order l. If there exists a zero eigenvalue with two blocks of order 1, then it has the corresponding submatrices U_k, I_k.

Theorem 1. *Suppose that the zero eigenvalues of the Hamiltonian matrix (2) form blocks of even order in the matrix of the Jordan normal form (4) with the possible exception of one pair forming two blocks of the first order. Let the matrix T be such that $T^{-1}VT = \Phi$, and let $t_1, \ldots t_{2n}$ be its columns. Then the vectors $t_1, t_2, \ldots, t_n$ in the first n columns of the matrix T satisfy the symplectic equations*

$$(t_i)^T J t_j = 0,\ i,j = \overline{1,n};\ J = \begin{pmatrix} O & E \\ -E & O \end{pmatrix}, \tag{7}$$

where E is the $n \times n$ identity matrix.

Proof. The columns of the matrix T are eigenvectors and generalized eigenvectors of the matrix V. Let us consider those of them which correspond to the eigenvalue zero

$$t_i,\ Vt_i,\ \ldots,\ V^{2k_i-1}t_i,\ (V^{2k_i}t_i = 0),\ i = \overline{1,m-1};$$

$$t_m,\ (Vt_m = 0),\ t_{m+1},\ (Vt_{m+1} = 0),$$

where $m+1$ is the number of Jordan blocks corresponding to the zero eigenvalue. Since the matrix V is a Hamiltonian one, we have $JV = -V^T J$. Therefore

$$(V^l t_i)^T J V^p t_j = (-1)^p (V^{l+p} t_i)^T J t_j = (-1)^l (t_i)^T J V^{l+p} t_j = 0,$$

$$\text{if } l + p \geq \min(2k_i,\ 2k_j),\ \ i,j = \overline{1,m-1}.$$

$$(t_m)^T J V^p t_i = (-1)^p (V^p t_m)^T J t_i = 0 \text{ if } p \geq 1,\ \ i = \overline{1,m-1};\ \ (t_m)^T J t_m = 0.$$

The vectors $V^l t_i$ in the first n columns of the matrix T satisfy the conditions $l \geq k_i \geq 1$. This means that we proved the theorem for the case $\lambda = 0$. In [6], the conditions (7) are proved for the case where the eigenvalues corresponding to the vectors satisfy (6). Hence we proved the theorem for all cases. ◀

Corollary 1. *Let*

$$T = \begin{pmatrix} F & H \\ G & W \end{pmatrix},$$

be a matrix where $T^{-1}VT = \Phi$ *and the square submatrices* F, G, H, W *are of order* n. *Then the equality* $F^TG = G^TF$ *holds. If the submatrix* F *of the matrix* T *is non-degenerate, then the matrix* $X = GF^{-1}$ *is a symmetric solution of the matrix Riccati equation (1).*

Theorem 2. *Let* V *be the Hamiltonian matrix (2) and let* $\det C \neq 0$. *Let*

$$t_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix},\ t_2 = \begin{pmatrix} f_2 \\ g_2 \end{pmatrix},\ \ldots,\ t_k = \begin{pmatrix} f_k \\ g_k \end{pmatrix},$$

be linearly independent eigenvectors corresponding to an eigenvalue λ *of the matrix* V. *Then the vectors* f_1, f_2, $\ldots$, f_k *are linearly independent as well.*

Proof. Suppose that the vectors $f_1, f_2, \ldots, f_k$ are linearly dependent:

$$c_1f_1 + c_2f_2 + \ldots + c_kf_k = 0.$$

Then

$$(B^T - \lambda E)(c_1f_1 + c_2f_2 + \ldots + c_kf_k) + C(c_1g_1 + c_2g_2 + \ldots + c_kg_k) =$$
$$= C(c_1g_1 + c_2g_2 + \ldots + c_kg_k) = 0.$$

The matrix C is non-degenerate and thus has no zero eigenvalue. Therefore

$$c_1g_1 + c_2g_2 + \ldots + c_kg_k = 0.$$

Hence

$$c_1t_1 + c_2t_2 + \ldots + c_kt_k = 0.$$

Therefore $c_1 = c_2 = \ldots = c_k = 0$. This proves the theorem. ◀

Corollary 2. *Let* V *be the Hamiltonian matrix (2),* $\det C \neq 0$, *and let all eigenvalues be zero eigenvalues with blocks of order one or two. Then there exists a symmetric solution of the matrix Riccati equation.*

Corollary 3. *Let* V *be a Hamiltonian matrix of order four with positive definite submatrix* C. *Then there exists a symmetric solution of the matrix Riccati equation.*

Corollary 4. *Let* V *be a Hamiltonian matrix with positive definite submatrix* C. *Let the equality* $CB = B^TC$ *hold. Then there exists a symmetric solution of the matrix Riccati equation (this was shown in [9]).*

Using a symmetric solution of the matrix Riccati equation, one can find a transformation which normalizes the canonical system of differential equations with a quadratic Hamiltonian [1, 3, 7, 9, 10].

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