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A Study of Meromorphically Univalent Functions Defined by a Linear Operator Associated with the λ-Generalized Hurwitz-Lerch Zeta Function

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Abstract. By using a linear operator associated with the λ-generalized Hurwitz-Lerch zeta function, which is defined here by means of the Hadamard product (or convolution), the authors introduce and investigate certain sufficient conditions for this meromorphic functions to satisfy a subordination. In fact, these results extend known results of starlikeness, convexity, and close to convexity.

Key Words and Phrases: analytic functions, univalent functions, meromorphic functions, λ-generalized Hurwitz-Lerch zeta function, Srivastava-Attiya operator, Dziok- Srivastava and Srivastava-Wright operators, Hadamard product (or convolution).

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1. Introduction, Definitions and Preliminaries

Let Σ denote the class of meromorphic functions $f(z)$ normalized by

$$f(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_k z^k,$$

which are analytic in the punctured unit disk

$$\mathbb{U}^* = \{z : z \in \mathbb{C} \quad \text{and} \quad 0 < |z| < 1\} = \mathbb{U} \setminus \{0\},$$

$\mathbb{C}$ being (as usual) the set of complex numbers. We denote by $\Sigma\mathcal{S}^*(\beta)$ and $\Sigma\mathcal{K}(\beta)$ ($\beta \geqq 0$) the subclasses of Σ consisting of all meromorphic functions which are, respectively, starlike of order β and convex of order β in $\mathbb{U}^*$ (see also the recent works [43] and [42]).

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For functions $f_j(z)$ $(j = 1, 2)$ defined by

$$f_j(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,j} z^k \qquad (j = 1, 2),$$

we denote the Hadamard product (or convolution) of $f_1(z)$ and $f_2(z)$ by

$$(f_1 * f_2)(z) = \frac{1}{z} + \sum_{k=1}^{\infty} a_{k,1} a_{k,2} z^k.$$

Let us consider the function $\widetilde{\phi}(\alpha, \beta; z)$ defined by

$$\widetilde{\phi}(\alpha, \beta; z) = \frac{1}{z} + \sum_{k=0}^{\infty} \frac{(\alpha)_{k+1}}{(\beta)_{k+1}} a_k z^k$$

$$\left(\beta \in \mathbb{C} \setminus \mathbb{Z}_0^-;\ \alpha \in \mathbb{C}\right),$$

where

$$\mathbb{Z}_0^- = \{0, -1, -2, \cdots\} = \mathbb{Z}^- \cup \{0\}.$$

Here, and in the remainder of this paper, $(\lambda)_\kappa$ denotes the general Pochhammer symbol defined, in terms of the Gamma function, by

$$(\lambda)_\kappa := \frac{\Gamma(\lambda + \kappa)}{\Gamma(\lambda)} = \begin{cases} \lambda(\lambda+1)\cdots(\lambda+n-1) & (\kappa = n \in \mathbb{N};\ \lambda \in \mathbb{C}) \\ 1 & (\kappa = 0;\ \lambda \in \mathbb{C} \setminus \{0\}), \end{cases}$$

it being understood *conventionally* that $(0)_0 := 1$ and assumed *tacitly* that the Γ-quotient exists (see, for details, [39, p. 21 *et seq.*]), $\mathbb{N}$ being the set of positive integers.

Very recently, Ghanim ([8]; see also [9]) made use of the Hadamard product for functions $f(z) \in \Sigma$ in order to introduce a new linear operator $L_a^s(\alpha, \beta)$ defined on Σ by

$$\begin{aligned} L_a^s(\alpha, \beta)(f)(z) &= \widetilde{\phi}(\alpha, \beta; z) * G_{s,a}(z) \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{(\alpha)_{n+1}}{(\beta)_{n+1}} \left(\frac{a+1}{a+k}\right)^s a_k z^k \quad \left(z \in \mathbb{U}^*\right), \end{aligned}$$

where

$$G_{s,a}(z) := (a+1)^s \left[\Phi(z, s, a) - a^s + \frac{1}{z(a+1)^s}\right]$$

$$= \frac{1}{z} + \sum_{k=1}^{\infty} \left(\frac{a+1}{a+k} \right)^{s} z^{k} \qquad \big(z \in \mathbb{U}^{*}\big) \tag{1}$$

and the function $\Phi(z,s,a)$ is the well-known Hurwitz-Lerch zeta function defined by (see, for example, [28, p. 121 *et seq.*]; see also [23], [29, p. 194 *et seq.*], [34] and [35])

$$\Phi(z,s,a) := \sum_{n=0}^{\infty} \frac{z^{n}}{(n+a)^{s}}$$

$$(a \in \mathbb{C} \setminus \mathbb{Z}_0^{-};\ s \in \mathbb{C} \quad \text{when} \quad |z|<1; \Re(s)>1 \quad \text{when} \quad |z|=1).$$

We recall that the following new family of the λ-generalized Hurwitz-Lerch zeta functions was introduced and investigated systematically by Srivastava [26] (see also [24, 25, 30, 32, 33, 34, 35, 36, 38, 41]):

$$\Phi_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}(z,s,a;b,\lambda) = \frac{1}{\lambda\,\Gamma(s)}$$

$$\cdot \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{(a+n)^{s} \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \, H_{0,2}^{2,0}\left[(a+n) b^{\frac{1}{\lambda}} \,\middle|\, \begin{matrix} \overline{\qquad\qquad} \\ (s,1), \big(0, \frac{1}{\lambda}\big) \end{matrix} \right] \frac{z^{n}}{n!} \tag{2}$$

$$\big(\min\{\Re(a), \Re(s)\} > 0;\ \Re(b) > 0;\ \lambda > 0\big),$$

where

$$\Big(\lambda_j \in \mathbb{C} \ \ (j=1,\cdots,p) \quad \text{and} \quad \mu_j \in \mathbb{C}\backslash\mathbb{Z}_0^{-} \ \ (j=1,\cdots,q);\ \rho_j > 0 \ \ (j=1,\cdots,p);$$

$$\sigma_j > 0 \ \ (j=1,\cdots,q);\ 1 + \sum_{j=1}^{q} \sigma_j - \sum_{j=1}^{p} \rho_j \geqq 0\Big)$$

and the equality in the convergence condition holds true for suitably bounded values of $|z|$ given by

$$|z| < \nabla := \left(\prod_{j=1}^{p} \rho_j^{-\rho_j} \right) \cdot \left(\prod_{j=1}^{q} \sigma_j^{\sigma_j} \right).$$

Definition 1. *The H-function involved in the right-hand side of (2) is the well-known Fox's H-function [14, Definition 1.1] (see also [37, 39]) defined by*

$$H^{m,n}_{\mathfrak{p},\mathfrak{q}}(z) = H^{m,n}_{\mathfrak{p},\mathfrak{q}}\left[z \left| \begin{matrix} (a_1, A_1), \cdots, (a_{\mathfrak{p}}, A_{\mathfrak{p}}) \\ (b_1, B_1), \cdots, (b_{\mathfrak{q}}, B_{\mathfrak{q}}) \end{matrix} \right. \right] = \frac{1}{2\pi \mathrm{i}} \int_{\mathcal{L}} \Xi(s) z^{-s}\, \mathrm{d}s \qquad \big(z \in \mathbb{C} \setminus \{0\};\ |\arg(z)| < \pi\big),$$

where

$$\Xi(s) = \frac{\prod_{j=1}^{m} \Gamma(b_j + B_j s) \cdot \prod_{j=1}^{n} \Gamma(1 - a_j - A_j s)}{\prod_{j=n+1}^{\mathfrak{p}} \Gamma(a_j + A_j s) \cdot \prod_{j=m+1}^{\mathfrak{q}} \Gamma(1 - b_j - B_j s)},$$

an empty product is interpreted as 1, $m, n, \mathfrak{p}$ *and* $\mathfrak{q}$ *are integers such that*

$$1 \leqq m \leqq \mathfrak{q} \qquad \text{and} \qquad 0 \leqq n \leqq \mathfrak{p},$$

$$A_j > 0 \quad (j = 1, \cdots, \mathfrak{p}) \qquad \text{and} \qquad B_j > 0 \quad (j = 1, \cdots, \mathfrak{q}),$$

$$a_j \in \mathbb{C} \quad (j = 1, \cdots, \mathfrak{p}) \qquad \text{and} \qquad b_j \in \mathbb{C} \quad (j = 1, \cdots, \mathfrak{q})$$

and $\mathcal{L}$ *is a suitable Mellin-Barnes type contour separating the poles of the gamma functions*

$$\{\Gamma(b_j + B_j s)\}_{j=1}^{m}$$

from the poles of the gamma functions

$$\{\Gamma(1 - a_j + A_j s)\}_{j=1}^{n}.$$

It is worthy of mention here that, by using the fact that [26, p. 1496, Remark 7]

$$\lim_{b \to 0} \left\{ H^{2,0}_{0,2} \left[(a+n) b^{\frac{1}{\lambda}} \left| \begin{matrix} \overline{\qquad\qquad} \\ (s, 1), \big(0, \frac{1}{\lambda}\big) \end{matrix} \right. \right] \right\} = \lambda\, \Gamma(s) \qquad (\lambda > 0),$$

the equation (1) reduces to the following form:

$$\Phi^{(\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q)}_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}(z, s, a; 0, \lambda) := \Phi^{(\rho_1, \cdots, \rho_p, \sigma_1, \cdots, \sigma_q)}_{\lambda_1, \cdots, \lambda_p; \mu_1, \cdots, \mu_q}(z, s, a) = \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p} (\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^{q} (\mu_j)_{n\sigma_j}} \frac{z^n}{n!}. \tag{3}$$

Definition 2. *The function* $\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a)$ *involved in (3) is the multiparameter extension and generalization of the Hurwitz-Lerch zeta function* $\Phi(z,s,a)$ *introduced by Srivastava et al. [41, p. 503, Eq. (6.2)] defined by*

$$\Phi^{(\rho_1,\cdots,\rho_p,\sigma_1,\cdots,\sigma_q)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a) := \sum_{n=0}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_{n\rho_j}}{(a+n)^s \cdot \prod_{j=1}^{q}(\mu_j)_{n\sigma_j}} \frac{z^n}{n!}$$

$$\Bigg(p,q \in \mathbb{N}_0;\ \lambda_j \in \mathbb{C}\ (j=1,\cdots,p);\ a,\mu_j \in \mathbb{C}\setminus\mathbb{Z}_0^-\ (j=1,\cdots,q);$$

$$\rho_j,\sigma_k \in \mathbb{R}^+\ (j=1,\cdots,p;\ k=1,\cdots,q);$$

$$\Delta > -1 \textit{ when } s,z \in \mathbb{C};$$

$$\Delta = -1 \textit{ and } s \in \mathbb{C} \textit{ when } |z| < \nabla^*;$$

$$\Delta = -1 \textit{ and } \Re(\Xi) > \frac{1}{2} \textit{ when } |z| = \nabla^*\Bigg)$$

with

$$\nabla^* := \left(\prod_{j=1}^{p}\rho_j^{-\rho_j}\right)\cdot\left(\prod_{j=1}^{q}\sigma_j^{\sigma_j}\right),$$

$$\Delta := \sum_{j=1}^{q}\sigma_j - \sum_{j=1}^{p}\rho_j \quad \textit{and} \quad \Xi := s + \sum_{j=1}^{q}\mu_j - \sum_{j=1}^{p}\lambda_j + \frac{p-q}{2}.$$

By applying this new family of the λ-generalized Hurwitz-Lerch zeta functions, Srivastava and Gaboury [31] introduced a new linear operator which consists in a generalization of the largely- (and widely-) studied Srivastava-Attiya operator [27] (see also [3, 20, 40]). This new operator contains, as its special cases, the operators investigated earlier by Prajapat and Bulboacă [19, p. 571, Eq. (1.8)], Noor and Bukhari [15, p. 2, Eq. (1.3)], Choi *et al.* [5], Cho and Srivastava [4], Jung *et al.* [13], Bernardi [1], Carlson and Shaffer [2], Owa and Srivastava [16] and Dziok and Srivastava [6, 7]. The Dziok-Srivastava convolution operator studied by Dziok and Srivastava [6, 7] is a generalization of the Hohlov operator [11] and the Ruscheweyh operator [21]. In fact, the Dziok-Srivastava convolution operator is itself a special case of the so-called Srivastava-Wright operator (see, for details, [12] and [22]; see also the other closely-related works cited in each of these recent publications).

In this paper, we consider the following linear operator:

$$J^{\alpha} f(z) \equiv J^{s,a,\lambda,\alpha,\beta}_{(\lambda_p),(\mu_q),b} f(z) : \Sigma \to \Sigma,$$

which is defined by

$$J^{\alpha} f(z) = G^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(z) * \widetilde{\phi}(\alpha,\beta;z), \tag{4}$$

where $*$ denotes the Hadamard product (or convolution) of analytic functions and the function $G^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(z)$ is given by

$$\begin{aligned} G^{s,a,\lambda}_{(\lambda_p),(\mu_q),b}(z) &:= (a+1)^s \cdot \Bigg[\Phi^{(1,\cdots,1,1,\cdots,1)}_{\lambda_1,\cdots,\lambda_p;\mu_1,\cdots,\mu_q}(z,s,a;b,\lambda) \\ &\qquad - \frac{a^{-s}}{\lambda\,\Gamma(s)}\Lambda(a,b,s,\lambda) + \frac{(a+1)^{-s}}{z}\Bigg] \\ &= \frac{1}{z} + \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{p}(\lambda_j)_k}{\prod_{j=1}^{q}(\mu_j)_k}\left(\frac{a+1}{a+k}\right)^s \frac{\Lambda(a+k,b,s,\lambda)}{\lambda\,\Gamma(s)}\,\frac{z^k}{k!} \end{aligned} \tag{5}$$

with

$$\Lambda(a,b,s,\lambda) := H^{2,0}_{0,2}\left[ab^{\frac{1}{\lambda}}\,\middle|\, \begin{array}{c}\overline{\qquad\qquad} \\ (s,1),\left(0,\frac{1}{\lambda}\right)\end{array}\right].$$

By combining (4) and (5), we obtain

$$\begin{aligned} J^{\alpha} f(z) &= \frac{1}{z} \\ &\quad + \sum_{k=1}^{\infty} \frac{(\alpha)_{k+1}\prod_{j=1}^{p}(\lambda_j)_k}{(\beta)_{k+1}\prod_{j=1}^{q}(\mu_j)_k}\left(\frac{a+1}{a+k}\right)^s \frac{\Lambda(a+k,b,s,\lambda)}{\lambda\,\Gamma(s)}\,a_k\,\frac{z^k}{k!} \end{aligned} \tag{6}$$

$$\left(z \in \mathbb{U}^*;\ \alpha,\lambda_j \in \mathbb{C}\ \ (j=1,\cdots,p);\ \beta,\mu_j \in \mathbb{C}\setminus\mathbb{Z}_0^-\ \ (j=1,\cdots,q);\ p \leqq q+1\right)$$

with

$$\min\{\Re(a),\Re(s)\} > 0;\ \lambda > 0 \quad \text{if} \quad \Re(b) > 0$$

and

$$s \in \mathbb{C} \qquad \text{and} \qquad a \in \mathbb{C}\setminus\mathbb{Z}_0^- \quad \text{if} \quad b = 0,$$

see Srivastava et al. [34] and [35]). Clearly, upon setting $p-1=q=0$ and $\lambda_1 = 1$ in (6) and taking the limit as $b \to 0$, we obtain the operator $L_a^s(\alpha,\beta)(f)(z)$ studied earlier by Ghanim [8].

Let the functions f and g be analytic in $\mathbb{U}$. Then we say that f is subordinate to g in $\mathbb{U}$, and write $f \prec g$; if there exists a Schwarz function w analytic in $\mathbb{U}$

such that $|w(z)| < 1$, $z \in \mathbb{U}$; and $w(0) = 0$ with $f(z) = g(w(z))$ in $\mathbb{U}$ (see [10]); further, if g is univalent in $\mathbb{U}$; then $f(z) \prec g(z) \Leftrightarrow f(0) = g(0)$ and $f(\mathbb{U}) \subset g(\mathbb{U})$.

In this paper, we investigate various properties of certain subclasses of the meromorphically analytic function class Σ in the punctured unit disk $\mathbb{U}^*$. We first introduce one of these function classes and investigate the properties of the linear operator

$$J^{\alpha+1} \equiv J^{s,a,\lambda,\alpha+1,\beta}_{(\lambda_p),(\mu_q),b} f(z).$$

and obtain certain sufficient conditions for a function $f \in \Sigma$ to satisfy either of the following subordinations:

$$\frac{J^{\alpha+1}f(z)}{J^{\alpha}f(z)} \prec \frac{\lambda(1-z)}{\lambda - z}, \quad \frac{J^{\alpha}f(z)}{z} \prec \frac{1+Az}{1-z}, \quad \frac{J^{\alpha}f(z)}{z} \prec \frac{\lambda(1-z)}{\lambda - z}.$$

Our results extend corresponding previously known results on starlikeness, convexity, and close to convexity.

To prove our main results, we need the following:

Lemma 1. *(cf. Miller and Mocanu [17, Theorem 3.4h, p.132]). Let $q(z)$ be univalent in the unit disk $\mathbb{U}$ and let ϑ and φ be analytic in a domain $D \supset q(\mathbb{U})$, with $\varphi(w) \neq 0$ when $w \in q(\mathbb{U})$. Set*

$$Q(z) := zq'(z)\varphi(q(z)), \quad h(z) := \vartheta(q(z)) + Q(z).$$

Suppose that
(1) $Q(z)$ is starlike univalent in $\mathbb{U}$, and
(2) $\Re\left(\frac{zh'(z)}{Q(z)}\right) > 0$ for $z \in \mathbb{U}$.
If $p(z)$ is analytic in $\mathbb{U}$ with $p(0) = q(0)$, $p(\mathbb{U}) \subset D$ and

$$\vartheta(p(z)) + zp'(z)\varphi(p(z)) \prec \vartheta(q(z)) + zq'(z)\varphi(q(z)). \tag{7}$$

Then $p(z) \prec q(z)$ and $q(z)$ is the best dominant.

2. Main results

Theorem 1. *Let $\alpha > 0$, $\mu \in \mathbb{R}$ satisfy $|\mu| \leq 1$ and $\lambda > 1$. If $f \in \Sigma$ satisfies $J^{\alpha}f(z) \neq 0$ in $\mathbb{U}^*$ and*

$$\left(\frac{J^{\alpha+1}f(z)}{J^{\alpha}f(z)}\right)^{\mu}\left((\alpha+1)\frac{J^{\alpha+2}f(z)}{J^{\alpha+1}f(z)} - 1\right) \prec h(z), \tag{8}$$

where

$$h(z)=\left(\frac{\lambda(1-z)}{\lambda-z}\right)^{\mu+1}\left(\alpha-\frac{(\lambda-1)z}{\lambda(1-z)^2}\right),$$

then

$$\frac{J^{\alpha+1}f(z)}{J^{\alpha}f(z)}\prec\frac{\lambda(1-z)}{\lambda-z}.$$

Proof. The condition (8) and $J^{\alpha}f(z)\neq 0$ in $\mathbb{U}^*$ imply that $J^{\alpha+1}f(z)\neq 0$ in $\mathbb{U}^*$. Define the function $p(z)$ by

$$p(z):=\frac{J^{\alpha+1}f(z)}{J^{\alpha}f(z)}.$$

Clearly $p(z)$ is analytic in $\mathbb{U}^*$. A computation shows that

$$\frac{zp'(z)}{p(z)}=\frac{z\left(J^{\alpha+1}f(z)\right)'}{J^{\alpha+1}f(z)}-\frac{z\left(J^{\alpha}f(z)\right)'}{J^{\alpha}f(z)}. \tag{9}$$

By using the identity

$$z\left(J^{\alpha}f(z)\right)'=\alpha\left(J^{\alpha+1}f(z)\right)-(\alpha+1)J^{\alpha}f(z), \tag{10}$$

we get from (9)

$$(\alpha+1)\frac{J^{\alpha+2}f(z)}{J^{\alpha+1}f(z)}=1+\alpha p(z)+\frac{zp'(z)}{p(z)}. \tag{11}$$

Using (11) in (8), we get

$$\alpha(p(z))^{\mu+1}+zp'(z)(p(z))^{\mu-1}\prec h(z). \tag{12}$$

Let $q(z)$ be the function defined by

$$q(z):=\frac{\lambda(1-z)}{\lambda-z}.$$

It is clear that q is convex univalent in $\mathbb{U}^*$. Since

$$h(z)=\alpha(q(z))^{\mu+1}+zq'(z)(q(z))^{\mu-1}.$$

We see that (12) can be written as (7) when ϑ and φ are given by

$$\vartheta(w) = \alpha w^{\mu+1} \qquad \text{and} \qquad \varphi(w) = w^{\mu-1}.$$

Clearly φ and ϑ are analytic in $\mathbb{C}\backslash\{0\}$. Now

$$Q(z) := zq'(z)\varphi(q(z)) = zq'(z)(q(z))^{\mu-1} = \frac{(1-\lambda)z\lambda^{\mu}(1-z)^{\mu-1}}{(\lambda-z)^{1+\mu}},$$

$$h(z) := \vartheta(q(z)) + Q(z) = \left(\frac{\lambda(1-z)}{\lambda-z}\right)^{1+\mu}\left(\alpha - \frac{(\lambda-1)z}{\lambda(1-z)^2}\right).$$

By our assumptions on the parameters μ and λ, we see that

$$\Re\left(\frac{zQ'(z)}{Q(z)}\right) = \Re\left(1 + \frac{z(1-\mu)}{1-z} + (1-\mu)\frac{z}{\lambda-z}\right)$$

$$> -1 + \frac{1}{2}(1-\mu) + \frac{(1+\mu)\lambda}{1+\lambda}$$

$$= \frac{(1+\mu)(\lambda-1)}{2(1+\lambda)} > 0,$$

and therefore $Q(z)$ is starlike. Also we have

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \alpha(1+\mu)\Re\left(\frac{\lambda(1-z)}{\lambda-z}\right) + \Re\left(\frac{zQ'(z)}{Q(z)}\right) \geq 0.$$

By an application of Lemma 1, we have $p(z) \prec q(z)$ or

$$\frac{J^{\alpha+1}f(z)}{J^{\alpha}f(z)} \prec \frac{\lambda(1-z)}{\lambda-z}.$$

This completes the proof of Theorem 1. ◄

By taking $\mu = 0$, $\alpha = 1$ and $\beta = 1$ in Theorem 1, we get the following corollary:

Corollary 1. *Let $f \in \Sigma$ and $f(z) \neq 0$ in $\mathbb{U}^*$. If $\lambda > 1$ and*

$$1 + \frac{zf''(z)}{f'(z)} \prec \frac{\lambda(1-z)}{\lambda-z} - \frac{(\lambda-1)z}{(\lambda-z)(1-z)},$$

then

$$\frac{zf'(z)}{f(z)} \prec \frac{\lambda(1-z)}{\lambda-z}.$$

Remark 1. *The function*

$$h(z)=\frac{\lambda\,(1-z)}{\lambda-z}-\frac{(\lambda-1)\,z}{(\lambda-z)\,(1-z)}=\frac{z}{\lambda-z}+\frac{1}{1-z},$$

takes real value for real value of z, $h(0)=1$ *and* $h(\mathbb{U})$ *is the region* $\Re\,(h(z))<\frac{(\lambda+1)}{2(\lambda-1)}$ *for* $1<\lambda\leq 2$ *and* $\Re\,(h(z))<\frac{(5\lambda-1)}{2(\lambda+1)}$ *for* $2<\lambda$. *Hence this result generalizes the result obtained by Owa et al. [18].*

We note that the image of the function

$$h(z)=1-\frac{(\lambda-1)\,z}{\lambda\,(1-z)^2}$$

is

$$h(\mathbb{U}^*)=\mathbb{C}-\left[\frac{5\lambda-1}{4\lambda},\infty\right].$$

Hence by taking $\mu=-1$, $\alpha=1$ and $\beta=1$ in Theorem 1, we get the following corollary:

Corollary 2. *Let* $\lambda>1$, $f\in\Sigma$ *and* $f(z)\neq 0$ *in* $\mathbb{U}$. *If* f *satisfies*

$$\Re\left(\frac{1+\frac{zf''(z)}{f'(z)}}{\frac{zf'(z)}{f(z)}}\right)<\frac{5\lambda-1}{4\lambda},$$

then

$$\frac{zf'(z)}{f(z)}\prec\frac{\lambda\,(1-z)}{\lambda-z}.$$

Theorem 2. *Let* $\alpha>0$, $-1\leq\mu<0$ *and* $-1\leq A<1$. *If* $f\in\Sigma$ *satisfies the condition* $J^\alpha f(z)/z\neq 0$ *in* $\mathbb{U}^*$ *and*

$$\left(\frac{J^\alpha f(z)}{z}\right)^\mu\left(\alpha\frac{J^{\alpha+1}f(z)}{z}\right)\prec h\,(z)\,, \tag{13}$$

where

$$h(z)=\left(\frac{1+Az}{1-z}\right)^\mu\left(\alpha\frac{1+Az}{1-z}+\frac{(1+A)\,z}{(1-z)^2}\right),$$

then

$$\frac{J^\alpha f(z)}{z}\prec\frac{1+Az}{1-z}.$$

Proof. Define the function $p(z)$ by

$$p(z) := \frac{J^{\alpha} f(z)}{z}. \tag{14}$$

It is clear that p is analytic in $\mathbb{U}^{*}$. By using the identity (10), we get from (14)

$$\alpha \left(J^{\alpha+1} f(z)\right)' = zp'(z) - (\alpha - 1)\, p(z). \tag{15}$$

Using (15) in (13), we see that the subordination becomes

$$\alpha p(z)^{1+\mu} + p(z)^{\mu} z p'(z) \prec h(z).$$

Define the function $q(z)$ by

$$q(z) := \frac{1 + Az}{1 - z}.$$

It is clear that $q(z)$ is univalent in $\mathbb{U}$ and $q(\mathbb{U})$ is the region $\Re\left(q(z)\right) > (1 - A)/2$. By defining the functions ϑ and φ by

$$\vartheta(w) = \alpha\, w^{\mu+1} \qquad \text{and} \qquad \varphi(w) = w^{\mu}.$$

we observe that (13) can be written as (7). Note that φ and ϑ are analytic in $\mathbb{C}\setminus\{0\}$. Also we see that

$$Q(z) := zq'(z)\varphi\left(q(z)\right) = \frac{(1 + A)\, z\, (1 + Az)^{\mu}}{(1 - z)^{2+\mu}},$$

and

$$h(z) := \vartheta\left(q(z)\right) + Q(z) = \left(\frac{1 + Az}{1 - z}\right)^{\mu} \left(\alpha \frac{1 + Az}{1 - z} + \frac{(1 + A)\, z}{(1 - z)^{2}}\right).$$

By our assumptions, we have

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left[1 + \mu \frac{Az}{1 + Az} + (2 + \mu)\, \frac{z}{1 - z}\right]$$

$$> 1 - \frac{\mu\, |A|}{1 + |A|} - \frac{2 + \mu}{2} = \frac{-\mu\, (1 - |A|)}{2\, (1 + |A|)} > 0,$$

and

$$\Re\left(\frac{zh'(z)}{Q(z)}\right) = \Re\left[\frac{\vartheta'\left(q(z)\right)}{\varphi\left(q(z)\right)} + \frac{zQ'(z)}{Q(z)}\right] = \alpha\left(1+\mu\right) + \Re\left(\frac{zQ'(z)}{Q(z)}\right) \geq 0.$$

The result now follows by an application of Lemma 1. ◀

The function $h(z) = \alpha + \frac{1+Az}{(1-z)(1+Az)}$ takes real values for real values of z with $h(0) = \alpha$ and $h(\mathbb{U})$ is symmetric with respect to the real axis and

$$\Re\left(h(z)\right) > \alpha + \frac{1}{2} - \frac{1}{1-|A|}, \qquad z \in \mathbb{U}^*.$$

Consequently, by letting $\mu = -1$ in Theorem 2, we obtain the following corollary:

Corollary 3. *Let $-1 < A < 1$, $\alpha > 0$ and $f \in \Sigma$ with $J^{\alpha}f(z)/z \neq 0$ in $\mathbb{U}^*$ and*

$$\Re\left(\frac{J^{\alpha+1}f(z)}{J^{\alpha}f(z)}\right) > 1 + \frac{1}{2\alpha} - \frac{1}{\alpha\left(1-|A|\right)}.$$

Then

$$\frac{J^{\alpha}f(z)}{z} \prec \frac{1+Az}{1-z}.$$

Theorem 3. *Let $\mu \geq -1$, $\lambda > 1$, $f \in \Sigma$ and $J^{\alpha}f(z)/z \neq 0$ in $\mathbb{U}^*$. If f satisfies*

$$\left(\frac{J^{\alpha}f(z)}{z}\right)^{\mu}\left(\alpha\frac{J^{\alpha+1}f(z)}{z}\right) \prec \frac{\lambda^{1+\mu}\left(1-z\right)^{\mu}}{\left(\lambda-z\right)^{1+\mu}}\left(\alpha\left(1-z\right) - \frac{\lambda\left(1-z\right)}{\lambda-z}\right),$$

then

$$\frac{J^{\alpha}f(z)}{z} \prec \frac{\lambda\left(1-z\right)}{\lambda-z}.$$

Proof. The proof of Theorem 3, also based upon Lemma 1, is similar to that of Theorem 1. Indeed, in this case, the result follows from Lemma 1 when we define the functions φ and ϑ by $\vartheta(w) = \alpha\, w^{-(1+\mu)}$ and $\varphi(w) = -w^{-(2+\mu)}$. ◀

Finally we note that $\Re\left(1 - \frac{(\lambda-1)z}{(\lambda-z)(1-z)}\right) < \frac{3\lambda-1}{2(\lambda-1)}$ for $z \in \mathbb{U}^*$ and so from above Theorem by choosing $\alpha = \beta = 1$ we can get the following corollary:

Corollary 4. *Let $\lambda > 1$, $f \in \Sigma$ and $f'(z) \neq 0$ in $\mathbb{U}^*$. If f satisfies*

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) < \frac{3\lambda-1}{2\left(\lambda-1\right)},$$

then

$$f'(z) \prec \frac{\lambda\left(1-z\right)}{\lambda-z}.$$

3. Concluding Remarks and Observations

In our present investigation, we have successfully applied a remarkably general family of linear operators which are associated with the λ-generalized Hurwitz-Lerch zeta function. By means of this general linear operator, we have introduced and investigated various properties of some new subclasses of meromorphically univalent functions in the punctured unit disk $\mathbb{U}^*$. We have also considered several closely-related (known or new) corollaries and consequences of the main results (Theorems 1, 2 and 3) presented in this paper.

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