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Parseval Equality for Non-Self-Adjoint Differential Operator with Block-Triangular Potential

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Abstract. Parseval equality is proved for Sturm-Liouville equation with block-triangular, increasing at infinity operator potential.

Key Words and Phrases: differential operator, block-triangular operator potential, Parseval equality.

2010 Mathematics Subject Classifications: 34K11, 47A10

1. Introduction

In the study of the connection between spectral and oscillation properties of non-self-adjoint differential operators with block-triangular operator coefficients (see [1] – [2]) the question arises of the structure of the spectrum of such operators. For scalar non- self- adjoint differential operators these questions have been studied by M.A. Naimark [3], [4], V.E. Lyantse [5], V.A. Marchenko [6], [7], F.S. Rofe-Beketov [8], J.T. Schwartz [9]. In the context of inverse scattering problem, for a differential operator with a triangular matrix potential decreasing at infinity and having a bounded first moment it was proved in [10, 11] that the discrete spectrum of the operator consists of a finite number of negative eigenvalues and essential spectrum covers the positive half. For the operator with block-triangular matrix potential that increases at infinity these questions have been considered in [12] based on the construction of the Green's function, the resolvent and the proof of Parseval equality. Later, in [13]-[15] these results were generalized to the equations with block-triangular operator coefficients increasing at infinity. In those works, using an operator solution decreasing at infinity, a

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Green's function and a resolvent have been constructed, and a series expansion for a Green's function has been obtained.

In this paper, we prove the Parseval equality for an equation with block-triangular operator coefficients. It is a logical continuation of the papers [13]-[15] and to some extent completes the research on this topic.

2. Preliminaries

Let $H_k,\ k = 1, 2, ..., r$ be finite-dimensional or infinite-dimensional separable Hilbert spaces with inner product $(\cdot, \cdot)$ and norm $|\cdot|$, $\dim H_k \leq \infty$. Denote $\mathbf{H} = H_1 \oplus H_2 \oplus ... \oplus H_r$. Element $h \in \mathbf{H}$ will be written in the form $h = \mathrm{col}\,(h_1, h_2, ..., h_r)$, where $h_k \in H_k,\ k = \overline{1, r},\ I_k,\ I$ are identity operators in H_k and $\mathbf{H}$, respectively.

We denote by $L_2(\mathbf{H}, (0, \infty))$ the Hilbert space of vector-valued functions $y(x)$ with values in $\mathbf{H}$, inner product $\langle y, z\rangle = \int\limits_0^\infty (y(x), z(x))\,dx$ and the corresponding norm $\|\cdot\|$.

Now let us consider the equation with block-triangular operator potential in $B(\mathbf{H})$

$$l[y] = -y'' + V(x)\,y = \lambda y, \quad 0 \leqslant x < \infty\ , \tag{1}$$

where

$$V(x) = v(x)\cdot I + U(x)\,, \quad U(x) = \begin{pmatrix} U_{11}(x) & U_{12}(x) & ... & U_{1r}(x) \\ 0 & U_{22}(x) & ... & U_{2r}(x) \\ ... & ... & ... & ... \\ 0 & 0 & ... & U_{rr}(x) \end{pmatrix}, \tag{2}$$

$v(x)$ is a real scalar function with a monotone absolutely continuous derivative, and $0 < v(x) \to \infty$ monotonically as $x \to \infty$. Also, $U(x)$ is a relatively small perturbation, e.g. $|U(x)|\cdot v^{-1}(x) \to 0$ as $x \to \infty$ or $|U|\,v^{-1} \in L^\infty(\mathbb{R}_+)$. The diagonal blocks $U_{kk}(x)\,,\quad k = \overline{1, r}$ are assumed to be bounded self-adjoint operators in H_k, $U_{kl}:\ H_l \to H_k$.

In case where

$$v(x) \geqslant Cx^{2\alpha},\ C > 0,\ \alpha > 1, \tag{3}$$

we suppose that the coefficients of the equation (1) satisfy the relations

$$\int\limits_0^\infty |U(t)|\cdot v^{-\frac{1}{2}}(t)\,dt < \infty,$$

$$\int\limits_0^\infty v'^2(t)\cdot v^{-\frac{5}{2}}(t)\,dt<\infty,\quad \int\limits_0^\infty v''(t)\cdot v^{-\frac{3}{2}}(t)\,dt<\infty. \tag{4}$$

In case where $v(x)=x^{2\alpha},\ 0<\alpha\leqslant 1$, we suppose that the coefficients of the equation (1) satisfy the relation

$$\int\limits_a^\infty |U(t)|\cdot t^{-\alpha}dt<\infty,\quad a>0. \tag{5}$$

In [16], for equation with block-triangular, increasing at infinity operator potential, a fundamental system of solutions is constructed, one of which, $\Phi(x,\lambda)$, is decreasing at infinity, while the other, $\Psi(x,\lambda)$, is increasing.

Let the following boundary condition be given at $x=0$:

$$\cos A\cdot y'(0)-\sin A\cdot y(0)=0, \tag{6}$$

where A is a block-triangular operator of the same structure as the potential $V(x)$ (2) of the differential equation (1), and $A_{kk},\ k=\overline{1,r}$ are the bounded self-adjoint operators in H_k, which satisfy the conditions

$$-\frac{\pi}{2}I_k<<A_{kk}\leqslant\frac{\pi}{2}I_k. \tag{7}$$

Together with the problem (1), (6), we consider the separated system

$$l_k[y_k]=-y_k''+(w(x)I_k+U_{kk}(x))\,y_k=\lambda y_k,\quad k=\overline{1,r} \tag{8}$$

with the boundary conditions

$$\cos A_{kk}\cdot y_k'(0)-\sin A_{kk}\cdot y_k(0)=0\ . \tag{9}$$

Let L' denote the minimal differential operator generated by differential expression $l[y]$ (1) and the boundary condition (6), and let $L_k',\ k=\overline{1,r}$ denote the minimal differential operator on $L_2(H_k,(0,\infty))$ generated by differential expression $l_k[y_k]$ (8) and the boundary conditions (9). Taking into account the conditions on coefficients, as well as sufficient smallness of perturbations $U_{kk}(x)$, and conditions (7), we conclude that, for every symmetric operator $L_k',\ k=\overline{1,r}$, there is a case of limit point at infinity. Hence their self-adjoint extensions L_k are the closures of operators L_k', respectively. The operators L_k are semi-bounded below, and their spectra are discrete.

Let L denote the extension of the operator L',with a requirement that $L_2(\mathbf{H},(0,\infty))$ is the domain of operator L.

Along with the equation (1), we consider the equation

$$l_1[y] = -y'' + V^*(x)\, y = \lambda y \tag{10}$$

($V^*(x)$ is adjoint to the operator $V(x)$). If the space $\mathbf{H}$ is finite-dimensional, then the equation (10) can be rewritten as

$$\tilde{l}[\tilde{y}] = -\tilde{y}'' + \tilde{y}V(x) = \lambda\tilde{y},$$

where $\tilde{y} = (\tilde{y}_1 \ \ \tilde{y}_2 ... \tilde{y}_r)$ and the equation is called the left.

For operator-functions $Y(x,\lambda)\,,\, Z(x,\lambda) \in B(\mathbf{H})$ let

$$W\{Z^*, Y\} = Z^{*\prime}\left(x,\bar{\lambda}\right) Y(x,\lambda) - Z^*\left(x,\bar{\lambda}\right) Y'(x,\lambda)\,.$$

If $Y(x,\lambda)$ is an operator solution of the equation (1), and $Z(x,\lambda)$ is an operator solution of equation (10), then the Wronskian does not depend on x.

Now we denote by $Y(x,\lambda)$ and $Y_1(x,\lambda)$ the solutions of the equations (1) and (10), respectively, satisfying the initial conditions

$$Y(0,\lambda) = \cos A,\ Y'(0,\lambda) = \sin A,$$

$$Y_1(0,\lambda) = (\cos A)^*,\ {Y_1}'(0,\lambda) = (\sin A)^*,\ \ \lambda \in \mathbb{C}.$$

As the operator function $Y_1^*\left(x,\bar{\lambda}\right)$ satisfies the equation

$$-Y_{1^*}''\left(x,\bar{\lambda}\right) + Y_1^*\left(x,\bar{\lambda}\right)\cdot V(x) = \lambda Y_1^*\left(x,\bar{\lambda}\right),$$

the operator function $\tilde{Y}(x,\lambda) =: Y_1^*\left(x,\bar{\lambda}\right)$ is a solution of the equation

$$-\tilde{Y}''(x,\lambda) + \tilde{Y}(x,\lambda)\cdot V(x) = \lambda\tilde{Y}(x,\lambda) \tag{11}$$

and satisfies the initial conditions $\tilde{Y}(0,\lambda) = \cos A,\ \ \tilde{Y}'(0,\lambda) = \sin A,\ \ \lambda \in \mathbb{C}$.

Operator solutions of equation (10) decreasing and increasing at infinity will be denoted by $\Phi_1(x,\lambda)$, $\Psi_1(x,\lambda)$, respectively, and the corresponding solutions of the equation (11) will be denoted by $\tilde{\Phi}(x,\lambda)$ and $\tilde{\Psi}(x,\lambda)$, respectively. For the system of operator solutions $Y(x,\lambda)\,,\tilde{\Phi}(x,\lambda) \in B(\mathbf{H})$ of the equations (1) and (11), respectively, the corresponding Wronskian has the following form: $W\left\{\tilde{\Phi}, Y\right\} = \tilde{\Phi}'(x,\lambda)\, Y(x,\lambda) - \tilde{\Phi}(x,\lambda)\, Y'(x,\lambda)$.

Denote

$$G(x,\,t,\,\lambda) = \begin{cases} Y(x,\,\lambda)\left(W\left\{\tilde{\Phi}, Y\right\}\right)^{-1}\tilde{\Phi}(t,\lambda) & 0 \leqslant x \leqslant t, \\ -\Phi(x,\,\lambda)\left(W\left\{\tilde{Y}, \Phi\right\}\right)^{-1}\tilde{Y}\,(t,\lambda) & x \geqslant t \end{cases}.$$

It is proved in [13] that the operator function $G(x, t, \lambda)$ is the Green's function of the differential operator L, i.e. it possesses all the classical properties of the Green's function. In particular, for a fixed t, the function $G(x, t, \lambda)$ of the variable x is an operator solution of equation (1) on each of the intervals $[0, t)$, (t, ∞), and satisfies the boundary condition (6), and for a fixed x, the function $G(x, t, \lambda)$ satisfies equation (11) in the variable t on each of the intervals $[0, x)$, (x, ∞), and it satisfies the boundary condition $(\cos A)^* \cdot y'(0) - (\sin A)^* \cdot y(0) = 0$. We consider the operator R_λ defined in $L_2(\mathbf{H}, (0, \infty))$ by the relation

$$
\begin{aligned}
(R_\lambda f)(x) = \int_0^\infty G(x, t, \lambda) f(t)\, dt = \\
= -\int_0^x \Phi(x, \lambda) \left(W\left\{\tilde{Y}, \Phi\right\}\right)^{-1} \tilde{Y}(t, \lambda) f(t)\, dt + \\
+ \int_x^\infty Y(x, \lambda) \left(W\left\{\tilde{\Phi}, Y\right\}\right)^{-1} \tilde{\Phi}(t, \lambda) f(t)\, dt.
\end{aligned} \tag{12}
$$

The operator R_λ is the resolvent of the operator L (see [14]).

Similar to [17] and [10], we define the normalizing polynomials by the formulas

$$
\begin{aligned}
N_j(t) = \\
= e^{-\lambda_j t} \operatorname{Re} s_{\lambda_j} \left\{ e^{\lambda t} \left(W\left\{\tilde{Y}, \Phi\right\}\right)^{-1} W\left\{\tilde{Y}, \Psi\right\} \right\}
\end{aligned}
$$

or

$$
\begin{aligned}
N_j(t) = \\
= \sum_{k=0}^{r_j - 1} \left(\sum_{l=0}^{r_j - (k+1)} \operatorname{Re} s_{\lambda_j} \left\{ \left(W\left\{\tilde{Y}, \Phi\right\}\right)^{-1} (\lambda - \lambda_j)^{l+k} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} W\left\{\tilde{Y}, \Psi\right\} \Big|_{\lambda = \lambda_j} \right) \frac{t^k}{k!}.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\frac{d^k}{dt^k} (N_j(t)) \Big|_{t=0} = \\
= \sum_{l=0}^{r_j - (k+1)} \operatorname{Re} s_{\lambda_j} \left\{ \left(W\left\{\tilde{Y}, \Phi\right\}\right)^{-1} (\lambda - \lambda_j)^{l+k} \right\} \frac{1}{l!} \frac{d^l}{d\lambda^l} W\left\{\tilde{Y}, \Psi\right\} \Big|_{\lambda = \lambda_j}.
\end{aligned} \tag{13}
$$

Lemma 1. *(see [15]). If the operators $A(\lambda)$ and $C(\lambda)$ are the entire functions and the operator $B(\lambda)$ has a pole of order r at the point λ_0, then the residue of the operator $A(\lambda)B(\lambda)C(\lambda)$ at λ_0 can be calculated as follows:*

$$\operatorname{Re}s_{\lambda_0}\{A(\lambda)B(\lambda)C(\lambda)\} = \sum_{k=0}^{r-1}\frac{1}{k!}\frac{d^k}{d\lambda^k}A(\lambda)\Big|_{\lambda=\lambda_0}\times$$

$$\times\sum_{l=0}^{r-(k+1)}\operatorname{Re}s_{\lambda_j}\left\{B(\lambda)(\lambda-\lambda_j)^{k+l}\right\}\frac{1}{l!}\frac{d^l}{d\lambda^l}C(\lambda)\Big|_{\lambda=\lambda_0}.$$

Equality (13), by virtue of Lemma 1, can be rewritten as

$$\frac{d^k}{dt^k}(N_j(t))\Big|_{t=0} = \operatorname{Re}s_{\lambda_j}\left\{\left(W\left\{\tilde{Y},\Phi\right\}\right)^{-1}(\lambda-\lambda_j)^k W\left\{\tilde{Y},\Psi\right\}\right\}.$$

It is proved in [15] proved that the Green function $G(x, t, z)$ has the form

$$G(x,t,z) = \sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1}\frac{1}{k!}\frac{d^k}{d\lambda^k}\left(\frac{1}{\lambda-z}\Phi(x,\lambda)\right)\Big|_{\lambda=\lambda_j}\times$$

$$\times\sum_{l=0}^{r_j-(k+1)}\frac{1}{l!}\frac{d^{k+l}}{dt^{k+l}}N_j(t)\Big|_{t=0}\frac{d^l}{d\lambda^l}\left(\tilde{\Phi}(t,\lambda)\right)\Big|_{\lambda=\lambda_j}. \qquad (14)$$

3. Parseval equality

Let $S(x)$, $T(x)$ be arbitrary operator functions of $L_2(\mathbf{H},(0,\infty))$. Denote

$$E(S,\lambda) = \int_0^\infty S(t)\Phi(t,\lambda)\,dt, \qquad (15)$$

$$\tilde{E}(S,\lambda) = \int_0^\infty \tilde{\Phi}(t,\lambda)S(t)\,dt.$$

Theorem 1. *Suppose that the coefficients of the problem* (1), (6) *satisfy the conditions* (3), (4) *for $\alpha > 1$ or the condition* (5) *for $0 < \alpha \leqslant 1$. Then, for arbitrary operator functions $S(x)$, $T(x) \in L_2(\mathbf{H},(0,\infty))$, the following expansion* *with respect to the solutions* $\Phi(x,\lambda)$ *and* $\tilde{\Phi}(x,\lambda)$ *of the equations* (1) *and* (11), *respectively, hold:*

$$S(x)=\sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1}\frac{1}{k!}\frac{d^k}{d\lambda^k}\left(E(S,\lambda)\right)\Big|_{\lambda=\lambda_j}\times$$

$$\times\sum_{l=0}^{r_j-(k+1)}\frac{1}{l!}\frac{d^{k+l}}{dt^{k+l}}N_j(t)\Big|_{t=0}\frac{d^l}{d\lambda^l}\left(\tilde{\Phi}(x,\lambda)\right)\Big|_{\lambda=\lambda_j}, \tag{16}$$

$$S(x)=\sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1}\frac{1}{k!}\frac{d^k}{d\lambda^k}\left(\Phi(x,\lambda)\right)\Big|_{\lambda=\lambda_j}\times$$

$$\times\sum_{l=0}^{r_j-(k+1)}\frac{1}{l!}\frac{d^{k+l}}{dt^{k+l}}N_j(t)\Big|_{t=0}\frac{d^l}{d\lambda^l}\left(\tilde{E}(S,\lambda)\right)\Big|_{\lambda=\lambda_j}, \tag{17}$$

and the Parseval equality

$$\int_0^{\infty}S(x)\,T(x)\,dx=\sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1}\frac{1}{k!}\frac{d^k}{d\lambda^k}E(S,\lambda)|_{\lambda=\lambda_j}\times$$

$$\times\sum_{l=0}^{r_j-(k+1)}\frac{1}{l!}\frac{d^{k+l}}{dt^{k+l}}N_j(t)\Big|_{t=0}\frac{d^l}{d\lambda^l}\tilde{E}(T,\lambda)\Big|_{\lambda=\lambda_j}, \tag{18}$$

is true.

Proof. Since $\left(\tilde{l}-zI\right)\left[\tilde{\Phi}(x,\lambda)\right]=(\lambda-z)\,\tilde{\Phi}(x,\lambda)$, we conclude that $\tilde{\Phi}(x,\lambda)=\frac{1}{\lambda-z}\left(\tilde{l}-zI\right)\left[\tilde{\Phi}(x,\lambda)\right]$ for $\lambda\neq z$. It follows that

$$\tilde{E}(R_z[T],\lambda)=\int_0^{\infty}\tilde{\Phi}(x,\lambda)\,R_z[T](x)\,dx=$$

$$=\frac{1}{\lambda-z}\int_0^{\infty}\left(\tilde{l}-zI\right)\left[\tilde{\Phi}(x,\lambda)\right]R_z[T](x)\,dx.$$

For a finite function $T(x) \in L_2(\mathbf{H}, (0,\infty))$, by integrating by parts twice, we get

$$\tilde{E}(R_z[T], \lambda) = \frac{1}{\lambda - z}\int_0^\infty \tilde{\Phi}(x,\lambda)\ (l - zI)\, R_z[T](x)\, dx =$$

$$= \frac{1}{\lambda - z}\int_0^\infty \tilde{\Phi}(x,\lambda)\, T(x)\, dx = \frac{1}{\lambda - z}\tilde{E}(T,\lambda). \tag{19}$$

By (12), (14) and (15), for an arbitrary operator $T(x) \in L_2(\mathbf{H}, (0,\infty))$ we have

$$(R_z[T])(x) = \sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1} \frac{1}{k!}\frac{d^k}{d\lambda^k}\left(\frac{1}{\lambda - z}\Phi(x,\lambda)\right)\Big|_{\lambda=\lambda_j} \times$$

$$\times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!}\frac{d^{k+l}}{dt^{k+l}} N_j(t)\Big|_{t=0} \frac{d^l}{d\lambda^l}\tilde{E}(T,\lambda)\Big|_{\lambda=\lambda_j}.$$

Denoting inner sum over l by $a_k(\lambda_j)$, we rewrite the formula in the form

$$(R_z[T])(x) = \sum_{j=1}^{\infty}\sum_{k=0}^{r_j-1}\sum_{s=0}^{k} \frac{1}{s!}\frac{1}{(\lambda_j - z)^{k-s+1}}\frac{d^s}{d\lambda^s}(\Phi(x,\lambda))\Big|_{\lambda=\lambda_j} a_k(\lambda_j).$$

We change the summation limits by k and s:

$$(R_z[T])(x) = \sum_{j=1}^{\infty}\sum_{s=0}^{r_j-1} \frac{1}{s!}\frac{d^s}{d\lambda^s}(\Phi(x,\lambda))\Big|_{\lambda=\lambda_j} \sum_{k=s}^{r_j-1}\frac{1}{(\lambda_j - z)^{k-s+1}} a_k(\lambda_j).$$

In what follows, values of the function $\Phi(x,\lambda)$ and its derivatives in λ will be considered at the point $\lambda = \lambda_j$, and the values of the function $N_j(t)$ and its derivatives will be considered at $t = 0$. Therefore, in order to simplify the notation, we will omit specifying the point where the function is considered. Denoting $k - s = u$, we obtain

$$(R_z[T])(x) = \sum_{j=1}^{\infty}\sum_{s=0}^{r_j-1} \frac{1}{s!}\frac{d^s}{d\lambda^s}(\Phi(x,\lambda)) \sum_{u=0}^{r_j-(s+1)}\frac{1}{(\lambda_j - z)^{u+1}}\times$$

$$\times \sum_{l=0}^{r_j-(s+u+1)} \frac{1}{l!} \frac{d^{s+u+l}}{dt^{s+u+l}} N_j(t) \frac{d^l}{d\lambda^l} \tilde{E}(T,\lambda).$$

We change the summation limits by u and l:

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (\Phi(x,\lambda)) \times$$

$$\times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \left(\sum_{u=0}^{r_j-(k+l+1)} \frac{1}{(\lambda_j - z)^{u+1}} \frac{d^{k+u+l}}{dt^{k+u+l}} N_j(t) \right) \frac{d^l}{d\lambda^l} \tilde{E}(T,\lambda).$$

Making the change of $u + l = p$, we obtain

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (\Phi(x,\lambda)) \times$$

$$\times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \left(\sum_{p=l}^{r_j-(k+1)} \frac{1}{(\lambda_j - z)^{p-l+1}} \frac{d^{k+p}}{dt^{k+p}} N_j(t) \right) \frac{d^l}{d\lambda^l} \tilde{E}(T,\lambda).$$

We change the summation limits by l and p:

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (\Phi(x,\lambda)) \times$$

$$\times \sum_{p=0}^{r_j-(k+1)} \frac{d^{k+p}}{dt^{k+p}} N_j(t) \left(\sum_{l=0}^{p} \frac{1}{l!} \frac{1}{(\lambda_j - z)^{p-l+1}} \frac{d^l}{d\lambda^l} \tilde{E}(T,\lambda) \right).$$

Here

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (\Phi(x,\lambda)) \times$$

$$\times \sum_{p=0}^{r_j-(k+1)} \frac{1}{p!} \frac{d^{k+p}}{dt^{k+p}} N_j(t) \frac{d^p}{d\lambda^p} \left(\frac{1}{\lambda - z} \tilde{E}(T, \lambda) \right).$$

In view of the formula (19), we have

$$(R_z[T])(x) = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (\Phi(x,\lambda)) \Big|_{\lambda=\lambda_j} \times$$

$$\times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{E}(R_z[T], \lambda) \right) \Big|_{\lambda=\lambda_j}.$$

It follows that

$$\int_0^{\infty} S(x)(R_z[T])(x)\,dx = \sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (E(S,\lambda)) \Big|_{\lambda=\lambda_j} \times$$

$$\times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{E}(R_z[T], \lambda) \right) \Big|_{\lambda=\lambda_j}.$$

Thus, for any finite function $T(x) \in L_2(\mathbf{H}, (0,\infty))$, we have

$$\int_0^{\infty} S(x)(R_z[T])(x)\,dx = \int_0^{\infty} \left(\sum_{j=1}^{\infty} \sum_{k=0}^{r_j-1} \frac{1}{k!} \frac{d^k}{d\lambda^k} (E(S,\lambda)) \Big|_{\lambda=\lambda_j} \times \right.$$

$$\left. \times \sum_{l=0}^{r_j-(k+1)} \frac{1}{l!} \frac{d^{k+l}}{dt^{k+l}} N_j(t) \Big|_{t=0} \frac{d^l}{d\lambda^l} \left(\tilde{\Phi}(x,\lambda) \right) \Big|_{\lambda=\lambda_j} \right) (R_z[T])(x)\,dx$$

Since the range of resolvent is dense in $L_2(\mathbf{H}, (0,\infty))$, we obtain the formula (16). The formula (17) is proved similarly. By multiplying both sides of (16) by $T(x)$ and then integrating, we obtain the Parseval equality (18). ◄

4. Conclusion

This work actually completes a series of investigations for non-self-adjoint differential operator with block-triangular operator potential increasing at infinity. We construct a fundamental system of solutions, one of which is decreasing at infinity, and the other is increasing. Green's function and resolution of the operator are constructed. Structure of the spectrum is established. The series expansion of the Green's function is obtained. Parseval equality is proved for a differential operator with block-triangular operator potential.

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