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On Approximation of Hexagonal Fourier Series

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Abstract. Let the function f belong to the Hölder class $H^\alpha\left(\overline{\Omega}\right)$, $0<\alpha\leq 1$, where Ω is the spectral set of the hexagonal lattice in the Euclidean plane. Also, let $p=(p_n)$ and $q=(q_n)$ be two sequences of non-negative real numbers such that $p_n<q_n$ and $q_n\to\infty$ as $n\to\infty$. The order of approximation of f by deferred Cesàro means $D_n(p,q;f)$ of its hexagonal Fourier series is estimated in the uniform and Hölder norms.

Key Words and Phrases: deferred Cesàro mean, hexagonal Fourier series, Hölder class.

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1. Introduction

In the study of trigonometric approximation of $2\pi-$periodic functions on the real line, the partial sums and various means (Cesàro means, Abel-Poisson means, de la Vallèe-Poussin means, etc.) of Fourier series are most useful tools. Especially, there are many results on the order of approximation of continuous $2\pi-$periodic functions in the uniform norm by partial sums of trigonometric (or equivalently exponential) Fourier series and means of these sums. These results can be found in the monographs [3, 11, 13] and in the survey [7]. Also, there are several theorems about approximation of continuous $2\pi-$periodic functions by partial sums, Cesàro means and their generalizations in the Hölder norm (see, for example, [9] and [2]).

The order of approximation of functions defined on cubes of the d-dimensional Euclidean space $\mathbb{R}^d$ was studied by several authors. The common point of these studies is the assumption that the functions are $2\pi-$periodic with respect to each of their variables (see, for example [11, Sections 5.3 and 6.3] and [13, Vol. II, Chapter XVII]). But, in the case of non-tensor product domains of $\mathbb{R}^d$, other types of periodicity are needed to study approximation problems. The most notable periodicity is the periodicity defined by lattices. The discrete Fourier analysis on lattices was developed in [8].

A lattice is the discrete subgroup $A\mathbb{Z}^d = \{Ak : k \in \mathbb{Z}^d\}$ of the Euclidean space $\mathbb{R}^d$, where A is a non-singular $d \times d$ matrix$-$the generator matrix of the lattice. The lattice $A^{-tr}\mathbb{Z}^d$, where A^{-tr} is the transpose of the inverse matrix A^{-1}, is called the dual lattice of $A\mathbb{Z}^d$. A bounded open set $\Omega \subset \mathbb{R}^d$ is said to tile $\mathbb{R}^d$ with the lattice $A\mathbb{Z}^d$ if

$$\sum_{\alpha \in A\mathbb{Z}^d} \chi_\Omega (x + \alpha) = 1,$$

for almost all $x \in \mathbb{R}^d$. In this case the set Ω is called a spectral set for the lattice $A\mathbb{Z}^d$, written as $\Omega + A\mathbb{Z}^d = \mathbb{R}^d$. The spectral set Ω is not unique. It is specified that it contains 0 as an interior point and tiles $\mathbb{R}^d$ with the lattice $A\mathbb{Z}^d$ without overlapping and without gap, i.e.

$$\sum_{k \in \mathbb{Z}^d} \chi_\Omega (x + Ak) = 1,$$

for all $x \in \mathbb{R}^d$, and $\Omega + Ak$ and $\Omega + Aj$ are disjoint if $k \neq j$. For example, we can take $\Omega = \left[-\frac{1}{2}, \frac{1}{2}\right)^d$ for the standard lattice $\mathbb{Z}^d$ (the lattice generated by the identity matrix).

Let Ω be the spectral set of the lattice $A\mathbb{Z}^d$. $L^2(\Omega)$ becomes a Hilbert space with respect to the inner product

$$\langle f, g \rangle_\Omega := \frac{1}{|\Omega|} \int_\Omega f(x) \overline{g(x)} dx,$$

where $|\Omega|$ is the $d-$dimensional Lebesgue measure of Ω. A theorem of Fuglede states that the set $\{e^{2\pi i \langle \alpha, x \rangle} : \alpha \in A^{-tr}\mathbb{Z}^d\}$ is an orthonormal basis of the Hilbert space $L^2(\Omega)$, where $\langle \alpha, x \rangle$ is the usual Euclidean inner product of α and x ([4]). This theorem suggests that, by using the exponentials $e^{2\pi i \langle \alpha, x \rangle}$ $\left(\alpha \in A^{-tr}\mathbb{Z}^d\right)$ one can study Fourier series and approximation on the spectral set of the lattice $A\mathbb{Z}^d$.

A function f is said to be periodic with respect to the lattice $A\mathbb{Z}^d$ if

$$f(x + Ak) = f(x)$$

for all $k \in \mathbb{Z}^d$.

If we consider the standard lattice $\mathbb{Z}^d$ and its spectral set $\left[-\frac{1}{2}, \frac{1}{2}\right)^d$, it is clear that Fourier series with respect to this lattice coincide with usual multiple Fourier series of functions of d variables.

2. Hexagonal Fourier Series

In the Euclidean plane $\mathbb{R}^2$, besides the standard lattice $\mathbb{Z}^2$ and the rectangular domain $\left[-\frac{1}{2},\frac{1}{2}\right)^2$, the simplest lattice is the hexagonal lattice and the simplest spectral set is the regular hexagon. Also, it is well known that the hexagonal lattice offers the densest packing of $\mathbb{R}^2$ with unit balls. Thus, the hexagonal lattice and hexagonal Fourier series have great importance in Fourier analysis.

The generator matrix and the spectral set of the hexagonal lattice $H\mathbb{Z}^2$ are given by

$$H=\begin{bmatrix}\sqrt{3} & 0\\ -1 & 2\end{bmatrix}$$

and

$$\Omega_H=\left\{(x_1,x_2)\in\mathbb{R}^2 : -1\leq x_2,\frac{\sqrt{3}}{2}x_1\pm\frac{1}{2}x_2<1\right\}.$$

It is more convenient to use the homogeneous coordinates (t_1,t_2,t_3) that satisfy $t_1+t_2+t_3=0$. If we define

$$t_1:=-\frac{x_2}{2}+\frac{\sqrt{3}x_1}{2},\ t_2:=x_2,\ t_3:=-\frac{x_2}{2}-\frac{\sqrt{3}x_1}{2}, \tag{1}$$

the hexagon Ω_H becomes

$$\Omega=\left\{(t_1,t_2,t_3)\in\mathbb{R}^3 : -1\leq t_1,t_2,-t_3<1,\ t_1+t_2+t_3=0\right\},$$

which is the intersection of the plane $t_1+t_2+t_3=0$ with the cube $[-1,1]^3$.

We use bold letters $\mathbf{t}$ for homogeneous coordinates and we denote by $\mathbb{R}^3_H$ the plane $t_1+t_2+t_3=0$, that is

$$\mathbb{R}^3_H=\left\{\mathbf{t}=(t_1,t_2,t_3)\in\mathbb{R}^3 : t_1+t_2+t_3=0\right\}.$$

Also we use the notation $\mathbb{Z}^3_H$ for the set of points in $\mathbb{R}^3_H$ with integer components, that is $\mathbb{Z}^3_H=\mathbb{Z}^3\cap\mathbb{R}^3_H$.

In the homogeneous coordinates, the inner product on $L^2(\Omega)$ becomes

$$\langle f,g\rangle_H=\frac{1}{|\Omega|}\int_{\Omega} f(\mathbf{t})\overline{g(\mathbf{t})}d\mathbf{t},$$

where $|\Omega|$ denotes the area of Ω, and the orthonormal basis of $L^2(\Omega)$ becomes

$$\left\{\phi_{\mathbf{j}}(\mathbf{t})=e^{\frac{2\pi i}{3}\langle\mathbf{j},\mathbf{t}\rangle} : \mathbf{j}\in\mathbb{Z}^3_H,\ \mathbf{t}\in\mathbb{R}^3_H\right\}.$$

Also, a function f is periodic with respect to the hexagonal lattice (or $H-$periodic) if and only if $f(\mathbf{t}) = f(\mathbf{t}+\mathbf{s})$ whenever $\mathbf{s} \equiv \mathbf{0}\ (mod 3)$, where $\mathbf{t} \equiv \mathbf{s}\ (mod 3)$ defined as

$$t_1 - s_1 \equiv t_2 - s_2 \equiv t_3 - s_3 \ (mod 3).$$

It is clear that the functions $\phi_{\mathbf{j}}(\mathbf{t})$ are $H-$periodic, and if the function f is $H-$periodic, then

$$\int_{\Omega} f(\mathbf{t}+\mathbf{s})\, d\mathbf{t} = \int_{\Omega} f(\mathbf{t})\, d\mathbf{t}. \ \left(\mathbf{s} \in \mathbb{R}_H^3\right)$$

For every natural number n, we define a subset of $\mathbb{Z}_H^3$ by

$$\mathbb{H}_n := \left\{\mathbf{j} = (j_1, j_2, j_3) \in \mathbb{Z}_H^3 : -n \le j_1, j_2, j_3 \le n\right\}.$$

Note that, $\mathbb{H}_n$ consists of all points with integer components inside the hexagon $n\overline{\Omega}$. Members of the set

$$\mathcal{H}_n := \operatorname{span}\{\phi_{\mathbf{j}} : \mathbf{j} \in \mathbb{H}_n\}, \ \ (n \in \mathbb{N})$$

are called hexagonal trigonometric polynomials. It is clear that the dimension of $\mathcal{H}_n$ is $\#\mathbb{H}_n = 3n^2 + 3n + 1$.

The hexagonal Fourier series of an $H-$periodic function $f \in L^1(\Omega)$ is

$$f(\mathbf{t}) \sim \sum_{\mathbf{j} \in \mathbb{Z}_H^3} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}), \tag{2}$$

where

$$\widehat{f}_{\mathbf{j}} = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t})\, e^{-\frac{2\pi i}{3}\langle \mathbf{j}, \mathbf{t}\rangle} d\mathbf{t}, \ \ \left(\mathbf{j} \in \mathbb{Z}_H^3\right).$$

The nth partial sum of the series (2) is defined by

$$S_n(f)(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \widehat{f}_{\mathbf{j}} \phi_{\mathbf{j}}(\mathbf{t}). \ \ (n \in \mathbb{N})$$

It is clear that

$$S_n(f)(\mathbf{t}) = \frac{1}{|\Omega|} \int_{\Omega} f(\mathbf{t}-\mathbf{s})\, D_n(\mathbf{s})\, d\mathbf{s}, \tag{3}$$

where D_n is the Dirichlet kernel, defined by

$$D_n(\mathbf{t}) := \sum_{\mathbf{j} \in \mathbb{H}_n} \phi_{\mathbf{j}}(\mathbf{t}).$$

It is known that the Dirichlet kernel can be expressed as

$$D_n(\mathbf{t}) = \Theta_n(\mathbf{t}) - \Theta_{n-1}(\mathbf{t}), \quad (n \in \mathbb{N}), \tag{4}$$

where

$$\Theta_n(\mathbf{t}) := \frac{\sin\frac{(n+1)(t_1-t_2)\pi}{3}\sin\frac{(n+1)(t_2-t_3)\pi}{3}\sin\frac{(n+1)(t_3-t_1)\pi}{3}}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}} \tag{5}$$

for $\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3_H$ ([10, 8]).

We refer to [8] and [12] for more detailed information about Fourier analysis on lattices and hexagonal Fourier series.

We denote by $C_H(\overline{\Omega})$ the Banach space of $H-$periodic continuous functions on $\mathbb{R}^3_H$, equipped with the uniform norm

$$\|f\|_{C_H(\overline{\Omega})} = \sup_{\mathbf{t}\in\overline{\Omega}} |f(\mathbf{t})|,$$

and by $H^\alpha(\overline{\Omega})$ $(0 < \alpha \le 1)$ the Hölder class, that is the class of functions $f \in C_H(\overline{\Omega})$ for which

$$\sup_{\mathbf{t}\neq\mathbf{s}} \frac{|f(\mathbf{t}) - f(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^\alpha} < \infty$$

holds, where

$$\|\mathbf{t}\| := \max\{|t_1|, |t_2|, |t_3|\}, \quad \left(\mathbf{t} = (t_1, t_2, t_3) \in \mathbb{R}^3_H\right).$$

$H^\alpha(\overline{\Omega})$ $(0 < \alpha \le 1)$ becomes a Banach space with respect to the norm

$$\|f\|_{H^\alpha(\overline{\Omega})} := \|f\|_{C_H(\overline{\Omega})} + \sup_{\mathbf{t}\neq\mathbf{s}} \frac{|f(\mathbf{t}) - f(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^\alpha}.$$

Y. Xu [12] proved that the Abel-Poisson means and the sequence of Cesàro $(C, 1)$ means of the Fourier series of a function $f \in C_H(\overline{\Omega})$ converge to this function uniformly on $\overline{\Omega}$. Later, the order of approximation by Abel-Poisson and $(C, 1)$ means of Fourier series of functions belonging to the class $H^\alpha(\overline{\Omega})$, $(0 < \alpha \le 1)$ was investigated in uniform norm ([6]) and in the Hölder norm ([5]).

In this work, we will estimate the order of approximation by deferred Cesàro means of Fourier series of functions belonging to the Hölder class $H^\alpha(\overline{\Omega})$ in uniform and Hölder norms, and generalize some results of [6] and [5].

3. Main results

Let $p = (p_n)$ and $q = (q_n)$ be two sequences of non-negative integers such that

$$p_n < q_n \text{ and } \lim_{n\to\infty} q_n = \infty. \tag{6}$$

The deferred Cesàro means of the series (2) are defined by

$$D_n(p,q;f)(\mathbf{t}) := \frac{1}{q_n - p_n}\sum_{k=p_n+1}^{q_n} S_k(f)(\mathbf{t}).$$

It is known that the $D_n(p,q)$ summability method is regular under conditions (6) and generalizes the Cesàro $(C,1)$ method if and only if $p_n \ll q_n - p_n$ ([1]).

By considering (3) and (4) we obtain

$$\begin{aligned} D_n(p,q;f)(\mathbf{t}) &= \frac{1}{|\Omega|}\int_\Omega f(\mathbf{t}-\mathbf{s})\left(\frac{1}{q_n-p_n}\sum_{k=p_n+1}^{q_n} D_k(\mathbf{s})\right)d\mathbf{s} \\ &= \frac{1}{q_n-p_n}\frac{1}{|\Omega|}\int_\Omega f(\mathbf{t}-\mathbf{s})\left(\sum_{k=p_n+1}^{q_n}\Theta_k(\mathbf{s})-\Theta_{k-1}(\mathbf{s})\right)d\mathbf{s} \\ &= \frac{1}{q_n-p_n}\frac{1}{|\Omega|}\int_\Omega f(\mathbf{t}-\mathbf{s})\left(\Theta_{q_n}(\mathbf{s})-\Theta_{p_n}(\mathbf{s})\right)d\mathbf{s}. \end{aligned}$$

Hence we have

$$f(\mathbf{t}) - D_n(p,q;f)(\mathbf{t}) =$$

$$= \frac{1}{(q_n-p_n)}\frac{1}{|\Omega|}\int_\Omega (f(\mathbf{t})-f(\mathbf{t}-\mathbf{s}))\left(\Theta_{q_n}(\mathbf{s})-\Theta_{p_n}(\mathbf{s})\right)d\mathbf{s}, \tag{7}$$

for each $f \in L^1(\Omega)$ and $\mathbf{t} \in \mathbb{R}^3_H$.

If we take $q_n = n$ and $p_n = 0$ for $n = 1,2,...$, $D_n(p,q;f)$ becomes the $(C,1)$ means $S_n^{(1)}(f)$.

Hereafter, we shall write $A \ll B$ for the quantities A and B, if there exists a constant $K > 0$ such that $A \leq KB$ holds.

The rate of approximation by $(C,1)$ means of hexagonal Fourier series was estimated as follows:

Theorem A ([6]). *Let* $f \in H^\alpha\left(\overline{\Omega}\right)$, $0 < \alpha \leq 1$. *Then*

$$\left\|f - S_n^{(1)}(f)\right\|_{C_H\left(\overline{\Omega}\right)} \ll \begin{cases} \frac{1}{n^\alpha}, & \alpha < 1, \\ \frac{(\log n)^2}{n}, & \alpha = 1, \end{cases} \tag{8}$$

holds.

In this work we generalize Theorem A. Our main theorem is the following.

Theorem 1. *For each* $f \in H^{\alpha}\left(\overline{\Omega}\right)$, $0 < \alpha \leq 1$, *the estimate*

$$\|f - D_n(p,q;f)\|_{C_H(\overline{\Omega})} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \begin{cases} \frac{1}{(q_n-p_n)^{\alpha}}, & \alpha < 1, \\ \frac{(\log(2(q_n-p_n)))^2}{q_n-p_n}, & \alpha = 1, \end{cases} \tag{9}$$

holds.

Proof. Since $f \in H^{\alpha}\left(\overline{\Omega}\right)$, by (7) we have

$$|f(\mathbf{t}) - D_n(p,q;f)(\mathbf{t})| \ll \frac{1}{(q_n - p_n)} \frac{1}{|\Omega|} \int\limits_{\Omega} \|\mathbf{s}\|^{\alpha} \left|\Theta_{q_n}(\mathbf{s}) - \Theta_{p_n}(\mathbf{s})\right| d\mathbf{s}.$$

Since the integrated function is symmetric with respect to its variables, it is sufficient to estimate the integral

$$I_n := \int\limits_{\Delta} \|\mathbf{t}\|^{\alpha} \left|\Theta_{q_n}(\mathbf{t}) - \Theta_{p_n}(\mathbf{t})\right| d\mathbf{t},$$

where

$$\begin{aligned} \Delta \quad : \quad &= \left\{\mathbf{t} = (t_1,t_2,t_3) \in \mathbb{R}^3_H : 0 \leq t_1, t_2, -t_3 \leq 1\right\} \\ &= \left\{(t_1,t_2) : t_1 \geq 0,\ t_2 \geq 0,\ t_1 + t_2 \leq 1\right\}, \end{aligned}$$

which is one of the six equilateral triangles in $\overline{\Omega}$. By (5) and some simple trigonometric identities

$$\Theta_{q_n}(\mathbf{t}) - \Theta_{p_n}(\mathbf{t}) =$$

$$= 2\cos\left(\left(\frac{q_n + p_n}{2} + 1\right)\frac{(t_3 - t_1)\pi}{3}\right)\sin\left(\left(\frac{q_n - p_n}{2}\right)\frac{(t_3 - t_1)\pi}{3}\right) \times$$

$$\times \frac{\sin\left((q_n+1)\frac{(t_1-t_2)\pi}{3}\right)\sin\left((q_n+1)\frac{(t_2-t_3)\pi}{3}\right)}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}} +$$

$$+2\cos\left(\left(\frac{q_n + p_n}{2} + 1\right)\frac{(t_2 - t_3)\pi}{3}\right)\sin\left(\left(\frac{q_n - p_n}{2}\right)\frac{(t_2 - t_3)\pi}{3}\right) \times$$

$$\times \frac{\sin\left((q_n+1)\frac{(t_1-t_2)\pi}{3}\right)\sin\left((p_n+1)\frac{(t_3-t_1)\pi}{3}\right)}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}} +$$

$$+2\cos\left(\left(\frac{q_n+p_n}{2}+1\right)\frac{(t_1-t_2)\pi}{3}\right)\sin\left(\left(\frac{q_n-p_n}{2}\right)\frac{(t_1-t_2)\pi}{3}\right)\times$$
$$\times\frac{\sin\left((p_n+1)\frac{(t_2-t_3)\pi}{3}\right)\sin\left((p_n+1)\frac{(t_3-t_1)\pi}{3}\right)}{\sin\frac{(t_1-t_2)\pi}{3}\sin\frac{(t_2-t_3)\pi}{3}\sin\frac{(t_3-t_1)\pi}{3}}.$$

If we use the change of variables

$$s_1:=\frac{t_1-t_3}{3},\ s_2:=\frac{t_2-t_3}{3}, \tag{10}$$

we obtain

$$I_n\le 3\int\limits_{\widetilde{\Delta}}(s_1+s_2)^\alpha\left(|L_1(s_1,s_2)|+|L_2(s_1,s_2)|+|L_3(s_1,s_2)|\right)ds_1ds_2,$$

where

$$\widetilde{\Delta}:=\left\{(s_1,s_2):0\le s_1\le 2s_2, 0\le s_2\le 2s_1, s_1+s_2\le 1\right\},$$

and

$$L_1(s_1,s_2) \ : \ = \frac{\sin\left(\left(\frac{q_n-p_n}{2}\right)(s_1\pi)\right)\sin\left((q_n+1)(s_1-s_2)\pi\right)\sin\left((q_n+1)s_2\pi\right)}{\sin\left((s_1-s_2)\pi\right)\sin(s_2\pi)\sin(s_1\pi)}$$
$$L_2(s_1,s_2) \ : \ = \frac{\sin\left(\left(\frac{q_n-p_n}{2}\right)(s_2\pi)\right)\sin\left((q_n+1)(s_1-s_2)\pi\right)\sin\left((p_n+1)(s_1\pi)\right)}{\sin\left((s_1-s_2)\pi\right)\sin(s_2\pi)\sin(s_1\pi)}$$
$$L_3(s_1,s_2) \ : \ = \frac{\sin\left(\left(\frac{q_n-p_n}{2}\right)(s_1-s_2)\pi\right)\sin\left((p_n+1)(s_2\pi)\right)\sin\left((p_n+1)(s_1\pi)\right)}{\sin\left((s_1-s_2)\pi\right)\sin(s_2\pi)\sin(s_1\pi)}.$$

Since the integrated function is symmetric with respect to s_1 and s_2, we have

$$I_n\le 6\int\limits_{\Delta^*}(s_1+s_2)^\alpha\left(|L_1(s_1,s_2)|+|L_2(s_1,s_2)|+|L_3(s_1,s_2)|\right)ds_1ds_2,$$

where

$$\Delta^*:=\left\{(s_1,s_2)\in\widetilde{\Delta}:s_1\le s_2\right\},$$

i. e. the half of $\widetilde{\Delta}$. The change of variables

$$s_1:=\frac{u_1-u_2}{2},\ s_2:=\frac{u_1+u_2}{2} \tag{11}$$

transforms the triangle Δ^* onto triangle

$$\Gamma:=\left\{(u_1,u_2):0\le u_2\le\frac{u_1}{3}, 0\le u_1\le 1\right\},$$

and hence

$$I_n \leq 3 \int_{\Gamma} u_1^{\alpha} \left(\left| L_1^* (u_1, u_2) \right| + \left| L_2^* (u_1, u_2) \right| + \left| L_3^* (u_1, u_2) \right| \right) du_1 du_2,$$

where

$$L_1^* (u_1, u_2) := \frac{\sin \left(\left(\frac{q_n - p_n}{2} \right) \frac{(u_1 - u_2)\pi}{2} \right) \sin \left((q_n + 1) u_2 \pi \right) \sin \left((q_n + 1) \frac{(u_1 + u_2)\pi}{2} \right)}{\sin (u_2 \pi) \sin \left(\frac{(u_1 + u_2)\pi}{2} \right) \sin \left(\frac{(u_1 - u_2)\pi}{2} \right)}$$

$$L_2^* (u_1, u_2) := \frac{\sin \left(\left(\frac{q_n - p_n}{2} \right) \frac{(u_1 + u_2)\pi}{2} \right) \sin \left((q_n + 1) u_2 \pi \right) \sin \left((p_n + 1) \frac{(u_1 - u_2)\pi}{2} \right)}{\sin (u_2 \pi) \sin \left(\frac{(u_1 + u_2)\pi}{2} \right) \sin \left(\frac{(u_1 - u_2)\pi}{2} \right)}$$

$$L_3^* (u_1, u_2) := \frac{\sin \left(\left(\frac{q_n - p_n}{2} \right) u_2 \pi \right) \sin \left((p_n + 1) \frac{(u_1 + u_2)\pi}{2} \right) \sin \left((p_n + 1) \frac{(u_1 - u_2)\pi}{2} \right)}{\sin (u_2 \pi) \sin \left(\frac{(u_1 + u_2)\pi}{2} \right) \sin \left(\frac{(u_1 - u_2)\pi}{2} \right)}.$$

If we divide the triangle Γ into three parts as

$$\Gamma_1 := \left\{ (u_1, u_2) \in \Gamma : u_1 \leq \frac{1}{2 (q_n - p_n)} \right\},$$

$$\Gamma_2 := \left\{ (u_1, u_2) \in \Gamma : u_1 \geq \frac{1}{2 (q_n - p_n)},\ u_2 \leq \frac{1}{6 (q_n - p_n)} \right\},$$

$$\Gamma_3 := \left\{ (u_1, u_2) \in \Gamma : u_1 \geq \frac{1}{2 (q_n - p_n)},\ u_2 \geq \frac{1}{6 (q_n - p_n)} \right\},$$

we get

$$I_n \ll I_n^{(1)} + I_n^{(2)} + I_n^{(3)},$$

where

$$I_n^{(j)} = \int_{\Gamma_j} u_1^{\alpha} \left(\left| L_1^* (u_1, u_2) \right| + \left| L_2^* (u_1, u_2) \right| + \left| L_3^* (u_1, u_2) \right| \right) du_1 du_2, \quad (j = 1, 2, 3).$$

We shall need the inequalities

$$\left| \frac{\sin nt}{\sin t} \right| \leq n, \quad (n \in \mathbb{N}), \tag{12}$$

and

$$\sin t \geq \frac{2}{\pi} t \quad \left(0 \leq t \leq \frac{\pi}{2} \right) \tag{13}$$

to estimate integrals $I_n^{(1)}, I_n^{(2)}$ and $I_n^{(3)}$.

For $(u_1,u_2)\in\Gamma$,

$$\sin\left(\frac{u_1+u_2}{2}\pi\right)\geq\frac{\sqrt{3}}{2}\sin\frac{u_1\pi}{2},$$

and by (13) we obtain

$$\frac{1}{\sin\left(\frac{u_1+u_2}{2}\pi\right)}\leq\frac{2}{\sqrt{3}}\frac{1}{u_1}\quad(u_1\neq 0)\,. \tag{14}$$

By the inequality (12) we get

$$\begin{aligned}\left|L_1^*(u_1,u_2)\right| &\ll (q_n-p_n)(q_n+1)^2,\\ \left|L_2^*(u_1,u_2)\right| &\ll (q_n-p_n)(q_n+1)(p_n+1),\end{aligned}$$

and

$$\left|L_3^*(u_1,u_2)\right|\ll(q_n-p_n)(p_n+1)^2$$

for $(u_1,u_2)\in\Gamma_1$. Hence

$$\int\limits_{\Gamma_1}u_1^{\alpha}\left|L_j^*(u_1,u_2)\right|du_1du_2\,(q_n-p_n)(q_n+1)^2\int\limits_{\Gamma_1}u_1^{\alpha}du_1du_2=$$

$$=(q_n-p_n)(q_n+1)^2\int\limits_{0}^{\frac{1}{6(q_n-p_n)}}\int\limits_{3u_2}^{\frac{1}{2(q_n-p_n)}}u_1^{\alpha}du_1du_2\leq\frac{(q_n+1)^2}{(q_n-p_n)^{1+\alpha}},$$

for $j=1,2,3$, which implies

$$I_n^{(1)}\ll\frac{q_n^2}{(q_n-p_n)^{1+\alpha}}. \tag{15}$$

Since $u_1-u_2\geq\frac{2u_1}{3}$, by (13) one can easily see that

$$\frac{1}{\sin\left(\frac{(u_1-u_2)\pi}{2}\right)}\leq\frac{3}{2u_1},\quad(u_1,u_2)\in\Gamma_2\cup\Gamma_3. \tag{16}$$

Thus, by (12), (14) and (16) we obtain

$$\begin{aligned}\left|L_1^*(u_1,u_2)\right| &\ll (q_n+1)\frac{1}{u_1^2},\\ \left|L_2^*(u_1,u_2)\right| &\ll (q_n+1)\frac{1}{u_1^2},\end{aligned}$$

and

$$\left|L_3^*(u_1,u_2)\right| \ll (q_n-p_n)\frac{1}{u_1^2}$$

for $(u_1,u_2)\in\Gamma_2$. For $j=1,2$,

$$\begin{aligned}\int_{\Gamma_2} u_1^{\alpha}\left|L_j^*(u_1,u_2)\right| du_1du_2 &\ll (q_n+1)\int_0^{\frac{1}{6(q_n-p_n)}}\int_{\frac{1}{2(q_n-p_n)}}^{1} u_1^{\alpha-2}du_1du_2 \\ &= \frac{q_n+1}{6(q_n-p_n)}\begin{cases}\frac{1}{1-\alpha}\left(\frac{1}{(2(q_n-p_n))^{\alpha-1}}-1\right), & \alpha<1, \\ \log(2(q_n-p_n)), & \alpha=1.\end{cases}\end{aligned}$$

Hence

$$\int_{\Gamma_2} u_1^{\alpha}\left|L_j^*(u_1,u_2)\right| du_1du_2 \ll \begin{cases}\frac{q_n}{(q_n-p_n)^{\alpha}}, & \alpha<1, \\ \frac{q_n\log 2(q_n-p_n)}{q_n-p_n}, & \alpha=1,\end{cases}$$

for $j=1,2$. Similarly

$$\begin{aligned}\int_{\Gamma_2} u_1^{\alpha}\left|L_3^*(u_1,u_2)\right| du_1du_2 &\ll (q_n-p_n)\int_0^{\frac{1}{6(q_n-p_n)}}\int_{\frac{1}{2(q_n-p_n)}}^{1} u_1^{\alpha-2}du_1du_2 \\ &\ll \begin{cases}\frac{1}{(q_n-p_n)^{\alpha-1}}, & \alpha<1, \\ \log(2(q_n-p_n)), & \alpha=1.\end{cases}\end{aligned}$$

Thus we get

$$I_n^{(2)} \ll \begin{cases}\frac{q_n}{(q_n-p_n)^{\alpha}}, & \alpha<1, \\ \frac{q_n\log(2(q_n-p_n))}{q_n-p_n}, & \alpha=1.\end{cases} \tag{17}$$

By (13), (14) and (16) we obtain

$$\left|L_j^*(u_1,u_2)\right| \ll \frac{1}{u_1^2u_2}\ (j=1,2,3)$$

for $(u_1,u_2)\in\Gamma_3$. Thus

$$\int_{\Gamma_3} u_1^{\alpha}\left|L_j^*(u_1,u_2)\right| du_1du_2 \ll \int_{\frac{1}{6(q_n-p_n)}}^{\frac{1}{3}}\int_{3u_2}^{1} u_1^{\alpha-2}\frac{1}{u_2}du_1du_2$$

$$\ll \begin{cases} \frac{1}{(q_n-p_n)^{\alpha-1}}, & \alpha<1, \\ (\log(2(q_n-p_n)))^2, & \alpha=1. \end{cases}$$

This last estimate, (15) and (17) yield

$$I_n \ll \begin{cases} \frac{q_n^2}{(q_n-p_n)^{\alpha+1}}, & \alpha<1, \\ \frac{q_n^2}{(q_n-p_n)^2}(\log(2(q_n-p_n)))^2, & \alpha=1, \end{cases}$$

and the proof is completed. ◀

The following approximation theorem in Hölder norm was obtained in ([5]) :

Theorem B. *Let* $0\le\beta<\alpha\le 1$. *Then for each* $f\in H^\alpha\left(\overline{\Omega}\right)$ *the estimate*

$$\left\|f-S_n^{(1)}(f)\right\|_{H^\beta\left(\overline{\Omega}\right)} \ll \begin{cases} \frac{1}{n^{\alpha-\beta}}, & \alpha<1, \\ \frac{(\log n)^2}{n^{1-\beta}}, & \alpha=1. \end{cases} \tag{18}$$

holds.

We generalize Theorem B as follows:

Theorem 2. Let $0\le\beta<\alpha\le 1$ and $f\in H^\alpha\left(\overline{\Omega}\right)$. Then

$$\|f-D_n(p,q;f)\|_{H^\beta\left(\overline{\Omega}\right)} \ll \left(\frac{q_n}{q_n-p_n}\right)^2 \begin{cases} \frac{1}{(q_n-p_n)^{\alpha-\beta}}, & \alpha<1, \\ \frac{(\log(2(q_n-p_n)))^2}{(q_n-p_n)^{1-\beta}}, & \alpha=1. \end{cases} \tag{19}$$

Proof. Set $e_n(\mathbf{t}) := f(\mathbf{t}) - D_n(p,q;f)(\mathbf{t})$. Hence

$$\|f-D_n(p,q;f)\|_{H^\beta\left(\overline{\Omega}\right)} = \|e_n\|_{C_H\left(\overline{\Omega}\right)} + \sup_{\mathbf{t}\ne\mathbf{s}} \frac{|e_n(\mathbf{t})-e_n(\mathbf{s})|}{\|\mathbf{t}-\mathbf{s}\|^\beta}.$$

By (7) we have

$$e_n(\mathbf{t})-e_n(\mathbf{s}) = \frac{1}{(q_n-p_n)}\frac{1}{|\Omega|}\int_\Omega \varphi_{\mathbf{t},\mathbf{s}}(\mathbf{u})\left(\Theta_{q_n}(\mathbf{u})-\Theta_{p_n}(\mathbf{u})\right)d\mathbf{u},$$

where

$$\varphi_{\mathbf{t},\mathbf{s}}(\mathbf{u}) := f(\mathbf{t})-f(\mathbf{t}-\mathbf{u})-f(\mathbf{s})+f(\mathbf{s}-\mathbf{u}).$$

Thus

$$|e_n(\mathbf{t})-e_n(\mathbf{s})| \ll \frac{1}{q_n-p_n}J_n,$$

where

$$J_n := \int_\Omega |\varphi_{\mathbf{t},\mathbf{s}}(\mathbf{u})|\left|\left(\Theta_{q_n}(\mathbf{u})-\Theta_{p_n}(\mathbf{u})\right)\right| d\mathbf{u}.$$

Since $f \in H^{\alpha}\left(\overline{\Omega}\right)$, we have

$$\left|\varphi_{\mathbf{t},\mathbf{s}}\left(\mathbf{u}\right)\right| \ll \left\|\mathbf{u}\right\|^{\alpha},$$

and by Theorem 1,

$$\begin{aligned} J_n^{1-\frac{\beta}{\alpha}} &= \left(\int\limits_{\Omega} \left|\varphi_{\mathbf{t},\mathbf{s}}\left(\mathbf{u}\right)\right| \left|\left(\Theta_{q_n}\left(\mathbf{u}\right)-\Theta_{p_n}\left(\mathbf{u}\right)\right)\right| d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}} \\ &\ll \left(\int\limits_{\Omega} \left\|\mathbf{u}\right\|^{\alpha} \left|\left(\Theta_{q_n}\left(\mathbf{u}\right)-\Theta_{p_n}\left(\mathbf{u}\right)\right)\right| d\mathbf{u}\right)^{1-\frac{\beta}{\alpha}} \\ &\ll \begin{cases} \left(\frac{q_n^2}{(q_n-p_n)^{\alpha+1}}\right)^{1-\frac{\beta}{\alpha}}, & \alpha<1, \\ \left[\frac{q_n^2}{(q_n-p_n)^2}\left(\log\left(2\left(q_n-p_n\right)\right)\right)^2\right]^{1-\beta}, & \alpha=1. \end{cases} \end{aligned}$$

We also have

$$\left|\varphi_{\mathbf{t},\mathbf{s}}\left(\mathbf{u}\right)\right| \ll \left\|\mathbf{t}-\mathbf{s}\right\|^{\alpha},$$

and hence

$$\begin{aligned} J_n^{\frac{\beta}{\alpha}} &\ll \left(\int\limits_{\Omega} \left\|\mathbf{t}-\mathbf{s}\right\|^{\alpha} \left|\left(\Theta_{q_n}\left(\mathbf{u}\right)-\Theta_{p_n}\left(\mathbf{u}\right)\right)\right| d\mathbf{u}\right)^{\frac{\beta}{\alpha}} \\ &= \left\|\mathbf{t}-\mathbf{s}\right\|^{\beta} \left(\int\limits_{\Omega} \left|\left(\Theta_{q_n}\left(\mathbf{u}\right)-\Theta_{p_n}\left(\mathbf{u}\right)\right)\right| d\mathbf{u}\right)^{\frac{\beta}{\alpha}}. \end{aligned}$$

As in proof of Theorem 1, it is sufficient to estimate the integral

$$\int\limits_{\Delta} \left|\left(\Theta_{q_n}\left(\mathbf{t}\right)-\Theta_{p_n}\left(\mathbf{t}\right)\right)\right| d\mathbf{t}.$$

By the transforms (10) and (11),

$$\int\limits_{\Delta} \left|\left(\Theta_{q_n}\left(\mathbf{t}\right)-\Theta_{p_n}\left(\mathbf{t}\right)\right)\right| d\mathbf{t} \ll$$

$$\ll \int\limits_{\Gamma} \left(\left|L_1^*\left(u_1,u_2\right)\right| + \left|L_2^*\left(u_1,u_2\right)\right| + \left|L_3^*\left(u_1,u_2\right)\right|\right) du_1 du_2.$$

For $j = 1, 2, 3$,

$$\int\limits_{\Gamma_1} \left| L_j^* (u_1, u_2) \right| du_1 du_2 \ll (q_n - p_n)(q_n + 1)^2 \int\limits_0^{\frac{1}{6(q_n - p_n)}} \int\limits_{3u_2}^{\frac{1}{2(q_n - p_n)}} du_1 du_2 \ll \frac{q_n^2}{q_n - p_n}.$$

For $j = 1, 2$,

$$\int\limits_{\Gamma_2} \left| L_j^* (u_1, u_2) \right| du_1 du_2 \ll (q_n + 1) \int\limits_0^{\frac{1}{6(q_n - p_n)}} \int\limits_{\frac{1}{2(q_n - p_n)}}^{1} \frac{1}{u_1^2} du_1 du_2 \ll q_n,$$

and

$$\int\limits_{\Gamma_2} \left| L_3^* (u_1, u_2) \right| du_1 du_2 \ll (q_n - p_n) \int\limits_0^{\frac{1}{6(q_n - p_n)}} \int\limits_{\frac{1}{2(q_n - p_n)}}^{1} \frac{1}{u_1^2} du_1 du_2 \ll q_n - p_n.$$

Also,

$$\int\limits_{\Gamma_3} \left| L_j^* (u_1, u_2) \right| du_1 du_2 \ll \int\limits_{\frac{1}{6(q_n - p_n)}}^{\frac{1}{3}} \int\limits_{3u_2}^{1} \frac{1}{u_1^2 u_2} du_1 du_2 \ll q_n - p_n$$

for $j = 1, 2, 3$. By combining these inequalities we obtain

$$\int\limits_{\Delta} \left| \left(\Theta_{q_n} (\mathbf{t}) - \Theta_{p_n} (\mathbf{t}) \right) \right| d\mathbf{t} \ll \frac{q_n^2}{q_n - p_n}.$$

Hence

$$J_n^{\frac{\beta}{\alpha}} \ll \|\mathbf{t} - \mathbf{s}\|^{\beta} \left(\frac{q_n^2}{q_n - p_n} \right)^{\frac{\beta}{\alpha}}.$$

Let $\alpha < 1$.

$$\begin{aligned} J_n &= J_n^{\frac{\beta}{\alpha}} J_n^{1 - \frac{\beta}{\alpha}} \ll \|\mathbf{t} - \mathbf{s}\|^{\beta} \left(\frac{q_n^2}{q_n - p_n} \right)^{\frac{\beta}{\alpha}} \left(\frac{q_n^2}{(q_n - p_n)^{\alpha + 1}} \right)^{1 - \frac{\beta}{\alpha}} \\ &= \|\mathbf{t} - \mathbf{s}\|^{\beta} \frac{q_n^2}{(q_n - p_n)^{\alpha - \beta + 1}}. \end{aligned}$$

This implies

$$|e_n(\mathbf{t}) - e_n(\mathbf{s})| \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \frac{1}{(q_n - p_n)^{\alpha-\beta}} \|\mathbf{t} - \mathbf{s}\|^\beta,$$

and hence

$$\frac{|e_n(\mathbf{t}) - e_n(\mathbf{s})|}{\|\mathbf{t} - \mathbf{s}\|^\beta} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \frac{1}{(q_n - p_n)^{\alpha-\beta}}$$

for every $\mathbf{t}, \mathbf{s} \in \mathbb{R}_H^3$ with $\mathbf{t} \neq \mathbf{s}$. This and Theorem 1 give

$$\|f - D_n(p, q; f)\|_{H^\beta(\overline{\Omega})} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \frac{1}{(q_n - p_n)^{\alpha-\beta}}.$$

Now let $\alpha = 1$. In this case,

$$\begin{aligned} J_n &= J_n^{\frac{\beta}{\alpha}} J_n^{1-\frac{\beta}{\alpha}} \ll \|\mathbf{t} - \mathbf{s}\|^\beta \left(\frac{q_n^2}{q_n - p_n}\right)^\beta \left[\frac{q_n^2}{(q_n - p_n)^2} (\log(2(q_n - p_n)))^2\right]^{1-\beta} \\ &= \|\mathbf{t} - \mathbf{s}\|^\beta \frac{q_n^2}{(q_n - p_n)^{2-\beta}} (\log(2(q_n - p_n)))^{2(1-\beta)}, \end{aligned}$$

which implies

$$\frac{|e_n(\mathbf{t}) - e_n(\mathbf{s})|}{\|\mathbf{t} - \mathbf{s}\|^\beta} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \frac{(\log(2(q_n - p_n)))^{2(1-\beta)}}{(q_n - p_n)^{1-\beta}}, \ \mathbf{t} \neq \mathbf{s}.$$

By this inequality and by Theorem 1 we obtain

$$\|f - D_n(p, q; f)\|_{H^\beta(\overline{\Omega})} \ll \left(\frac{q_n}{q_n - p_n}\right)^2 \frac{(\log(2(q_n - p_n)))^{2(1-\beta)}}{(q_n - p_n)^{1-\beta}},$$

which finishes the proof. ◄

Remark. If we take $p_n = n - 1$ and $q_n = n + k - 1$, where $k \in \mathbb{N}$ satisfies $n \ll k$, then the summability method $D_n(p, q)$ generalizes the $(C, 1)$ method and $D_n(p, q; f)$ becomes

$$D_n(p, q; f) = S_{n,k}^{(1)}(f) := \left(1 + \frac{n}{k}\right) S_{n+k-1}^{(1)}(f) - \frac{n}{k} S_{n-1}^{(1)}(f),$$

which is called the delayed arithmetic mean ([13, Vol. I, p.80]). These means give the same approximation order as Cesàro $(C, 1)$ means.

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