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New Stability Conditions for the Delayed Liénard Nonlinear Equation via Fixed Point Technique

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Abstract. The class of second order nonlinear neutral differential equations having the form

$$\ddot{x} + f\left(t, x, \dot{x}\right)\dot{x} + b\left(t\right)g\left(x\left(t - \tau\left(t\right)\right)\right) = 0, \text{ for } t \geq t_0,$$

is studied by means of contraction mappings. We give some new conditions ensuring that the zero solution is asymptotically stable. Our results are strong and do not require conditions that have been indispensable in previous investigations such as, $\frac{g(x)}{x} \geq \beta > 0$ and $\lim \frac{g(x)}{x}$ exists as $x \to 0$, the delay $\tau\left(t\right)$ is differentiable, the map $t \mapsto t - \tau\left(t\right)$ is strictly increasing (see [16]) or the function $b : \mathbb{R}^+ \to \mathbb{R}^+$ is bounded and there exists a function $c : \mathbb{R}^+ \to \mathbb{R}^+$ such that $f\left(t, x, y\right) \leq F\left(x, y\right)c\left(t\right)$ for all $t \geq 0$ and $x, y \in \mathbb{R}$. The results obtained improve those of D. Pi [16] and are very significant because, from practical point of view, it is hard to control the factors of nuclear reactors that ensure the delay is smoothly changed if we do not allow dependence between the functions and the time.

Key Words and Phrases: fixed points, Liénard equation, variable delay, stability, asymptotic stability.

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1. Introduction

Time-delay systems constitute basic mathematical models of real phenomenons such as nuclear reactors, chemical engineering systems, biological systems and population dynamics models. Such systems are often sources of instability and degradation in control performance in many control problems. For more than 100 years, the Lyapunov direct method has been the ultimate key tool to study stability problems. The direct method was widely used to study the stability of

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solutions of ordinary differential equations and functional differential equations. Nevertheless, the pointwise nature of this method and the construction of the Lyapunov functionals have been, and still are, an arduous task (see [7]). Recently, many authors have realized that the fixed points theory can be used to study the stability of the solution (see [1]-[11], [12], [15]-[18]). Levin and Nohel [14] studied the following nonlinear systems of differential equations of Liénard form:

$$\ddot{x} + h(t, x, \dot{x})\dot{x} + f(x) = a(t). \tag{1}$$

They obtained, by constructing a proper Lyapunov function, conditions under which all solutions of (1) tend to zero as $t \to \infty$. In [10], Burton considered the following delay equation:

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + b(t)g(x(t-L)) = 0, \tag{2}$$

where L is a positive constant. By using the fixed point theory, he gave sufficient conditions for each solution $x(t)$ to satisfy $(x(t), \dot{x}(t)) \longrightarrow 0$ as $t \to \infty$. D. Pi (see [16, 18]) studied the asymptotic stability of the following equations with variable delays:

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + b(t)g(x(t-\tau(t))) = 0. \tag{3}$$

Nevertheless, Pi results (see [16]) rely basically on the assumption that $t-\tau(t)$ is strictly increasing. Many other interesting results on fixed points and stability properties can be found in the references ([1]-[8]). In this paper, we consider the Liénard equation with delay

$$\ddot{x} + f(t, x, \dot{x})\dot{x} + b(t)g(x(t-\tau(t))) = 0, \tag{4}$$

with the initial condition

$$x(t) = \psi(t) \text{ for } t \in [m(t_0), t_0],$$

for $t \geq 0$, where $m(t_0) =: \inf\{t-\tau(t) : t \geq t_0\}$, $b : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$ is continuous, $\tau : \mathbb{R}^+ \longrightarrow \mathbb{R}^+$, $f : \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}^+$, $g : [-\tau(0), \infty) \longrightarrow \mathbb{R}$ are all continuous functions. We assume that

$$t - \tau(t) \longrightarrow \infty \text{ as } t \longrightarrow \infty. \tag{5}$$

For each $t_0 \geq 0$, let $C(t_0) := C([m(t_0), t_0], \mathbb{R})$ be the space of continuous functions endowed with the supremum norm $\|\cdot\|$. That is, for $\psi \in C(t_0)$, we let

$$\|\psi\| := \sup\{|\psi(s)| : m(t_0) \leq s \leq t_0\}.$$

Later, we will also use the same notation $\|\varphi\| := \sup\{|\varphi(t)| : t \in [m(t_0), \infty)\}$ to express the supremum of elements of the space C of continuous bounded functions on $[m(t_0), \infty)$, where the function

$$\rho(\varphi, \phi) := \|\varphi - \phi\| = \sup\{|\varphi(s) - \phi(s)| : m(t_0) \le s\}, \ \varphi, \phi \in C, \tag{6}$$

is the associated metric.

It is well known (see [13]) that, for a given continuous function ψ and a number $y(t_0)$, there exists a solution for equation (4) on an interval $[m(t_0), T)$, and if the solution remains bounded, then $T = \infty$. We denote by $(x(t), y(t))$ the solution $(x(t, t_0, \psi), y(t, t_0, \psi))$ of (4).

Define $A(t) := f((t, x(t), y(t))$. We can rewrite equation (4) as a system

$$\begin{cases} \dot{x}(t) = y(t), \\ \dot{y}(t) = -A(t)\, y(t) - b(t)\, g(x(t - \tau(t))). \end{cases} \tag{7}$$

Since 1990 a series of papers, including references [10] and [16, 17, 18], have been published to investigate stability of equation (4) subject to various conditions. Dingheng Pi [16] has recently considered equation (4) for the case of variable delay. The details are as follows. Suppose that $(\mathcal{A})$ $t - \tau(t)$ is strictly increasing and $\lim(t - \tau(t)) = \infty$ as $t \to \infty$. Then, the inverse of $t - \tau(t)$ exists and we denote it by $p(t)$. Let $0 \le b(t) \le M$, for some constant $M > 0$.

Theorem 1 ([16])**.** *Suppose $(\mathcal{A})$ holds and assume that*

B1) there exists constant $l > 0$ such that g satisfies a Lipschitz condition on $[-l, l]$. g is odd, strictly increasing on $[-l, l]$ and $x - g(x)$ is nondecreasing on $[0, l]$,

B2) there exists an $\alpha \in (0, 1)$ and a continuous function $a : \mathbb{R}^+ \to \mathbb{R}^+$ such that $a(t) \le f(t, x, y)$ for $t \ge 0$, $x, y \in \mathbb{R}$ and

$$\begin{aligned} &2 \sup_{t \ge 0} \int_t^{p(t)} \int_0^\infty e^{-\int_s^{w+s} a(v)dv} b(s)\, dw ds + \\ &+ 2 \sup_{t \ge 0} \int_0^t \int_{t-s}^\infty e^{-\int_s^{w+s} a(v)dv} b(s)\, dw ds \ \le \ \alpha, \end{aligned} \tag{8}$$

B3) there exist constants $a_0 > 0$ and Q such that for each $t \ge 0$ if $J \ge Q$ then

$$\int_t^{t+J} a(s)\, ds \ge a_0 J. \tag{9}$$

Then there exists $\delta \in (0, l)$ *such that for each initial function* $\psi : [m(t_0), t_0] \to \mathbb{R}$ *and* $\dot{x}(t_0)$ *satisfying* $|\dot{x}(t_0)| + \|\psi\| \leq \delta$*, there is a unique continuous function* $x : [m(t_0), \infty) \to \mathbb{R}$ *satisfying* $x(t) = \psi(t)$*,* $t \in [m(t_0), t_0]$*, which is a solution of (4) on* $[t_0, \infty)$*. Moreover, the zero solution of (4) is stable. Furthermore, if in addition*

$$\int_0^\infty a(t)\, dt = \infty \tag{10}$$

and there exist continuous functions $F : \mathbb{R} \times \mathbb{R} \to [0, \infty)$ *and* $c : [0, \infty) \to [0, \infty)$ *such that for all* $t > 0$*,* $x, y \in \mathbb{R}$ *we have*

$$f(t, x, y) \leq F(x, y)\, c(t), \tag{11}$$

$g'(x)$ *is continuous on* $[-l, l]$*,* $g'(0) \neq 0$*. Then,*

(i) if for each $\gamma > 0$*,*

$$\int_0^\infty \int_0^\infty e^{-\int_s^{u+s} \gamma c(v) dv} b(s)\, duds = \infty, \tag{12}$$

the zero solution of (4) is asymptotically stable.

(ii) if the zero solution of (4) is asymptotically stable, then

$$\int_0^\infty \int_0^\infty e^{-\int_s^{u+s} a(v) dv} b(s)\, duds = \infty. \tag{13}$$

Here we use fixed point technique to give, what we hope, an essential improvement to this problem of great and continuing interest.

Stability definitions, fixed point technique and more details on delay differential equations can be found in ([7, 13]) which also contain substantial references.

Definition 1. *The zero solution of (7) is stable if for each* $\varepsilon > 0$ *there exists* $\delta = \delta(\varepsilon, t_0) > 0$ *such that* $[\psi \in \mathcal{C}(t_0), y_0 \in \mathbb{R}, \|\psi\| + |y_0| < \delta]$ *implies that* $|x(t, t_0, \psi)| + |y(t, t_0, \psi)| < \varepsilon$ *for* $t \geq t_0$*.*

Definition 2. *The zero solution of (7) is asymptotically stable if it is stable and there is a* $\delta_1 = \delta_1(t_0) > 0$ *such that* $[\psi \in \mathcal{C}(t_0), y_0 \in \mathbb{R}, \|\psi\| + |y_0| < \delta_1]$ *implies that* $|x(t, \psi, y_0)| + |y(t, \psi, y_0)| \longrightarrow 0$ *as* $t \longrightarrow \infty$*.*

2. Statement of main results

Now we try to define a fixed point mapping from (4). For that purpose we begin first by domesticating the equation by rewriting it in another, but equivalent, form.

Lemma 1. *Let $\psi : [m(t_0), t_0] \to \mathbb{R}$ be a given continuous initial function. Then, $x(t)$ is a solution of equation (7) and hence solution of (4) on an interval $[t_0, T)$ satisfying the initial condition $x(t) = \psi(t)$ on $[m(t_0), t_0]$ and $y(t_0) = \dot{x}(t_0)$ if and only if $x(t)$ is a solution of the following integral equation:*

$$\begin{aligned}
x(t) &= \psi(t_0) e^{-\int_{t_0}^{t} D(v)dv} + \dot{x}(t_0) \int_{t_0}^{t} e^{-\int_{s}^{t} D(v)dv} e^{-\int_{t_0}^{s} A(v)dv} ds + \\
&+ \dot{x}(t_0) \int_{t_0}^{t} e^{-\int_{s}^{t} D(v)dv} \int_{t_0}^{s} e^{-\int_{u}^{s} A(v)dv} b(u) \int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du ds - \\
&- \int_{t_0}^{t} e^{-\int_{s}^{t} D(v)dv} \int_{t_0}^{s} e^{-\int_{u}^{s} A(v)dv} b(u) \times \\
&\times \int_{u-\tau(u)}^{s} \int_{t_0}^{\mu} e^{-\int_{\theta}^{\mu} A(v)dv} b(\theta) g(x(\theta - \tau(\theta))) d\theta d\mu du ds + \\
&+ \int_{t_0}^{t} e^{-\int_{s}^{t} D(v)dv} \int_{t_0}^{s} e^{-\int_{u}^{s} A(v)dv} b(u) [x(u - \tau(u)) - g(x(u - \tau(u)))] du ds.
\end{aligned} \tag{14}$$

Conversely, if a continuous function $x(\cdot)$ is equal to $\psi(\cdot)$ for $t \in [m(t_0), t_0]$ and is the solution of equation (14) on an interval $[t_0, T_1]$, then $x(\cdot)$ is a solution of (7) on $[t_0, T_1]$.

Proof. By applying the variation of parameters formula to the second equation of (7), we obtain

$$\dot{x}(t) = \dot{x}(t_0) e^{-\int_{t_0}^{t} A(v)dv} - \int_{t_0}^{t} e^{-\int_{s}^{t} A(v)dv} b(s) g(x(s - \tau(s))) ds. \tag{15}$$

Rewrite (15) as

$$\begin{aligned}
\dot{x}(t) &= \dot{x}(t_0) e^{-\int_{t_0}^{t} A(v)dv} - \int_{t_0}^{t} e^{-\int_{s}^{t} A(v)dv} b(s) x(s - \tau(s)) ds + \\
&+ \int_{t_0}^{t} e^{-\int_{s}^{t} A(v)dv} b(s) [x(s - \tau(s)) - g(x(s - \tau(s)))] ds.
\end{aligned} \tag{16}$$

Let

$$D(t) := \int_{t_0}^{t} b(s) e^{-\int_s^t A(v)dv} ds. \tag{17}$$

Obviously, we have

$$x(s-\tau(s)) = x(t) - \int_{s-\tau(s)}^{t} y(v)\, dv. \tag{18}$$

Inserting (17) and (18) into (16), we get

$$\begin{aligned}
\dot{x}(t) &= -x(t) D(t) + \dot{x}(t_0) e^{-\int_{t_0}^t A(v)dv} + \\
&+ \dot{x}(t_0) \int_{t_0}^{t} e^{-\int_s^t A(v)dv} b(s) \int_{s-\tau(s)}^{t} e^{-\int_{t_0}^u A(v)dv} duds - \\
&- \int_{t_0}^{t} e^{-\int_s^t A(v)dv} b(s) \int_{s-\tau(s)}^{t} \int_{t_0}^{u} e^{-\int_\theta^u A(v)dv} b(\theta) g(x(\theta-\tau(\theta)))\, d\theta duds + \\
&+ \int_{t_0}^{t} e^{-\int_s^t A(v)dv} b(s) \left[x(s-\tau(s)) - g(x(s-\tau(s)))\right] ds.
\end{aligned} \tag{19}$$

Applying the variation of parameters formula by multiplying both sides of the equation (19) by the factor $e^{\int_{t_0}^t D(v)dv}$ and integrating from t_0 to any $t \in [t_0, T]$, we can see that this last equation is exactly (14).

Conversely, suppose that a continuous function x is equal to ψ on $[m(t_0), t_0]$ and satisfies (14) on an interval $[t_0, T)$. Then, x is differentiable on $[t_0, T)$. Differentiating x with the aid of Leibniz's rule, we obtain (4). ◄

From (14) we shall derive a fixed point mapping P for (4). But the challenge here is to choose a suitable metric space of functions on which the map P can be defined. Moreover, we have to choose prudently a weighted metric so that P do not only maps this set into itself but also is a contraction.

Toward this, let S be the space of all continuous functions $\varphi : [m(t_0), \infty) \to \mathbb{R}$. For a given initial function $\psi : [m(t_0), t_0] \to [-l, l]$, $l > 0$ define the set

$$S_\psi^l := \{\varphi : [m(t_0), \infty) \to \mathbb{R} \mid \varphi \in S,\ \varphi \equiv \psi \text{ on } [m(t_0), t_0], |\varphi(t)| \le l\}.$$

Define the mapping P on S_ψ^l as follows, for $\varphi \in S_\psi^l$:

$$(P\varphi)(t) := \psi(t) \text{ if } t \in [m(t_0), t_0],$$

while for $t > t_0$

$$P\varphi(t) := \psi(t_0) e^{-\int_{t_0}^t D(v)dv} + \dot{x}(t_0) \int_{t_0}^{t} e^{-\int_s^t D(v)dv} e^{-\int_{t_0}^s A(v)dv} ds +$$

$$+\dot{x}(t_0)\int_{t_0}^{t} e^{-\int_s^t D(v)dv}\int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u)\int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du ds-$$
$$-\int_{t_0}^{t} e^{-\int_s^t D(v)dv}\int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u)\int_{u-\tau(u)}^{s}\int_{t_0}^{\mu} e^{-\int_\theta^\mu A(v)dv}\times$$
$$\times b(\theta) g(\varphi(\theta-\tau(\theta))) d\theta d\mu du ds+$$
$$+\int_{t_0}^{t} e^{-\int_s^t D(v)dv}\int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u)\left[\varphi(u-\tau(u))-g(\varphi(u-\tau(u)))\right] du ds. \tag{20}$$

Lemma 2. *Suppose that*

i) there exists a constant $l>0$ such that g satisfies a Lipschitz condition on $[-l,l]$ and let L be the Lipschitz constant for both $g(x)$ and $x-g(x)$ on $[-l,l]$.

Then there exists a metric d on S_ψ^l such that

ii) the metric space $\left(S_\psi^l, d\right)$ is complete,

iii) P is a contraction mapping on $\left(S_\psi^l, d\right)$ if P maps S_ψ^l into itself.

Proof. Let S be the space of all continuous functions $\varphi:[m(t_0),\infty)\to\mathbb{R}$ that satisfy

$$|\varphi|_h := \sup\left\{|\varphi(t)|\, e^{-h(t)} : t\in[m(t_0),\infty)\right\}<\infty.$$

where $h(t):=\int_{t_0}^{t}[b(v)+kL]\,dv$, k is a constant with $2<k$ and L is the above mentioned common Lipschitz constant for $x-g(x)$ and $g(x)$. Then, one can check that $(S,|\cdot|_h)$ is a Banach space by using Cauchy's criterion for uniform convergence. Thus, (S,d) is a complete metric space, where d denotes the induced metric $d(\phi,\varphi)=|\phi-\varphi|_h$, for $\phi,\varphi\in S$. Under this metric, the subset S_ψ^l is closed in S. Therefore, the metric space $\left(S_\psi^l, d\right)$ is complete. Consequently, (ii) is proved.

To see (iii), let $\varpi(\phi(u)):=\phi(u-\tau(u))-g(\phi(u-\tau(u)))$. Since $P:S_\psi^l\to S_\psi^l$ and g satisfies a Lipschitz condition on $[-l,l]$, we can obtain for $\varphi,\phi\in S_\psi^l$ and for $t>t_0$,

$$|P\varphi-P\phi|_h \ \le$$

$$\leq \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{\mu} e^{-\int_\theta^\mu A(v)dv} b(\theta) \times$$
$$\times \left| g(\varphi(\theta-\tau(\theta))) - g(\phi(\theta-\tau(\theta))) \right| e^{-h(\theta-\tau(\theta))+h(\theta-\tau(\theta))-h(t)} d\theta d\mu du ds +$$
$$+ \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \times$$
$$\times \left| \varpi(\varphi(u)) - \varpi(\phi(u)) \right| e^{-h(\theta-\tau(\theta))+h(\theta-\tau(\theta))-h(t)} du ds \leq$$
$$\leq \left\{ \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{\mu} e^{-\int_\theta^\mu A(v)dv} \times \right.$$
$$\times b(\theta) e^{h(\theta-\tau(\theta))-h(t)} d\theta d\mu du ds +$$
$$\left. + \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) e^{h(\theta-\tau(\theta))-h(t)} du ds \right\} L \left| \varphi - \phi \right|_h .$$

For $t \geq s \geq \mu \geq \theta$ we have

$$h(\theta-\tau(\theta)) - h(t) = \int_{t_0}^{\theta-\tau(\theta)} b(v)\,dv + kL(\theta-\tau(\theta)) - \int_{t_0}^{t} b(v)\,dv - kLt \leq$$
$$\leq - \int_{\theta}^{\mu} b(v)\,dv - kL(t-\theta) \leq - \int_{\theta}^{\mu} b(v)\,dv - kL(s-\mu).$$

For $t \geq s \geq u$,

$$h(u-\tau(u)) - h(t) = - \int_{u}^{t} b(v)\,dv - kL(t-u) \leq - \int_{u}^{s} b(v)\,dv - kL(t-s).$$

Making use of these inequalities, we have

$$\left| P\varphi - P\phi \right|_h \leq \left\{ \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \times \right.$$
$$\times \int_{u-\tau(u)}^{s} e^{-kL(s-\mu)} \int_{t_0}^{\mu} b(\theta) e^{-\int_\theta^\mu b(v)dv} d\theta d\mu du ds +$$
$$\left. + \int_{t_0}^{t} e^{-kL(t-s)} \int_{t_0}^{s} b(u) e^{-\int_u^s b(v)dv} du ds \right\} L \left| \varphi - \phi \right|_h \leq$$
$$\leq \left\{ \frac{1}{kL} + \frac{1}{kL} \right\} L \left| \varphi - \phi \right|_h \leq \frac{2}{k} \left| \varphi - \phi \right|_h ,$$

for all $t > t_0$. Obviously, this holds for all $t \geq m(t_0)$ by definition of the mapping P. Thus, $d(P\varphi, P\phi) \leq \frac{2}{k} d(\varphi, \phi)$. Since $k > 2$, we conclude that P is a contraction on $\left(S_\psi^l, d\right)$. ◄

Now, we prove an existence and uniqueness result for P whenever $\|\psi\|$ is well chosen.

Theorem 2. *Suppose g satisfies condition (i) in Lemma 2 and*

H1) g is odd and strictly increasing on $[-l,l]$;

H2) $x-g(x)$ is non-decreasing on $[0,l]$;

H3) there exists an $\alpha\in(0,1)$ and a continuous function $A_1:\mathbb{R}^+\to\mathbb{R}^+$ such that $A(t)=f(t,x,y)\geq A_1(t)$ for $t\geq 0$, $x,y\in\mathbb{R}$ and

$$\int_{t_0}^{t} e^{-\int_s^t D(v)dv}\int_{t_0}^{s} e^{-\int_u^s A(v)dv}b(u)\int_{u-\tau(u)}^{s}\int_{t_0}^{\mu} e^{-\int_\theta^\mu A(v)dv}b(\theta)\,d\theta d\mu ds\leq\alpha,$$

H4) there exist constants $a_0>0$ and $J_0>0$ such that

$$\int_{t_0}^{t} A_1(s)\,ds\geq a_0 t,\ \text{for } t\geq J_0.$$

Then, a δ exists such that for each initial function $\psi:[m(t_0),t_0]\to\mathbb{R}$ and for each $\dot{x}(t_0)$ satisfying $|\dot{x}(t_0)|+\|\psi\|<\delta$, there is a unique continuous function $x:[m(t_0),\infty)\to\mathbb{R}$ satisfying $x(t)=\psi(t)$, on $[m(t_0),t_0]$, which is a solution of (4) on $[t_0,\infty)$. Furthermore, the zero solution of (4) is stable at $t=t_0$.

Proof. First, for the fixed value $|\dot{x}(t_0)|$, we examine the second and the third terms containing $\dot{x}$ on the right hand side of (20) and show that each term is bounded. Indeed, obviously, by hypotheses (H1) and (H4), the second term is bounded because

$$\begin{aligned}&|\dot{x}(t_0)|\int_{t_0}^{t} e^{-\int_s^t D(v)dv}e^{-\int_{t_0}^s A(v)dv}ds\ \leq\\ &\leq|\dot{x}(t_0)|\left(\int_{t_0}^{J_0} e^{-\int_{t_0}^s A(v)dv}ds+\frac{e^{-a_0J_0}}{a_0}\right)\leq\\ &\leq|\dot{x}(t_0)|\,M_1,\end{aligned}\tag{21}$$

where

$$M_1:=\int_{t_0}^{J_0} e^{-\int_{t_0}^s A(v)dv}ds+\frac{e^{-a_0J_0}}{a_0}.$$

For the third term we see that

$$|\dot{x}(t_0)|\int_{u-\tau(u)}^{s} e^{-\int_{t_0}^\theta A(v)dv}d\theta\ \leq$$

$$\leq |\dot{x}(t_0)| \left(\int_{-\tau(t_0)}^{J_0} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta + \frac{e^{-a_0 J_0}}{a_0} \right) \leq$$
$$\leq |\dot{x}(t_0)| M_2,$$

where

$$M_2 := \int_{-\tau(t_0)}^{J_0} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta + \frac{e^{-a_0 J_0}}{a_0}.$$

Thus,

$$\begin{aligned}
&|\dot{x}(t_0)| \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du ds \leq \\
&\leq |\dot{x}(t_0)| M_2 \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) du ds \leq \\
&\leq |\dot{x}(t_0)| M_2. \qquad (22)
\end{aligned}$$

Now, since g is odd and satisfies the Lipschitz condition on $[-l,l]$, $g(0)=0$ and g is (uniformly) continuous on $[-l,l]$, one can choose a δ that satisfies the inequality

$$\delta + |\dot{x}(t_0)| (M_1 + M_2) \leq (1-\alpha) g(l). \qquad (23)$$

Let $\psi : [m(t_0), t_0] \to (-\delta, \delta)$ be an initial continuous function. Note that condition (23) implies $\delta < l$ since $g(l) \leq l$ by (H2). Thus, $|\psi| \leq l$ for $m(t_0) \leq t \leq t_0$. We declare that, for such a ψ, $P : S_\psi^l \to S_\psi^l$. Indeed, for an arbitrary $\varphi \in S_\psi^l$, it follows from conditions (H1) and (H2) that

$$\begin{aligned}
&|P\varphi(t)| \leq \\
&\leq \delta + |\dot{x}(t_0)| (M_1 + M_2) + \\
&+ \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{\mu} e^{-\int_\theta^\mu A(v)dv} b(\theta) \times \\
&\times |g(\varphi(u-\tau(u)))| d\theta d\mu du ds + \\
&+ \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) |\varphi(u-\tau(u)) - g(\varphi(u-\tau(u)))| du ds.
\end{aligned}$$

for $t > t_0$. By (H3), (23), (21) and (22), this implies

$$\begin{aligned}
|P\varphi(t)| &\leq \delta + |\dot{x}(t_0)| (M_1 + M_2) + \alpha g(l) + (l - g(l)) \leq \\
&\leq (1-\alpha) g(l) + \alpha g(l) + (l - g(l)) = l. \qquad (24)
\end{aligned}$$

Hence, $|P\varphi(t)| \leq l$ for all $t \in [m(t_0), \infty)$ since we have $|P\varphi(t)| = \psi(t) \leq l$ for $t \in [m(t_0), t_0]$. Therefore, $P\varphi \in S_\psi^l$. By Lemma 2, P is a contraction on the complete metric space $\left(S^l_\psi, d\right)$. Then P has a unique fixed point $x \in S^l_\psi$, which is a solution of (4) on $[m(t_0), \infty)$ by Lemma 2 and $x(t) \leq l$ for all $t \geq m(t_0)$. Hence, x is the only continuous function satisfying (4) for $t \geq t_0$ with $x \equiv \psi$ on $[m(t_0), t_0]$.

Since g is bounded, the boundedness of $x(t)$ in (15) yields that $y(t)$ is bounded too. More precisely, while $A(t) \geq A_1(t)$, assume that

$$\int_{t_0}^{t} e^{-\int_s^t A_1(v)dv} b(s)\, ds \leq \sigma. \tag{25}$$

Then, from (15) we have

$$\begin{aligned} |y(t)| &\leq \delta + g(l) \int_{t_0}^{t} e^{-\int_s^t A(v)dv} b(s)\, ds \leq \\ &\leq \delta + g(l)\, M \leq l(1+\sigma). \end{aligned}$$

It follows that

$$|x(t)| + |y(t)| \leq l(2+\sigma).$$

To show the stability at $t = t_0$, let $\varepsilon > 0$ be given. By choosing $r < \min(\varepsilon, l(2+\sigma))$ and replacing l with r we see that there is a $\delta > 0$ satisfying (23) such that $\|\psi\| < \delta$ implies that the unique continuous solution coinciding with ψ on $[m(t_0), t_0]$ implies $|x(t)| + |y(t)| < r < \varepsilon$ for all $t \geq m(t_0)$. ◄

Now, supposing that the conditions in Lemma 2 and Theorem 2 hold for some $l > 0$, we try to investigate asymptotic stability with necessary and sufficient conditions. This can be possible if we deal with those functions of S^l_ψ that tends to zero at infinity. Unfortunately, this subset of functions of S^l_ψ is not complete with respect to the weighted metric d. Nevertheless, one can take another way to get round this. So, let C be the set of real continuous bounded functions on $[m(t_0), \infty)$ and let

$$\begin{aligned} C^0_\psi \; : &= \{\varphi : [m(t_0), \infty) \to \mathbb{R} |\ \varphi \in C, \varphi(t) = \psi(t) \\ &\quad \text{for } t \in [m(t_0), t_0], \varphi(t) \to 0 \text{ as } t \to \infty\}. \end{aligned}$$

Endowed with the metric ρ (6) induced by the supremum norm, the space $\left(S^0_\psi, \rho\right)$ is complete. Define the subset

$$S^{l,0}_\psi := \left\{\varphi \in C^0_\psi |\ |\varphi(t)| \leq l,\ \varphi(t) \to 0 \text{ as } t \to \infty\right\}.$$

Clearly, $S^{l,0}_\psi$ is a closed subset of C^0_ψ. The metric space $\left(S^{l,0}_\psi, \rho\right)$ is then complete. We are ready to show that, under the conditions of the next theorem, the zero solution of (4) is asymptotically stable in the sense of Definition 2.

Theorem 3. *Suppose that all of the conditions in Lemma 2 and Theorem 2 hold. Furthermore, suppose that g is continuously differentiable on $[-l,l]$ and $g'(0) \neq 0$. Then, the zero solution of (4) is asymptotically stable if and only if*

$$\int_0^t D(s)\,ds \to \infty \text{ as } t \to \infty. \tag{26}$$

Proof. Suppose condition (26) is fulfilled. Set

$$K := \sup_{t \geq 0} e^{-\int_0^t D(s)ds}. \tag{27}$$

Denote the five terms on the right hand side of (20) by I_1, I_2, I_3, I_4, I_5, respectively. If $\varphi \in S_\psi^{l,0}$, we prove that each term of $P\varphi(t)$ tends to 0 as $t \to \infty$. Obviously, the first term I_1 of (20) tends to zero by condition (26) as $t \to \infty$. For the second term I_2 of (20), let $\varepsilon > 0$ be given and choose $T \geq J_0$ large enough so that $e^{-a_0 T} \leq a_0 \varepsilon$. Then,

$$\begin{aligned} |I_2| &\leq |\dot{x}(t_0)| \left\{ e^{-\int_T^t D(v)dv} \int_{t_0}^T e^{-\int_s^T D(v)dv} e^{-\int_{t_0}^s A(v)dv} ds + \frac{e^{-a_0 T}}{a_0} \right\} \leq \\ &\leq |\dot{x}(t_0)| \, e^{-\int_T^t D(v)dv} \int_{t_0}^T e^{-\int_s^T D(v)dv} e^{-\int_{t_0}^s A(v)dv} ds + |\dot{x}(t_0)| \, \varepsilon. \end{aligned} \tag{28}$$

By condition (26) the first factor on the r.h.s of (28) tends to 0, as $t \to \infty$, while the second is arbitrarily small. Thus, I_2 tends to 0, as $t \to \infty$.

Now, to investigate I_3, we begin by rewriting it as follows:

$$\begin{aligned} |I_3| = |\dot{x}(t_0)| \Bigg\{ &\int_{t_0}^T e^{-\int_s^t D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^s e^{-\int_{t_0}^\theta A(v)dv} d\theta du ds + \\ &+ \int_T^t e^{-\int_s^t D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^s e^{-\int_{t_0}^\theta A(v)dv} d\theta du ds \Bigg\}. \end{aligned} \tag{29}$$

We check that the decomposition on the right hand side of (29) tends to zero at infinity. We see that

$$\begin{aligned} &\int_{t_0}^T e^{-\int_s^t D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^s e^{-\int_{t_0}^\theta A(v)dv} d\theta du ds = \\ &= e^{-\int_T^t D(v)dv} \int_{t_0}^T e^{-\int_s^T D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^s e^{-\int_{t_0}^\theta A(v)dv} d\theta du ds. \end{aligned} \tag{30}$$

By (26), the first factor on the r.h.s. of (30) tends to zero as $t \longrightarrow \infty$ and so does (30). The remaining term of on the r.h.s of (29) needs further consideration. So, let $s \geq u - \tau(u) \geq T \geq J_0$. Then we have

$$\int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta \leq \int_{u-\tau(u)}^{s} e^{-a\theta} d\theta \leq \frac{1}{a_0} e^{-a_0(u-\tau(u))} \to 0 \text{ as } u \to \infty.$$

Thus, for $\varepsilon > 0$ given there exists $J > T \geq J_0$ such that

$$\int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta \leq \varepsilon, \text{ whenever } u \geq J.$$

Hence, for $J > T \geq J_0$, we have

$$\int_{T}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du ds \leq$$
$$\leq \int_{T}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{J} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du ds + \varepsilon. \tag{31}$$

Also, let $S^* > J$ and choose $\sigma > S^* + J_0$ so that $\frac{\hat{M}}{a_0} e^{-a_0\sigma} \leq \varepsilon$. Consequently, it comes

$$\int_{T}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{J} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du ds =$$
$$= \int_{T}^{t} e^{-\int_s^t D(v)dv} e^{-\int_{S^*}^s A(v)dv} \int_{t_0}^{J} e^{-\int_u^{S^*} A(v)dv} b(u) \int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du ds =$$
$$= \left(\int_{t_0}^{J} e^{-\int_u^{S^*} A(v)dv} b(u) \int_{u-\tau(u)}^{u} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du \right) \int_{T}^{t} e^{-\int_s^t D(v)dv} e^{-\int_{S^*}^s A(v)dv} ds =$$
$$= \hat{M} \int_{T}^{\sigma} e^{-\int_s^t D(v)dv} e^{-\int_{S^*}^s A(v)dv} ds + \hat{M} \int_{\sigma}^{t} e^{-\int_s^t D(v)dv} e^{-\int_{S^*}^s A(v)dv} ds \leq$$
$$\leq \hat{M} \int_{T}^{\sigma} e^{-\int_s^t D(v)dv} e^{-\int_{S^*}^s A(v)dv} ds + \hat{M} \int_{\sigma}^{t} e^{-\int_{S^*}^s A_1(v)dv} ds \leq$$
$$\leq \hat{M} e^{-\int_\sigma^t D(v)dv} \int_{T}^{\sigma} e^{-\int_s^\sigma D(v)dv} e^{-\int_{S^*}^s A_1(v)dv} ds + \varepsilon, \tag{32}$$

where

$$\hat{M} := \int_{t_0}^{J} e^{-\int_u^{S^*} A(v)dv} b(u) \int_{u-\tau(u)}^{u} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du.$$

Clearly, the first term of (32) tends to zero as $t \to \infty$, while the second factor is arbitrarily small. Thus, we reach the conclusion that the whole third term goes to 0 at infinity. Now, consider the fourth term of (20). We observe that $|g(x)| \leq L|x|$, because $g(0) = 0$ and g is Lipschtizian with constant L on $[-l, l]$. It follows that

$$\begin{aligned} |I_4| \quad &\leq \quad L \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{\mu} e^{-\int_\theta^\mu A(v)dv} \times \\ &\times b(\theta) |\varphi(\theta - \tau(\theta))| \, d\theta d\mu du ds. \end{aligned}$$

Having (5) in mind, for a given $\varepsilon > 0$, there exists a J such that $\theta \geq J$ implies $L|\varphi(\theta - \tau(\theta))| < \frac{\varepsilon}{\alpha}$. So,

$$\begin{aligned} |I_4| \quad &\leq \quad L \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{J} e^{-\int_\theta^\mu A(v)dv} \times \\ &\times b(\theta) |\varphi(\theta - \tau(\theta))| \, d\theta d\mu du ds + \varepsilon. \end{aligned}$$

By letting $T \geq J$, we can write

$$\begin{aligned} &\int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{J} e^{-\int_\theta^\mu A(v)dv} \times \\ &\times b(\theta) |\varphi(\theta - \tau(\theta))| \, d\theta d\mu du ds = \int_{t_0}^{T} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} \times \\ &\times b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{J} e^{-\int_\theta^\mu A(v)dv} b(\theta) |\varphi(\theta - \tau(\theta))| \, d\theta d\mu du ds + \\ &+ \int_{T}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{J} e^{-\int_\theta^\mu A(v)dv} \times \\ &\times b(\theta) |\varphi(\theta - \tau(\theta))| \, d\theta d\mu du ds. \end{aligned} \tag{33}$$

By (26), the first integral on the r.h.s of (33) is as

$$\begin{aligned} &\int_{t_0}^{T} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{J} e^{-\int_\theta^\mu A(v)dv} \times \\ &\times b(\theta) |\varphi(\theta - \tau(\theta))| \, d\theta d\mu du ds \leq \alpha \|\varphi\|_{[m(t_0),T]} e^{-\int_T^t D(v)dv}, \end{aligned}$$

which, by (26), tends to zero at infinity.

On the other hand, for $\mu \geq J$ we can choose $J^* > J$ such that, by taking,

$$M_3 := \int_{t_0}^{J} e^{-\int_\theta^{J^*} A(v)dv} b(\theta) |\varphi(\theta - \tau(\theta))| \, d\theta,$$

we see that the second term on the r.h.s of (33) is as

$$\int_T^t e^{-\int_s^t D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) \times$$

$$\times \int_{u-\tau(u)}^s \int_{t_0}^J e^{-\int_\theta^\mu A(v)dv} b(\theta) |\varphi(\theta - \tau(\theta))| d\theta d\mu du ds =$$

$$= \int_T^t e^{-\int_s^t D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^s e^{-\int_{J^*}^\mu A(v)dv} \times$$

$$\times \int_{t_0}^J e^{-\int_\theta^{J^*} A(v)dv} b(\theta) |\varphi(\theta - \tau(\theta))| d\theta d\mu du ds =$$

$$= M_3 \int_T^t e^{-\int_s^t D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^s e^{-\int_{J^*}^\mu A(v)dv} d\mu du ds =$$

$$= M_3 e^{\int_{t_0}^{J^*} A(v)dv} \int_T^t e^{-\int_s^t D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^s e^{-\int_{t_0}^\mu A(v)dv} d\mu du ds \le$$

$$\le M_3 e^{\int_{t_0}^{J^*} A(v)dv} \int_{t_0}^t e^{-\int_s^t D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^s e^{-\int_{t_0}^\mu A(v)dv} d\mu du ds \tag{34}$$

The first factor of (34) is simply a finite number while the second factor tends to zero as $t \to \infty$ by using a similar technique like that used for (29). Thus, I_4 ends at zero for large time. It remains to show that the last term of (20) goes to zero as time tends to infinity. Using the hypothesis that $x - g(x)$ satisfies an L-Lipschtiz condition on $[-l, l]$, we obtain

$$\begin{aligned} |I_5| &\le L \int_{t_0}^t e^{-\int_s^t D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) |\varphi(u - \tau(u))| du ds \le \\ &\le L |\varphi(\theta - \tau(\theta))|_{[m(t_0),T]} e^{-\int_T^t D(v)dv} + \\ &+ L \int_T^t e^{-\int_s^t D(v)dv} \int_{t_0}^s e^{-\int_u^s A(v)dv} b(u) |\varphi(u - \tau(u))| du ds. \end{aligned} \tag{35}$$

Since $\varphi \in S_\psi^{l,0}$, we have, by (5), $|\varphi(u - \tau(u))| \to 0$ as $u - \tau(u)$ tends to infinity and a similar argumentation like in (29) leads to the fact that the terms on the r.h.s. of (35) tend to zero at infinity and so does I_5. Consequently, $(P\varphi)(t) \to 0$ as $t \to \infty$. So, $P : S_\psi^{l,0} \to S_\psi^{l,0}$ when $l > 0$.

Next, we show that P is a contraction mapping on $S_\psi^{l.0}$. Let $\zeta \in [0, 1]$ and define $q(\zeta) := \min\{g'(x) \mid |x| \le \zeta\}$, $Q(\zeta) := \max\{g'(x) \mid |x| \le \zeta\}$. Since $\alpha < 1$, we have $\alpha = 1-\varepsilon$ for some $\varepsilon \in (0,1)$. We claim that if l is sufficiently small, then $q(l)$, $Q(l)$ will satisfy the inequality $\alpha Q(l) < q(l)$. In fact, $\lim_{\zeta\to 0} q(\zeta) = \lim_{\zeta\to 0} Q(\zeta) = g'(0) \neq 0$, then $\lim_{\zeta\to 0} \frac{Q(\zeta)}{q(\zeta)} = 1$. Since g is strictly increasing, $g'(0) \neq 0$ and $g'(x)$ is continuous, then there exists a neighborhood of $x=0$ on which $g'(x) > 0$. So $Q(\zeta) > 0$, for $0 < \zeta < l$. Then there exists $\gamma \in (0,l]$ such that $\left| \frac{Q(\zeta)}{q(\zeta)} - 1 \right| < \varepsilon$, for $0 < \zeta < \gamma$. Hence $\alpha Q(\zeta) = (1-\varepsilon) Q(\zeta) < q(\zeta)$. By replacing the original l by $l = \zeta$, we can obtain $\alpha Q(l) < q(l)$. Set $\varpi(x) = x - g(x)$. For $x \in [-l,l]$, $|\varpi'(x)| = 1 - g'(x) \leq 1 - q(l)$. For $\phi, \varphi \in S_\psi^{l,0}$, using (20) we have

$$
\begin{aligned}
&|P\varphi(t) - P\phi(t)| \;\leq \\
&\leq \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{\mu} e^{-\int_\theta^\mu A(v)dv} b(\theta) \times \\
&\times |g(\varphi(\theta - \tau(\theta))) - g(\phi(\theta-\tau(\theta)))| \, d\theta d\mu du ds + \\
&+ \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \, |\varpi(\varphi(u)) - \varpi(\phi(u))| \, du ds.
\end{aligned}
$$

By the mean value theorem, using the condition (H3) of Theorem 1, we have

$$
\begin{aligned}
|P\varphi - P\phi| &\leq Q(l)\alpha|\varphi - \phi| + (1 - q(l))|\varphi - \phi| < \\
&< [Q(l)\alpha + (1 - q(l))] \|\varphi - \phi\|,
\end{aligned}
$$

where $Q(t)\alpha + (1 - q(l)) < 1$. This implies that P is a contraction mapping, so P has a unique fixed point $x(t)$ in $S_\psi^{l,0}$. Thus, $x(t) \to 0$, as $t \to \infty$.

Next, we will prove that $y(t) \to 0$ as $t \to \infty$. Returning to (15), we see that its first term tends to zero as $t \to \infty$ by (H4). For any $\varepsilon > 0$, choose $T > t_0$ so that $|x(t - \tau(t))| < L\sigma\varepsilon$ for $t > T$. The second integral in (15) is estimated as

$$
\begin{aligned}
&\int_{t_0}^{t} e^{-\int_s^t A(v)dv} b(s) |g(x(s-\tau(s)))| \, ds \;\leq \\
&\leq Ll \int_{t_0}^{T} e^{-\int_s^t A_1(v)dv} b(s) \, ds + L\sigma\varepsilon \int_{T}^{t} e^{-\int_s^t A(v)dv} b(s) \, ds \leq \\
&\leq Ll \int_{t_0}^{T} e^{-\int_s^t A_1(v)dv} b(s) \, ds + \varepsilon.
\end{aligned}
$$

Also, by (H4), we can choose $T^* \geq T$ such that

$$
\int_{T}^{t} e^{-\int_s^t A(v)dv} b(s) |g(x(s-\tau(s)))| \, ds \;\leq
$$

$$\leq e^{-\int_{T^*}^{t} A_1(v)dv} Ll \int_{t_0}^{T} e^{-\int_{s}^{T^*} A_1(v)dv} b(s)\, ds + \varepsilon.$$

The first factor tends to zero as $t \to \infty$, while the second factor is simply a fixed number. Since we have obtained the stability of the zero solution in Theorem 2, it follows that the zero solution is asymptotically stable.

Conversely, we shall prove that $\int_{t_0}^{\infty} D(v)dv = \infty$. Contrary to this, there exists a sequence $\{t_n\}_{n\geq 1}$ with $t_n \longrightarrow \infty$ as $n \longrightarrow \infty$ such that $\int_{t_0}^{t_n} D(v)dv = C_0$ for a certain finite number $C_0 \in \mathbb{R}^+$. By condition (27), we may also choose $\mu > 0$ that satisfies the inequality $-\mu \leq \int_{t_0}^{t_n} D(v)dv \leq \mu$, for all $n \geq 1$. For convenience of notation, we set

$$\Theta(s) := \int_{t_0}^{s} e^{-\int_{u}^{s} A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{\mu} e^{-\int_{\theta}^{\mu} A(v)dv} b(\theta)\, d\theta d\mu du + \\ + \int_{t_0}^{s} e^{-\int_{u}^{s} A(v)dv} b(u)\, duds.$$

By conditions (H3), we have

$$\int_{t_0}^{t_n} e^{-\int_{t_0}^{t_n} D(v)dv} \Theta(s)\, ds \leq (\alpha+1).$$

This yields

$$e^{-\int_{t_0}^{t_n} D(v)dv} \int_{t_0}^{t_n} e^{\int_{t_0}^{s} D(v)dv} \Theta(s)\, ds \leq (\alpha+1).$$

Then

$$\int_{t_0}^{t_n} e^{\int_{t_0}^{s} D(v)dv} \Theta(s)\, ds \leq (\alpha+1) e^{\mu}. \tag{36}$$

The inequality (36) leads to the fact that the sequence

$$\int_{t_0}^{t_n} e^{\int_{t_0}^{s} D(v)dv} \Theta(s)\, ds,$$

is bounded, so there exists a convergent subsequence. For brevity, we assume that

$$\lim_{t \longrightarrow \infty} \int_{0}^{t_n} e^{\int_{0}^{s} D(v)dv} \Theta(s)\, ds = \sigma > 0.$$

Then, we can choose a positive integer n_0 large enough such that

$$\int_{t_{n_0}}^{t_n} \mathbf{e}^{\int_{t_0}^{u} D(v)dv} \Theta(s)\, ds < \frac{\delta_0}{8KL},$$

for $n \geq n_0$, where L is the common Lipschitz constant for $x - g(x)$ and $g(x)$ and $\epsilon > \delta_0 > 0$ satisfies

$$\begin{aligned}
&\delta_0 + |\dot{x}(t_0)| \int_{t_0}^{t} e^{-\int_s^t D(v)dv} e^{-\int_{t_0}^{s} A(v)dv} ds + \\
&+ |\dot{x}(t_0)| \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du ds \leq \\
&\leq (1-\alpha) g(l).
\end{aligned}$$

By a similar argument as in (24) and by replacing l by 1, this implies that $|x(t)| \leq 1$.

Now, we consider the solution $x(t) = x(t, \psi, \dot{x}(t_{n_0}))$ of equation (4) for the initial values ψ and $\dot{x}(t_{n_0})$ such that

$$\begin{aligned}
\psi(t_{n_0}) = \frac{3\delta_0}{4}&, \ \dot{x}(t_{n_0}) = \frac{\delta_0}{4}, \\
|\psi(s)| + |\dot{x}(s)| \leq \delta_0&, \qquad s \leq t_{n_0}.
\end{aligned}$$

Since $g(x)$ and $x - g(x)$ satisfy the Lipschitz condition, x is a fixed point of P and $|x(t)| = |\psi(t)| \leq \delta_0 \leq \epsilon < 1$ if $m(t_{n_0}) \leq t \leq t_{n_0}$, we have, for $n \geq n_0$

$$\begin{aligned}
|x(t_n)| \geq &\Bigg| e^{-\int_{t_{n_0}}^{t_n} D(v)dv} \psi(t_{n_0}) + \dot{x}(t_{n_0}) \int_{t_{n_0}}^{t_n} e^{-\int_u^{t_n} D(v)dv} e^{-\int_{t_{n_0}}^{u} A(v)dv} du + \\
&+ \dot{x}(t_{n_0}) \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} e^{-\int_{t_0}^{\theta} A(v)dv} d\theta du ds \Bigg| - \\
&- L \Bigg| \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) \int_{u-\tau(u)}^{s} \int_{t_0}^{\mu} e^{-\int_\theta^\mu A(v)dv} b(\theta) d\theta d\mu du ds + \\
&+ \int_{t_0}^{t} e^{-\int_s^t D(v)dv} \int_{t_0}^{s} e^{-\int_u^s A(v)dv} b(u) du ds \Bigg| \geq \\
&\geq e^{-\int_{t_{n_0}}^{t_n} D(v)dv} \frac{\delta_0}{4} - L \int_{t_{n_0}}^{t_n} e^{-\int_s^{t_n} D(v)dv} V(s) ds \geq \\
&\geq e^{-\int_{t_{n_0}}^{t_n} D(v)dv} \left[\frac{\delta_0}{4} - L e^{-\int_0^{t_{n_0}} D(v)dv} \int_{t_{n_0}}^{t_n} e^{\int_0^s D(v)dv} V(s) ds \right] \geq \frac{\delta_0}{8} e^{-2\mu} > 0.
\end{aligned}$$

On the other hand, if the zero solution is asymptotically stable, then $x(t) = x(t, \psi, \dot{x}(t_{n_0}))$ tends to zero at infinity. Which leads to a contradiction. This ends the proof of our claim. ◄

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