Azerbaijan Journal of Mathematics
V. 8, No 1, 2018, January
ISSN 2218-6816

Uncountable Frames in Non-Separable Hilbert Spaces and their Characterization

B.T. Bilalov*, M.I. Ismailov, Z.V. Mamedova

Abstract. The concepts of Bessel families and frames in non-separable Hilbert spaces are introduced in this work. Besselianness criterion for a family is found. Similar to the usual case, analysis, synthesis and frame operators are defined, their properties are studied. Many results related to usual frames are extended to new case. Examples are given.

Key Words and Phrases: non-separable space, Bessel family, uncountable frame.

2010 Mathematics Subject Classifications: 42C40, 42C15

1. Introduction

The concept of frame has been probably introduced by R.J. Duffin and A.C. Schaeffer in 1952 [1] in the study of non-harmonic Fourier series with respect to perturbed exponential systems. In this seminal work, Duffin and Schaefer established some properties of exponential frames. In the same work, they introduced the concept of abstract frame in separable Hilbert space and extended some properties of frames consisting of perturbed exponential systems to this concept. The interest to frames has grown in the 1980s due to wide applications of wavelet methods in various fields of natural science. Standing at the crossroads of theory and practice, the wavelets are widely used in processing and encoding of signals and different kinds of images (satellite images, roentgenograms of internal organs, etc), in pattern recognition, in the study of the properties of crystal surfaces and nano-objects, and in many other fields. Today, there are a lot of monographs and review articles dedicated to this direction of approximation theory. For theoretical aspects of this direction we refer the readers to Ch. Chui [2], Y. Meyer [3], I.

*Corresponding author.

Daubechies [4], S. Mallat [5], R. Young [6], Ch. Heil [7], O. Christensen [8, 9, 10], etc.

In subsequent years, the concept of frame has been generalized to various mathematical structures (for example, Banach frames, p-frames, etc) and new methods for establishing frames have been elaborated. One of these methods is a perturbation method. A lot of results have been obtained in this direction in the context of classical Paley-Wiener theorem on perturbation of an orthonormal basis (for more details see O. Christensen [8, 9, 10] and Ch.Heil [7]).

It should be noted that, unlike Hilbert's case, the definition of Banach frame does not, in general, provide the decomposition of arbitrary element of Banach space (or of arbitrary element of the closure of the linear span of the system under consideration). In special cases, such a decomposition exists. L^p-case has been considered by A. Aldroubi, Q. Sun, W.-Sh. Tang in [12] where the concept of p-frame has been introduced and the atomic decomposition with regard to L_p-subspaces invariant with respect to the shift operator has been obtained. This idea has been extended to the general Banach case by Christensen O. and Stoeva D. T. [10]. Also, the concept of q-Riesz basis for a Banach space has been introduced in these works, which generalizes the one of Riesz basis introduced by N.K. Bari [13]. Note that similar results have been obtained in [14, 15, 16, 17, 18, 19, 20, 47]. There are different generalizations of frames, and this research field has been continuously growing over the last years (see, e.g., [10, 11, 12, 21, 22, 23, 24, 25, 26, 27, 28, 29, 30, 31]).

Frames draw growing interest also from a theoretical point of view. As an example, we can mention the connection between the theory of frames and the well-known problem of Kadison and Singer (1959). Modified, but equivalent forms of this problem have been studied in different branches of mathematics such as theory of frames, theory of operators, time-frequency analysis, etc. (for more details see [32, 33, 34, 35, 36, 37, 38, 39, 44, 45, 46]).

In the context of applications to some problems of mechanics and mathematical physics, since recently there arose great interest in the study of different mathematical problems in non-standard function spaces such as Lebesgue spaces of variable summability, Morrey-type spaces, grand Lebesgue spaces, etc. (for more details see Cruz-Uribe [40], Kokilashvili V., Meskhi A., Rafeiro H., Samko S. [41], Adams D.R. [42], Bardaro C., Musileak J., Vinti G. [43], etc). It's worth noting that in most cases these spaces (like, for example, Morrey-type spaces, grand Lebesgue spaces, etc) are not separable. That's what makes the study of frames in non-separable spaces interesting.

In this work, we define Bessel families and frames in non-separable Hilbert spaces, and find Besselianness criterion for a family. Similar to the usual case, analysis, synthesis and frame operators are defined, their properties are studied.

Many results related to usual frames are extended to new case. Corresponding examples are given. Note that, as far as the authors know, this is the first time the non-separable case is considered.

2. Uncountable Bessel System

Let H be a non-separable Hilbert space and I be an index set equipotent with its topological dimension. Accept the following definition.

Definition 1. *A system $\{x_\alpha\}_{\alpha \in I}$ is called a Bessel system in H, if there exists an absolute constant $M > 0$ such that for $\forall \omega \subset I : card\omega \leq \theta_0$ (The cardinality of the set ω is at most countable):*

$$\sum_{\alpha \in \omega} |(x; x_\alpha)|^2 \leq M \|x\|^2 , \quad \forall x \in H, \tag{1}$$

where $(\cdot;\cdot)$ is a scalar product in H and $\|\cdot\| = \sqrt{(\cdot;\cdot)}$.

If $\{x_\alpha\}_{\alpha \in I}$ is a Bessel system in H, then it follows directly from (1) that the index set $I(x) = \{\alpha \in I : (x; x_\alpha) \neq 0\}$ is at most countable for $\forall x \in H$. In fact, take $\forall x \in H$. Let $I_n(x) = \left\{\alpha \in I : |(x; x_\alpha)| \geq \frac{1}{n}\right\}$. It follows directly from the convergence of the series

$$\sum_{\alpha \in I_n} |(x; x_\alpha)|^2 ,$$

that $card\, I_n < +\infty$. On the other hand, it is not difficult to see that $I(x) = \bigcup_{n=1}^{\infty} I_n(x)$. Hence, $I(x)$ is at most countable.

Consider the Hilbert space

$$l_2\left(I^C\right) = \left\{\lambda \in I^C : card\{\alpha \in I : \lambda_\alpha \neq 0\} \leq \theta_0 \wedge \sum_{\alpha \in I} |\lambda_\alpha|^2 < +\infty\right\},$$

equipped with the norm

$$\|\lambda\|_{l_2(I^C)} = \left(\sum_{\alpha \in I} |\lambda_\alpha|^2\right)^{1/2} .$$

Let $I(\lambda) = \{\alpha \in I : \lambda_\alpha \neq 0\}$, where $\lambda = \{\lambda_\alpha\}_{\alpha \in I}$. Scalar product in $l_2\left(I^C\right)$ is defined by the formula

$$(\lambda; \mu)_{l_2(I^C)} = \sum_{\alpha \in I} \lambda_\alpha \bar{\mu}_\alpha, \quad \forall \lambda; \mu \in l_2\left(I^C\right),$$

where $\lambda = \{\lambda_\alpha\}_{\alpha\in I}$, $\mu = \{\mu_\alpha\}_{\alpha\in I}$ (we will use these notations throughout this paper).

Let $\{x_\alpha\}_{\alpha\in I} \subset H$ be some system and the series $\sum_{\alpha\in I(\lambda)} \lambda_\alpha x_\alpha$ be convergent in H for $\forall\lambda \in l_2\left(I^C\right)$. Let's enumerate the elements of the set $I(\lambda)$ and denote $\left\{\alpha_n^\lambda\right\}_{n\in N} = I(\lambda)$. It is absolutely clear that the value of the series $\sum_{n=1}^{\infty} \lambda_{\alpha_n^\lambda} x_{\alpha_n^\lambda}$ does not depend on the method of enumeration.

Let's show that there exists an absolute constant $M > 0$ such that

$$\left\| \sum_{\alpha\in I(\lambda)} \lambda_\alpha x_\alpha \right\| \leq M \|\lambda\|_{l_2(I)}, \ \ \forall\lambda \in l_2\left(I^C\right). \tag{2}$$

Take $\forall\omega \subset I : card\omega \leq \theta_0$, so the cardinality of the set ω is at most countable. Denote

$$l_2(\omega) = \left\{ \{\lambda_\alpha\}_{\alpha\in\omega} : \sum_{\alpha\in\omega} |\lambda_\alpha|^2 < +\infty \right\}.$$

Obviously, $l_2(\omega) \subset l_2\left(I^C\right)$. Consider the operator $T_\omega : l_2(\omega) \to H$:

$$T_\omega \lambda = \sum_{\alpha\in\omega} \lambda_\alpha x_\alpha, \ \ \forall\lambda \in l_2(\omega).$$

It is absolutely clear that if $card\omega < +\infty$, then the operator T_ω is bounded. Therefore, without loss of generality, we will assume that $card\omega = \theta_0$. In this case we have $l_2(\omega) = l_2(N) = l_2$. Thus, the series

$$T_\omega \lambda = \sum_{\alpha\in\omega} \lambda_\alpha x_\alpha,$$

is convergent for $\forall\lambda \in l_2$. Then it is known (see, e.g., O. Christensen [8, 9], B.T. Bilalov [29]) that the operator T_ω is bounded, i.e.

$$\|T_\omega \lambda\| \leq \|T_\omega\| \|\lambda\|_{l_2}, \quad \forall\lambda \in l_2. \tag{3}$$

The sequence $\lambda = \{\lambda_\alpha\}$ is called finite if $card\{\alpha : \lambda_\alpha \neq 0\} < +\infty$. The definition of norm and the inequality (3) imply that for $\forall\varepsilon \in (0, \|T_\omega\|), \exists\lambda \in l_\Phi$:

$$\|T_\omega \lambda\| > (\|T_\omega\| - \varepsilon) \|\lambda\|_{l_2},$$

where l_Φ is a linear manifold of finite sequences.

Now let's prove the validity of the inequality (2). It means

$$\|T_\omega\| \leq M \Rightarrow \sup_\omega \|T_\omega\| < +\infty,$$

for $\forall \omega \in I : card\omega \leq \theta_0$. It is absolutely clear that if

$$\sup_{\omega \subset I:\, card\omega \leq \theta_0} \|T_\omega\| < +\infty, \tag{4}$$

then (2) is true. Assume that (4) does not hold, i.e.

$$\sup_{\omega \subset I:\, card\omega \leq \theta_0} \|T_\omega\| = +\infty. \tag{5}$$

Then, for $\forall n \in N$, $\exists \omega_n \subset I : card\omega_n \leq \theta_0$ such that $\|T_{\omega_n}\| > n$. Without loss of generality, we will assume that $\|T_{\omega_n}\| < \left\|T_{\omega_{n+1}}\right\|$, $\forall n \in N$. Let's show that the sets $\omega_n, n \in N$, can be chosen in such a way that they do not intersect pairwise, i.e. $\omega_k \bigcap \omega_j = \emptyset, \quad k \neq j$. In fact, let ω_1 be chosen. Put $I_2 = I \backslash \omega_1$. In the sequel, for simplicity we'll write $|\omega| = card\omega$. Let's show that

$$\sup_{\omega \subset I_2:\, |\omega| \leq \theta_0} \|T_\omega\| = +\infty. \tag{6}$$

Assume the contrary, i.e.

$$M_2 = \sup_{\omega \subset I_2:\, |\omega| \leq \theta_0} \|T_\omega\| < +\infty.$$

Take $\forall \omega \in I : |\omega| \leq \theta_0$. Let $\omega \bigcap \omega_1 = \omega^{(1)} \wedge \omega \bigcap I_2 = \omega^{(2)}$. It is clear that

$$\omega = \omega^{(1)} \bigcup \omega^{(2)} \wedge \omega^{(1)} \bigcap \omega^{(2)} = \emptyset.$$

We have

$$\left|\omega^{(1)}\right| \leq \theta_0 \wedge \left|\omega^{(2)}\right| \leq \theta_0.$$

Thus

$$T_\omega \lambda = \sum_{\alpha \in \omega} \lambda_\alpha x_\alpha = \sum_{\alpha \in \omega^{(1)}} \lambda_\alpha x_\alpha + \sum_{\alpha \in \omega^{(2)}} \lambda_\alpha x_\alpha = T_{\omega^{(1)}} \lambda^{(1)} + T_{\omega^{(2)}} \lambda^{(2)},$$

where

$$\lambda^{(1)} = \left\{\lambda_\alpha^{(1)}\right\}_{\alpha \in \omega}, \ \lambda_\alpha^{(1)} = \begin{cases} \lambda_\alpha, \ \alpha \in \omega^{(1)} \\ 0, \ \ \alpha \in \omega^{(2)} \end{cases},$$

$$\lambda^{(2)} = \left\{\lambda_\alpha^{(2)}\right\}_{\alpha \in \omega}, \ \lambda_\alpha^{(2)} = \begin{cases} 0, \ \ \alpha \in \omega^{(1)} \\ \lambda_\alpha, \ \alpha \in \omega^{(2)} \end{cases}.$$

Consequently, $\lambda = \lambda^{(1)} + \lambda^{(2)}$. So we obtain

$$\|T_\omega \lambda\| \leq \left\|T_{\omega^{(1)}} \lambda^{(1)}\right\| + \left\|T_{\omega^{(2)}} \lambda^{(2)}\right\| \leq$$

$$\leq \|T_{\omega^{(1)}}\| \left\|\lambda^{(1)}\right\|_{l_2(I^C)} + M_2 \left\|\lambda^{(2)}\right\|_{l_2(I^C)} \leq$$

$$\leq \tilde{M}_2 \left(\left\|\lambda^{(1)}\right\|_{l_2(I^C)} + \left\|\lambda^{(2)}\right\|_{l_2(I^C)} \right),$$

where $\tilde{M}_2 = \max\{\|T_{\omega^{(1)}}\| ; M_2\}$.
It is absolutely clear that

$$\|\lambda\|^2_{l_2(I^C)} = \left\|\lambda^{(1)}\right\|^2_{l_2(I^C)} + \left\|\lambda^{(2)}\right\|^2_{l_2(I^C)}.$$

From here it directly follows that

$$\left\|\lambda^{(1)}\right\|_{l_2(I_C)} + \left\|\lambda^{(2)}\right\|_{l_2(I^C)} \leq \sqrt{2}\, \|\lambda\|_{l_2(I^C)}.$$

As a result, we obtain

$$\|T_\omega \lambda\| \leq \sqrt{2}\tilde{M}_2 \|\lambda\|_{l_2(I^C)}, \quad \forall \lambda \in l_2\left(I^C\right).$$

But this contradicts our assumption, i.e. the inequality (4) holds. Thus, the relation (6) is true. It is absolutely clear that $|I_2| > \theta_0$. From (6) we obtain that $\exists \omega_2 \subset I_2 : |\omega_2| \leq \theta_0$ such that $\|T_{\omega_2}\| > 2$. Let $I_3 = I_2 \backslash \omega_2$. Similar to the previous case, we can show that

$$\sup_{\omega \subset I_3 :\, |\omega| \leq \theta_0} \|T_\omega\| = +\infty,$$

and, consequently, $\exists \omega_3 \subset I_3 : |\omega_3| \leq \theta_0$ such that $\|T_{\omega_3}\| > 3$. Continuing this procedure, we'll get what we need.

Denote $\omega_0 = \bigcup_{n=1}^{\infty} \omega_n$. Clearly, $|\omega_0| = \theta_0$. Let's show that the operator T_{ω_0} is unbounded. Assume the contrary. Then, $\exists M > 0$:

$$\|T_{\omega_0} \lambda\| \leq M \|\lambda\|_{l_2(I^C)}, \quad \forall \lambda \in l_2\left(I^C\right). \tag{7}$$

Let $n > M$ be an arbitrary natural number. We have $\|T_{\omega_n}\| > n$. Then it follows from previous considerations that $\exists \lambda^{(n)} \in l_\Phi$:

$$\left\|T_{\omega_n} \lambda^{(n)}\right\| > n \left\|\lambda^{(n)}\right\|_{l_2(I^C)}.$$

We have $\lambda^{(n)} = \left\{\lambda^{(n)}_\alpha\right\}_{\alpha \in \omega_n}$. Now let's consider the sequence $\lambda^0 = \left\{\lambda^0_\alpha\right\}_{\alpha \in \omega_0}$, where

$$\lambda^0_\alpha = \begin{cases} \lambda^{(n)}_\alpha, & \alpha \in \omega_n, \\ 0, & \alpha \in \omega_0 \backslash \omega_n. \end{cases}$$

It is absolutely clear that $T_{\omega_0}\lambda^0 = T_{\omega_n}\lambda^{(n)}$, and, consequently

$$\left\|T_{\omega_0}\lambda^0\right\| > n\left\|\lambda^{(n)}\right\|_{l_2(I^C)} > M\left\|\lambda^{(n)}\right\|_{l_2(I^C)} = M\left\|\lambda^0\right\|_{l_2(I^C)},$$

which contradicts (7). Thus, the following theorem is proved.

Theorem 1. *Let $\{x_\alpha\}_{\alpha\in I} \subset H$ be some system and the series*

$$\sum_{\alpha\in I(\lambda)} \lambda_\alpha x_\alpha,$$

be convergent for $\forall\lambda \in l_2\left(I^C\right)$. Then there exists an absolute constant $M > 0$ such that

$$\left\|\sum_{\alpha\in I(\lambda)} \lambda_\alpha x_\alpha\right\| \le M\|\lambda\|_{l_2(I^C)}, \quad \forall\lambda \in l_2\left(I^C\right). \tag{8}$$

Denote by $T : l_2\left(I^C\right) \to H$ the operator defined as follows:

$$T\lambda = \sum_{\alpha\in I(\lambda)} \lambda_\alpha x_\alpha \ \ \forall\lambda \in l_2\left(I^C\right).$$

From (8) it follows that the operator T is bounded, i.e. $T \in L\left(l_2\left(I^C\right); H\right)$. Let's find the operator $T^* \in L\left(H; l_2\left(I^C\right)\right)$ conjugate to T.

Let $(\cdot\,;\,\cdot)_H$ be a scalar product in H. Let $\forall\lambda \in l_2\left(I^C\right)$ and $x \in H$ be arbitrary elements. By the definition of conjugate operator we have

$$(T^*x;\lambda)_{l_2(I^C)} = (x;T\lambda)_H.$$

Consequently

$$\sum_{\alpha\in I}\mu_\alpha\bar{\lambda}_\alpha = \left(x;\sum_{\alpha\in I(\lambda)}\lambda_\alpha x_\alpha\right)_H = \sum_{\alpha\in I(\lambda)}(x;x_\alpha)_H\,\bar{\lambda}_\alpha, \quad \mu = T^*x. \tag{9}$$

Let $\omega \subset I : |\omega| \le \theta_0$ be an arbitrary set. Thus, the relation (9) is true for $\forall\lambda \in l_2(\omega)$. Then from (9) it follows that $\{(x;x_\alpha)_H\}_{\alpha\in\omega} \in l_2(\omega)$. Consequently, the series

$$\sum_{\alpha\in\omega}\left|(x;x_\alpha)_H\right|^2 < +\infty, \tag{10}$$

is convergent for $\forall\omega \subset I : |\omega| = \theta_0$. Let $I_x = \{\alpha \in I : (x;x_\alpha)_H \ne 0\}$ and $I_n = \left\{\alpha \in I : |(x;x_\alpha)_H| > \frac{1}{n}\right\}$, where $n \in N$ is an arbitrary number. It is absolutely clear that $I_x = \bigcup_{n=1}^{\infty} I_n$. From (11) it follows that $cardI_n < +\infty,\ \forall n \in N$. In fact, if $cardI_{n_0} = +\infty$ for some n_0, then we can choose the set $\omega_0 \subset I_{n_0} : |\omega_0| = \theta_0$. It is clear that

$$\sum_{\alpha\in\omega_0} \left|(x; x_\alpha)_H\right|^2 = +\infty,$$

which contradicts the relation (11). Thus, $cardI_x \leq \theta_0,\ \forall x \in H$. Then from (11) it follows immediately that

$$\mu_\alpha = \begin{cases} 0, & \alpha \notin I_x, \\ (x; x_\alpha)_H\,, & \alpha \in I_x\,, \end{cases}$$

in other words, $T^*x = \{(x; x_\alpha)_H\}_{\alpha\in I}$. It is absolutely clear that $\|T^*\| = \|T\|$. We have

$$\sum_{\alpha\in I_x} \left|(x; x_\alpha)_H\right|^2 = \|T^*x\|^2_{l_2(I\setminus C)} \leq \|T^*\|^2 \|x\|^2 =$$

$$= \|T\|^2 \|x\|^2\,, \quad \forall x \in H.$$

So the following theorem is true.

Theorem 2. *Let* $\{x_\alpha\}_{\alpha\in I} \subset H$ *be some system. If the series*

$$\sum_{\alpha\in I} \lambda_\alpha x_\alpha,$$

is convergent in H *for* $\forall\lambda \in l_2\left(I^C\right)$, *then* $\{x_\alpha\}_{\alpha\in I}$ *is a Bessel system in* H *for* $\forall x \in H$: $cardI_x \leq \theta_0$, *where* $I_x = \{\alpha \in I : (x; x_\alpha)_H \neq 0\}$. *Moreover, the inequality*

$$\sum_{\alpha\in I_x} \left|(x; x_\alpha)_H\right|^2 \leq \|T\|^2 \|x\|^2\,, \tag{11}$$

is true, where T *is the operator from Theorem 1.*

Accept the following definition.

Definition 2. *Let* $\{x_\alpha\}_{\alpha\in I} \subset H$ *be a Bessel system in* H. *The number* $\inf\{M :$ *satisfying the inequality* (1)$\}$ *is called a Bessel norm (B−norm) of the system* $\{x_\alpha\}_{\alpha\in I}$. *We denote it by* $B\left[\{x_\alpha\}_{\alpha\in I}\right]$.

Let's prove the following main theorem.

Theorem 3. *Let* $\{x_\alpha\}_{\alpha\in I} \subset H$ *be some system. In order for this system to be a Bessel system in* H, *it is necessary and sufficient that the operator* T *defined by*

$$T\lambda = \sum_{\alpha\in I(\lambda)} \lambda_\alpha x_\alpha, \tag{12}$$

act boundedly from $l_2\left(I^C\right)$ *to* H. *If so,* $B\left[\{x_\alpha\}_{\alpha\in I}\right] = \|T\|^2$.

Proof. First, we assume that the operator T defined by (12) belongs to the space $L\left(l_2\left(I^C\right);H\right)$. Then from Theorem 2 it follows that $\{x_\alpha\}_{\alpha\in I}$ is a Bessel system in H, and the relation (11) implies $B\left[\{x_\alpha\}_{\alpha\in I}\right] \le \|T\|^2$.

Now let's assume the contrary: let $\{x_\alpha\}_{\alpha\in I}$ be a Bessel system in H. Then it follows from Theorem 2 that for $\forall x \in H$ the relation $cardI_x \le \theta_0$ holds, where $I_x = \{\alpha \in I : (x;x_\alpha)_H \ne 0\}$. Let $\omega \subset I$ with $card\omega = \theta_0$ be an arbitrary set, and $\lambda \in l_2(\omega)$ be an arbitrary element. Let's prove that the series

$$\sum_{\alpha\in\omega} \lambda_\alpha x_\alpha,$$

is convergent in H. First, note that the inequality (1) implies the validity of the following relation:

$$\sum_{\alpha\in I} |(x;x_\alpha)|^2 \le M, \quad \forall x : \|x\| = 1,$$

i.e.

$$\sup_{\|x\|=1} \sum_{\alpha\in I} |(x;x_\alpha)|^2 \le M < +\infty.$$

Let $\omega = \{\alpha_1;\alpha_2;...\}$. We have

$$\left\|\sum_{k=n}^{m} \lambda_{\alpha_k} x_{\alpha_k}\right\| = \sup_{\|x\|=1} \left|\left(\sum_{k=n}^{m} \lambda_{\alpha_k} x_{\alpha_k};x\right)_H\right| =$$

$$= \sup_{\|x\|=1} \left|\sum_{k=n}^{m} \lambda_{\alpha_k} (x_{\alpha_k};x)_H\right| \le$$

$$\le \sup_{\|x\|=1} \left(\sum_{k=n}^{m} |(x;x_{\alpha_k})|^2\right)^{1/2} \left(\sum_{k=n}^{m} |\lambda_{\alpha_k}|^2\right)^{1/2} \le$$

$$\le M^{\frac{1}{2}} \left(\sum_{k=n}^{m} |\lambda_{\alpha_k}|^2\right)^{1/2} \to 0, \quad as \;\; n,m \to \infty.$$

From here it follows that the series $\sum_{\alpha\in\omega} \lambda_\alpha x_\alpha$ is convergent. The arbitrariness of ω and Theorem 1 imply $T \in L\left(l_2\left(I^C\right);H\right)$. Let $\omega = \{\alpha_k\}_{k\in N} \subset I$ be an arbitrary set and take $\forall\lambda \in l_2(\omega)$. We have

$$\left\|\sum_{n=1}^{m} \lambda_{\alpha_n} x_{\alpha_n}\right\| = \sup_{\|x\|=1} \left|\left(\sum_{n=1}^{m} \lambda_{\alpha_n} x_{\alpha_n};x\right)\right| \le$$

$$\leq \sup_{\|x\|=1} \left(\sum_{n=1}^{m} |(x;x_{\alpha_n})|^2 \right)^{1/2} \left(\sum_{n=1}^{m} |\lambda_{\alpha_n}|^2 \right)^{1/2} \leq$$

$$\leq \left(B\left[\{x_\alpha\}_{\alpha\in I}\right]\right)^{1/2} \|\lambda\|_{l_2(\omega)} .$$

Consequently

$$\|T\lambda\| \leq \left(B\left[\{x_\alpha\}_{\alpha\in I}\right]\right)^{1/2} \|\lambda\|_{l_2(\omega)} .$$

The arbitrariness of ω and $\lambda \in l_2(\omega)$ imply

$$\|T\lambda\| \leq \left(B\left[\{x_\alpha\}_{\alpha\in I}\right]\right)^{1/2} \|\lambda\|_{l_2(I^C)} , \forall \lambda \in l_2\left(I^C\right),$$

which in turn yields

$$\|T\|^2 \leq B\left[\{x_\alpha\}_{\alpha\in I}\right].$$

Taking into account the previous inequality, we obtain

$$\|T\|^2 = B\left[\{x_\alpha\}_{\alpha\in I}\right].$$

◀

The theorem below is an analogue to a result for the case of separable space.

Theorem 4. *Let $\{x_\alpha\}_{\alpha\in I} \subset H$ be some system and $V \subset H$ be a set everywhere dense in H. If there exists an absolute constant $B > 0$ such that the inequality*

$$\sum_{\alpha\in I} |(x;x_\alpha)_H|^2 \leq B\|x\|, \tag{13}$$

holds for $\forall x \in V : cardI_x \leq \theta_0$, where $I_x = \{\alpha \in I : (x;x_\alpha)_H \neq 0\}$, then $\{x_\alpha\}_{\alpha\in I} \subset H$ is a Bessel system in H.

Proof. Let's prove that the inequality (13) is true for $\forall x \in H$. Assume the contrary, i.e. assume $\exists x_0 \in H$ such that

$$\sum_{k=1}^{\infty} \left|(x_0;x_{\alpha_k})_H\right|^2 > B\|x_0\|,$$

for some index set $\{\alpha_k\}_{k\in N} \subset I$. Clearly, $\exists n_0 \in N$:

$$\sum_{k=1}^{n_0} \left|(x_0;x_{\alpha_k})_H\right|^2 > B\|x_0\|.$$

The continuity of scalar product and norm, and the density of V in H directly imply that $\exists y \in V$:

$$\sum_{k=1}^{n_0} \left|(y; x_{\alpha_k})_H\right|^2 > B \|y\| .$$

Thus, we have

$$\sum_{\alpha \in I_y} |(y; x_\alpha)_H|^2 \geq \sum_{k=1}^{n_0} \left|(y; x_{\alpha_k})_H\right|^2 > B \|y\| ,$$

which contradicts the inequality (13). ◄

3. Uncountable Hilbert Frame

Let H be a non-separable Hilbert space and $\{x_\alpha\}_{\alpha \in I} \subset H$ be some system.

Definition 3. *A system $\{x_\alpha\}_{\alpha \in I}$ is called an uncountable frame or simply a frame in H if for $\forall x \in H : cardI_x \leq \theta_0$, where $I_x = \{\alpha \in I : (x; x_\alpha) \neq 0\}$, there exist absolute constants $A; B > 0$ such that*

$$A \|x\|^2 \leq \sum_{\alpha \in I} |(x; x_\alpha)_H|^2 \leq B \|x\|^2 , \ \forall x \in H. \tag{14}$$

The numbers A and B are called the lower and upper frame bounds. A frame is called tight if we can take $A = B$ in (14).

A frame $\{x_\alpha\}_{\alpha \in I}$ in H is called exact if for $\forall \beta \in I$ the system $\{x_\alpha\}_{\alpha \in I;\, \alpha \neq \beta}$ stops being frame.

A system $\{x_\alpha\}_{\alpha \in I} \subset H$ is called a frame family if it forms a frame in $\overline{span \left[\{x_\alpha\}_{\alpha \in I}\right]}$.

Let $\{x_\alpha\}_{\alpha \in I} \subset H$ form a frame in H. Then it is clear that $\{x_\alpha\}_{\alpha \in I}$ is a Bessel system in H. From Theorem 3 it follows that the operator

$$T\lambda = \sum_{\alpha \in I} \lambda_\alpha x_\alpha, \forall \lambda \in l_2 \left(I^C\right),$$

is bounded, i.e. $T \in L\left(l_2\left(I^C\right); H\right)$. This operator is called a pre-frame operator. As established above, the conjugate operator $T^* \in L\left(H; l_2\left(I^C\right)\right)$ is defined by the formula

$$T^* x = \{(x; x_\alpha)_H\}_{\alpha \in I} , \ \ \forall x \in H.$$

The operator T^* is called a synthesis operator. Clearly, the operator $S = TT^*$ is self-adjoint and belongs to $L(H)$. We have

$$(Sx; x)_H = (T(T^* x); x)_H = \left(T\left(\{(x; x_\alpha)_H\}_{\alpha \in I}; x\right)\right)_H =$$

$$= \left(\sum_{\alpha \in I} (x; x_\alpha)_H \, x_\alpha; x \right)_H = \sum_{\alpha \in I} (x; x_\alpha)_H \, (x_\alpha; x)_H = \sum_{\alpha \in I} \left| (x; x_\alpha)_H \right|^2 .$$

From (14) we obtain

$$A \left\| x \right\|^2 \leq (Sx; x) \leq B \left\| x \right\|^2, \forall x \in H.$$

Consequently

$$AI \leq S \leq BI \Leftrightarrow 0 \leq I - B^{-1} S \leq \frac{B - A}{B} I.$$

As a result

$$\left\| I - B^{-1} S \right\| = \sup_{\|x\|=1} \left| \left(\left(I - B^{-1} S \right) x; x \right)_H \right| \leq$$

$$\leq \sup_{\|x\|=1} \left| \left(\frac{B - A}{B} x; x \right)_H \right| = \frac{B - A}{B} < 1.$$

From here it directly follows that S is boundedly invertible, i.e. $S^{-1} \in L(H)$. It is clear that $\left(S^{-1} \right)^* = S^{-1}$. Further, absolutely similar to the case of usual frame, we can show that the family $\left\{ S^{-1} x_\alpha \right\}_{\alpha \in I}$ also forms a frame in H with frame bounds B^{-1} and A^{-1}, i.e.

$$B^{-1} \left\| x \right\|^2 \leq \sum_{\alpha \in I} \left| \left(x; S^{-1} x_\alpha \right)_H \right|^2 \leq A^{-1} \left\| x \right\|^2, \forall x \in H.$$

The frame $\left\{ S^{-1} x_\alpha \right\}_{\alpha \in I}$ is called canonically dual to $\{ x_\alpha \}_{\alpha \in I}$.

The following theorem on decomposition of arbitrary element with respect to frame is true.

Theorem 5. *Let the family $\{ x_\alpha \}_{\alpha \in I} \subset H$ form a frame in H and S be the corresponding frame operator. Then*

$$x = \sum_{\alpha \in I} \left(x; S^{-1} x_\alpha \right)_H x_\alpha, \ \forall x \in H.$$

Proof. Let $y \in H$ be an arbitrary element. We have

$$Sy = T(T^* y) = T\left(\{ (y; x_\alpha)_H \}_{\alpha \in I} \right) = \sum_{\alpha \in I} (y; x_\alpha)_H \, x_\alpha. \tag{15}$$

Now take $\forall x \in H$ and consider $y = S^{-1} x$. Then from (15) we obtain

$$x = S\left(S^{-1} x \right) = \sum_{\alpha \in I} \left(S^{-1} x; x_\alpha \right)_H x_\alpha = \sum_{\alpha \in I} \left(x; S^{-1} x_\alpha \right)_H x_\alpha.$$

◄

The theorem below is proved in a quite similar way.

Theorem 6. *Let $\{x_\alpha\}_{\alpha\in I} \subset H$ be some system and $V \subset H$ be a set everywhere dense in H. Let for $\forall x \in V : card\,\{\alpha \in I : (x, x_\alpha)_H \neq 0\} \leq \theta_0$ and let there exist absolute constants $A; B > 0$ such that*

$$A\,\|x\|^2 \leq \sum_{\alpha\in I} |(x; x_\alpha)_H|^2 \leq B\,\|x\|^2\,, \forall x \in V.$$

Then $\{x_\alpha\}_{\alpha\in I}$ forms a frame in H.

4. Examples

Let $e_\lambda(t) = e^{i\lambda t}, t \in R$, where $\lambda \in R$, and consider the linear span $V = span\,\left[\{e_\lambda\}_{\lambda\in R}\right]$ over the field of complex numbers C. Define the scalar product in V as follows:

$$(x; y)_V = \lim_{T\to\infty} \frac{1}{2T} \int_{-T}^{T} x(t)\,\overline{y(t)}\,dt. \tag{16}$$

It is not difficult to verify that (16) is in fact a scalar product and the system $\{e_\lambda\}_{\lambda\in R}$ is orthonormal with respect to this scalar product. Let's complete V with respect to the norm generated by this scalar product and denote the resulting Hilbert space by $L_2^\Lambda(R)$. It is not difficult to see that $L_2^\Lambda(R)$ is non-separable.

It is absolutely clear that the system $\{e_\lambda\}_{\lambda\in R}$ is complete in $L_2^\Lambda(R)$ for $\forall x \in L_2^\Lambda(R) : card\,I_k \leq \theta_0$, where $I_x = \{\lambda \in R : (x; e_\lambda)_V \neq 0\}$. This assertion follows from Bessel's inequality for orthonormal systems. Moreover, Parseval's equality

$$\|x\|_V^2 = \sum_{\lambda\in I_x} |(x; e_\lambda)_V|^2\,,\ \forall x \in L_2^\Lambda(R)\,,$$

is true, where $\|\cdot\|_V^2 = (\cdot;\cdot)_V$.

4.1. Let $\lambda_0 \in R$ be an arbitrary number and $I = R\bigcup\{i\}$. Consider the system $\{\varphi_\lambda\}_{\lambda\in I}$ with

$$\varphi_\lambda = \begin{cases} e_\lambda, \lambda \in R, \\ e_{\lambda_0},\ \lambda = i. \end{cases}$$

Let's show that it forms a frame in $L_2^\Lambda(R)$. Take $\forall x\ \in L_2^\Lambda(R)$. Let $I_x = \{\lambda \in R : (x; e_\lambda)_V \neq 0\}$. We have

$$\|x\|_V^2 = \sum_{\lambda\in I_x} |(x; e_\lambda)_V|^2 \leq \sum_{\lambda\in I_x\bigcup i} |(x; \varphi_\lambda)_V|^2 \leq \left|(x; e_{\lambda_0})_V\right|^2 + \sum_{\lambda\in I_x} |(x; e_\lambda)_V|^2 =$$

$$= \left|(x; e_{\lambda_0})_V\right|^2 + \|x\|_V^2 \leq 2\,\|x\|_V\,.$$

Thus, we obtain

$$\|x\|_V^2 \leq \sum_{\lambda\in I} |(x;\varphi_\lambda)_V|^2 \leq 2\,\|x\|_V^2\,,\ \forall x \in L_2^\Lambda(R)\,.$$

Consequently, the system $\{\varphi_\lambda\}_{\lambda\in I}$ forms a frame in $L_2^\Lambda(R)$ with frame bounds $A = 1$, $B = 2$.

4.2. Let $\{\lambda_n\}_{n\in N} = \omega \subset R$ be some sequence of different numbers and $I = R\bigcup i\omega$. Consider the system $\{\varphi_\lambda\}_{\lambda\in I}$ with

$$\varphi_\lambda = \begin{cases} e_\lambda,\ \lambda \in R\,, \\ e_{\lambda_n},\ \lambda = i\lambda_n,\ n \in N. \end{cases}$$

Take $\forall x \in L_2^\Lambda(R)$ and let $I_x = \{\lambda \in R : (x;e_\lambda)_V \neq 0\}$. We have

$$\|x\|_V^2 = \sum_{\lambda\in I_x} |(x;e_\lambda)_V|^2 \leq \sum_{\lambda\in I_x} |(x;e_\lambda)_V|^2 + \sum_{\lambda\in\omega} |(x;\varphi_\lambda)_V|^2 =$$

$$= \sum_{\lambda\in I} |(x;\varphi_\lambda)_V|^2 \leq 2\sum_{\lambda\in I_x} |(x;e_\lambda)_V|^2 = 2\,\|x\|_V^2\,.$$

Consequently, the system $\{\varphi_\lambda\}_{\lambda\in I}$ forms a frame in $L_2^\Lambda(R)$ with frame bounds $A = 1$, $B = 2$.

4.3. Let $I = R\bigcup iR$ and consider the system $\{\varphi_\lambda\}_{\lambda\in I}$:

$$\varphi_\lambda = \begin{cases} e_\lambda, \lambda \in R, \\ e_\mu,\ \lambda = i\mu, \mu \in R \end{cases}.$$

Take $\forall x \in L_2^\Lambda(R)$ and let $I_x = \{\lambda \in R : (x;e_\lambda)_H \neq 0\}$. We have

$$2\,\|x\|_V^2 = \sum_{\lambda\in I_x} |(x;e_\lambda)_V|^2 + \sum_{\lambda_i\in I_x} |(x;\varphi_\lambda)_V|^2 = \sum_{\lambda\in I} |(x;\varphi_\lambda)_V|^2\,.$$

Consequently, $\{\varphi_\lambda\}_{\lambda\in I}$ forms a tight frame in $L_2^\Lambda(R)$ with frame bounds $A = B = 2$.

Similar examples can be given for an arbitrary non-separable Hilbert space with orthonormal basis $\{e_\alpha\}_{\alpha\in I}$ of the same cardinality.

5. Frame Family

In this section, we will need the concept of pseudo-inverse operator. This concept is based on the following lemma stated in the monograph by O. Christensen [8, 9].

Lemma 1 ([8]). *Let H_1 and H_2 be some Hilbert spaces and $U \in L(H_1; H_2)$ with $R(U) = \overline{R(U)}$, i.e. the range of the operator U is closed. Then $\exists U^+ \in L(H_2; H_1)$ such that*

$$UU^+ y = y, \ \forall y \in R(U).$$

The operator U^+ appearing in Lemma 1 is called a pseudo-inverse of the operator U. Pseudo-inverse operator has the following properties.

Lemma 2 ([8]). *Let $U \in L(H_1; H_2)$, $R(U) = \overline{R(U)}$ and U^+ be the pseudo-inverse of U. Then:*

(i) UU^+ is an orthogonal projection from H_2 on $R(U)$;
(ii) U^+U is an orthogonal projection from H_1 on $R(U^+)$;
(iii) $(U^*)^+ = (U^+)^*$ and $R(U^*) = \overline{R(U^*)}$;
(iv) on $R(U)$, the operator U^+ has the representation

$$U^+ = U^* (UU^*)^{-1}.$$

Now we proceed to the frame families. Let H be some non-separable Hilbert space and $\{x_\alpha\}_{\alpha \in I} \subset H$ be some frame family. Let

$$V = \overline{L\left[\{x_\alpha\}_{\alpha \in I}\right]}.$$

By definition, the family $\{x_\alpha\}_{\alpha \in I}$ forms a frame in V. Therefore, the analysis operator $T \in L\left(l_2\left(I^C\right); V\right)$, and, consequently, $T \in L\left(l_2\left(I^C\right); H\right) : R(T) = V$. As is known, $N_{T^*} = R(T)^\perp$. Consequently, $N_{T^*} = \{0\} \Leftrightarrow \exists (T^*)^{-1}$ only when $\overline{R(T)} = H$, i.e. $R(T)$ is everywhere dense in H.

So the following theorem is true.

Theorem 7. *Let $\{x_\alpha\}_{\alpha \in I} \subset H$ be a frame family and T be the corresponding analysis operator. The family $\{x_\alpha\}_{\alpha \in I}$ forms a frame in H only when T^* is injective, i.e. $KerT^* = \{0\}$.*

The theorem below can be proved quite similar to the case of usual frames.

Theorem 8. *Let the family $\{x_\alpha\}_{\alpha \in I} \subset H_1$ form a frame in H_1 and $F \in L(H_1; H_2)$ be some operator with closed range, i.e. $R_F = \overline{R}_F$, where H_k, $k = 1, 2$; is a Hilbert space with the scalar product $(\cdot; \cdot)_{H_k}$. Then $\{Fx_\alpha\}_{\alpha \in I}$ is a frame family in H_2 with frame bounds $A \|F^+\|^{-2}$, $B \|F\|^2$, where A and B are frame bounds of the family $\{x_\alpha\}_{\alpha \in I}$ in H_1.*

Proof. In fact, let $y \in H_2$ be an arbitrary element. We have

$$card \left\{\alpha \in I : (y; Fx_\alpha)_{H_2} \neq 0\right\} = card \left\{\alpha \in I : (F^* y; x_\alpha)_{H_1} \neq 0\right\} \leq \theta_0.$$

So

$$\sum_{\alpha\in I}\left|(y;Fx_\alpha)_{H_2}\right|^2=\sum_{\alpha\in I}\left|(F^*y;x_\alpha)_{H_1}\right|^2\le B\left\|F^*y\right\|_{H_1}^2\le B\left\|F\right\|^2\left\|y\right\|_{H_2}^2.$$

Consequently, the family $\{Fx_\alpha\}_{\alpha\in I}$ is a Bessel family in H_2. Let's show that the lower estimate also holds. Take $\forall y\in L\left[\{Fx_\alpha\}_{\alpha\in I}\right]$. Consequently, $y=Fx$, where $x\in L\left[\{x_\alpha\}_{\alpha\in I}\right]$. According to Lemma 2 of [8], the operator FF^+ is an orthogonal projection from H_2 on R_F, therefore it is self-adjoint. Taking into account that $FF^+y=y$, $\forall y\in R_F$, we have

$$y=Fx=\left(FF^+\right)^*Fx=\left(F^+\right)^*F^*Fx.$$

Consequently

$$\|y\|^2\le\left\|\left(F^+\right)^*\right\|^2\left\|F^*Fx\right\|^2\le\frac{\left\|(F^+)^*\right\|^2}{A}\sum_{\alpha\in I}\left|(F^*Fx;x_\alpha)_{H_1}\right|^2=$$

$$=\frac{\left\|(F^+)^*\right\|^2}{A}\sum_{\alpha\in I}\left|\left(Fx;\,Fx_\alpha\right)_{H_2}\right|^2.$$

As a result, for $\forall y\in R_F$ we obtain

$$\frac{A}{\left\|(F^+)^*\right\|^2}\|y\|^2\le\sum_{\alpha\in I}\left|(y;Fx_\alpha)_{H_2}\right|^2.$$

It is absolutely clear that this inequality is also true for $\forall y\in\overline{R_F}$. ◀

This theorem has the following direct corollary.

Corollary 1. *Let the family $\{x_\alpha\}_{\alpha\in I}$ form a frame in H_1 and $F\in L(H_1;H_2)$ be a surjective operator, i.e. $R_F=H_2$. Then the family $\{Fx_\alpha\}_{\alpha\in I}$ forms a frame in H_2 with the same frame bounds.*

Theorem below can be proved easily.

Theorem 9. *Let the family $\{x_\alpha\}_{\alpha\in I}\subset H$ form a frame in H and S be the corresponding frame operator. Then $\left\{S^{-\frac{1}{2}}x_\alpha\right\}_{\alpha\in I}$ forms a tight frame in H with frame bounds $A=B=1$ and*

$$x=\sum_{\alpha\in I}\left(x;S^{-\frac{1}{2}}x_\alpha\right)_H S^{-\frac{1}{2}}x_\alpha,\ \forall x\in H.$$

Let the family $\{x_\alpha\}_{\alpha\in I}\subset H$ form a frame in H and S be the corresponding frame operator. Then $\forall x\in H$ has a decomposition

$$x=\sum_{\alpha\in I}\left(x;S^{-1}x_\alpha\right)_H x_\alpha .$$

The family $\left\{\left(x;S^{-1}x_\alpha\right)_H\right\}_{\alpha\in I}$ are the frame coefficients of the element x. Denote them by $\{F_\alpha(x)\}_{\alpha\in I}$:

$$F_\alpha(x)=\left(x;S^{-1}x_\alpha\right)_H,\ \forall\alpha\in I.$$

Frame coefficients have the smallest norms among other decomposition coefficients. In other words, the following theorem is true.

Theorem 10. *Let the family $\{x_\alpha\}_{\alpha\in I}$ form a frame in H and $x\in H$ have a decomposition*

$$x=\sum_{\alpha\in I}\lambda_\alpha x_\alpha,\quad \textit{for}\ \ \{\lambda_\alpha\}_{\alpha\in I}\in l_2\left(I^C\right). \tag{17}$$

Then

$$\sum_{\alpha\in I}|\lambda_\alpha|^2=\sum_{\alpha\in I}|F_\alpha(x)|^2+\sum_{\alpha\in I}|\lambda_\alpha-F_\alpha(x)|^2 .$$

Proof. In fact, let the decomposition (17) hold. We have

$$\{\lambda_\alpha\}_{\alpha\in I}=\{\lambda_\alpha-F_\alpha(x)\}_{\alpha\in I}+\{F_\alpha(x)\}_{\alpha\in I}.$$

Obviously, $\{\lambda_\alpha-F_\alpha(x)\}_{\alpha\in I}\in l_2\left(I^C\right)$. Let $T\in L\left(l_2\left(I^C\right);H\right)$ be an analysis operator corresponding to the frame $\{x_\alpha\}_{\alpha\in I}$. We have

$$T\left(\{\lambda_\alpha-F_\alpha(x)\}_{\alpha\in I}\right)=\sum_{\alpha\in I}\left(\lambda_\alpha-F_\alpha(x)\right)x_\alpha=$$

$$=\sum_{\alpha\in}\lambda_\alpha x_\alpha-\sum_{\alpha\in I}F_\alpha(x)\,x_\alpha=x-x=0.$$

Consequently, $\{\lambda_\alpha-F_\alpha(x)\}_{\alpha\in I}\in KerT$. On the other hand

$$F_\alpha(x)=\left(x;S^{-1}x_\alpha\right)_H=\left(S^{-1}x;x_\alpha\right)_H,\ \forall\alpha\in I.$$

Hence, $\{F_\alpha(x)\}_{\alpha\in I}\in R_{T^*}$. As $KerT\perp R_{T^*}$, it is clear that $\{\lambda_\alpha-F_\alpha(x)\}_{\alpha\in I}\perp\{F_\alpha(x)\}_{\alpha\in I}$ in $l_2\left(I^C\right)$. The assertion of the theorem follows directly. ◄

Theorem below can be proved in a quite similar way to the case of usual frames.

Theorem 11. *Let the family $\{x_\alpha\}_{\alpha\in I}$ form a frame in a non-separable Hilbert space H, T be the corresponding analysis operator, S be a frame operator and T^+ be a pseudo-inverse of the operator T. Then the optimal (i.e. the best) frame bounds A and B are defined by*

$$A = \left\|S^{-1}\right\|^{-1} = \left\|T^+\right\|^{-2};\ B = \|S\| = \|T\|^2 .$$

6. Minimality and Biorthogonality in Non-Separable Case

Similar to the classical case, we accept the following definition.

Definition 4. *Let H be a non-separable H-space with the scalar product $(\cdot;\cdot)$. The families $\{x_\alpha; y_\alpha\}_{\alpha\in I} \subset H$ are called biorthogonal if*

$$(x_\alpha; y_\beta) = \begin{cases} 0, & \alpha \neq \beta, \\ \neq 0, & \alpha = \beta. \end{cases}$$

For $(x_\alpha; y_\beta) = \delta_{\alpha\beta}$ ($\delta_{\alpha\beta}$ is the Kronecker symbol), they are called biorthonormal.

Also accept the following definition.

Definition 5. *A family $\{x_\alpha\}_{\alpha\in I} \subset H$ in a non-separable H-space H is called minimal if $x_\beta \notin \overline{L\left[\{x_\alpha\}_{\alpha\neq\beta}\right]}$ for $\forall\beta \in I$, where $\overline{L[M]}$ is the closure of the linear span of the set $M \subset H$ in H.*

It is not difficult to see that if the family $\{x_\alpha\}_{\alpha\in I}$ has a biorthonormal family $\{y_\alpha\}_{\alpha\in I}$, then it is minimal. In fact, let $L_{\beta_0} = L\left[\{x_\alpha\}_{\alpha\neq\beta_0}\right]$ for $\forall\beta_0 \in I$. Clearly, $(x; y_{\beta_0}) = 0$, $\forall x \in L_{\beta_0}$. Let $x_{\beta_0} \in \overline{L_{\beta_0}}$ for some $\beta_0 \in I$. Consequently, $\exists\ \{u_n\}_{n\in N} \subset L_{\beta_0} : x_{\beta_0} = \lim_{n\to\infty} u_n$. We have

$$1 = (x_{\beta_0}; y_{\beta_0}) = \left(\lim_{n\to\infty} u_n; y_{\beta_0}\right) = \lim_{n\to\infty} (u_n; y_{\beta_0}) = 0.$$

The obtained contradiction proves that $x_{\beta_0} \notin \overline{L_{\beta_0}}$. Thus, the biorthogonality implies the minimality.

Now suppose that the family $\{x_\alpha\}_{\alpha\in I}$ is minimal in H. Take $\forall\beta \in I$. Then, $\rho\left(x_\beta; \overline{L}_\beta\right) = \inf_{x\in L_\beta} \|x_\beta - x\| = d > 0$, where $\|\cdot\| = \sqrt{(\cdot;\cdot)}$ is the norm in H. Let $M_\beta = L\left[x_\beta; \overline{L}_\beta\right]$. Every element $x \in M_\beta$ has a unique representation of the form

$$x = \lambda x_\beta + \tilde{x},\ \tilde{x} \in \overline{L}_\beta,\ \lambda \in C.$$

Consider the functional $\vartheta_\beta : M_\beta \to C$:$\vartheta_\beta(x) = \lambda$, $\forall x \in M_\beta$. It is not difficult to see that the functional $\vartheta_\beta(\cdot)$ is linear, and, moreover, $\vartheta_\beta(x_\beta) = 1, \vartheta_\beta(x) = 0$, $\forall x \in \overline{L}_\beta$. We have (for $\lambda \neq 0$)

$$\|x\| = \|\lambda x_\beta + \tilde{x}\| = |\lambda| \left\| x_\beta - \left(-\frac{\tilde{x}}{\lambda}\right) \right\| \geq |\lambda|\, d = d\, |\vartheta_\beta(x)| \Rightarrow |\vartheta_\beta(x)| \leq \frac{1}{d}\, \|x\|.$$

For $\lambda = 0$ this inequality is obvious. Therefore, the functional ϑ_β is bounded in M_β. According to Hahn-Banach theorem, ϑ_β can be continued to the whole H with norm-preserving. We denote the resulting functional also by ϑ_β. It is clear that the families $\{x_\alpha; \vartheta_\alpha\}_{\alpha\in I}$ are biorthonormal. So the following statement is true.

Statement 1. *A family $\{x_\alpha\}_{\alpha\in I} \subset H$ in a non-separable H-space H is minimal only when it has a biorthonormal family.*

Consider the following example which is of particular interest.

Example 1. *Let H be a non-separable H-space which has an orthonormal basis $\{e_\alpha\}_{\alpha\in I}$. Take $\omega \subset I$: $card\omega = \theta_0$. Let $\omega = \{\alpha_n\}_{n\in N}$. Consequently, the system $\{e_{\alpha_n}\}_{n\in N}$ forms an orthonormal basis for $H_\omega = \overline{L\left[\{e_{\alpha_n}\}_{n\in N}\right]}$. We have $H = H_\omega \dot{+} H_\omega^\perp$, where $H_\omega^\perp$ is an orthogonal complement of H_ω in H. Let*

$$\vartheta_{\alpha_n} = e_{\alpha_n} + e_{\alpha_{n+1}}, \forall n \in N,$$

and define the family $\{x_\alpha\}_{\alpha\in I}$:

$$x_\alpha = \begin{cases} e_\alpha, & \alpha \in I\backslash\omega, \\ \vartheta_\alpha, & \alpha \in \omega. \end{cases}$$

Let's show that $\{x_\alpha\}_{\alpha\in I}$ is a Bessel family. It is not difficult to see that $card\,\{\alpha \in I : (x; x_\alpha) \neq 0\} \leq \theta_0$ for $\forall x \in H$. We have

$$\sum_{\alpha\in I} |(x; x_\alpha)|^2 = \sum_{\alpha\in I\backslash\omega} |(x; e_\alpha)|^2 + \sum_{n=1}^{\infty} \left|\left(x; e_{\alpha_n} + e_{\alpha_{n+1}}\right)\right|^2 \leq$$

$$\leq 2 \sum_{\alpha\in I} |(x; e_\alpha)|^2 + 2 \sum_{n=1}^{\infty} \left|\left(x; e_{\alpha_{n+1}}\right)\right|^2 \leq 4\, \|x\|^2, \ \forall x \in H.$$

So, $\{x_\alpha\}_{\alpha\in I}$ is a Bessel family. Let
$f_{\alpha_k} = \sum_{n=1}^{k} (-1)^{n+1} e_{\alpha_n}$, if k is odd;
$f_{\alpha_k} = \sum_{n=1}^{k} (-1)^{n} e_{\alpha_n}$, if k is even,

and define

$$g_\alpha = \begin{cases} e_\alpha, \ \alpha \in I \backslash \omega, \\ f_\alpha, \ \alpha \in \omega. \end{cases}$$

It is not difficult to verify that the family $\{x_\alpha; g_\alpha\}_{\alpha\in I}$ *is biorthonormal.*

Let's show that $\{x_\alpha\}_{\alpha\in I}$ *is complete in* H*. Let* $(x; x_\alpha) = 0, \forall\alpha \in I$*, for some* $x \in H$*. Clearly,* $(x; e_\alpha) = 0, \ \forall\alpha \in I\backslash\omega$*.*

From $(x; \vartheta_{\alpha_n}) = 0, \forall n \in N$*, it follows that*

$$|(x; e_{\alpha_n})| = \left|(x; e_{\alpha_{n+1}})\right|, \ \forall n \in N.$$

Consequently, $|(x; e_{\alpha_n})| = const, \ \forall n \in N$*. As* $\lim\limits_{n\to\infty} (x; e_{\alpha_n}) = 0$*, it is clear that* $(x; e_{\alpha_n}) = 0, \ \forall n \in N$*. Thus,* $(x; e_\alpha) = 0, \ \forall\alpha \in I$*. It follows that* $x = 0$*. As a result, we obtain that the family* $\{x_\alpha\}_{\alpha\in I}$ *is complete and minimal in* H*. It is not difficult to see that the family* $\{g_\alpha\}_{\alpha\in I}$ *is also complete and minimal in* H*. The family* $\{x_\alpha\}_{\alpha\in I}$ *has no decomposition property, i.e. an arbitrary element cannot be decomposed with respect to this family. For example, it is not difficult to see that the element* $x = e_{\alpha_1}$ *cannot be decomposed with respect to* $\{x_\alpha\}_{\alpha\in I}$*.*

Using Theorem 10 on smallest norms of frame coefficients, the theorem below can be proved in a quite similar way to usual frames.

Theorem 12. *Let the family* $\{x_\alpha\}_{\alpha\in I} \subset H$ *form a frame in non-separable* H*-space* H*. Then, for* $\forall\beta \in I$*, the family* $\{x_\alpha\}_{\alpha\neq\beta}$ *either forms a frame in* H*, or is not complete in* H*. In other words: i) for* $\left(x_\beta; S^{-1}x_\beta\right) \neq 1$ *the family* $\{x_\alpha\}_{\alpha\neq\beta}$ *forms a frame in* H*; ii) for* $\left(x_\beta; S^{-1}x_\beta\right) = 1$ *it is not complete in* H*.*

A frame is called exact if after removing its arbitrary element the resulting family stops being a frame.

Let the family $\{x_\alpha\}_{\alpha\in I}$ form an exact frame in H. Then from Theorem 12 (and its proof) it follows that

$$\left(x_\alpha; S^{-1}x_\beta\right) = \delta_{\alpha\beta},$$

for $\forall\beta \in I$. As $\forall x \in H$ has a decomposition

$$x = \sum_{\alpha\in I} \left(x; S^{-1}x_\alpha\right) x_\alpha,$$

the minimality of the family $\{x_\alpha\}_{\alpha\in I}$ implies that such a decomposition is unique.

7. Riesz Bases in Non-Separable H-Space

Accept the following definition.

Definition 6. *A family $\{x_\alpha\}_{\alpha\in I} \subset H$ in a non-separable H-space H is called a Riesz basis for H if it is complete in H and $\exists A; B > 0$:*

$$A\,\|\lambda\|^2_{l_2(I^C)} \le \left\|\sum_{\alpha\in I}\lambda_\alpha x_\alpha\right\|^2 \le B\,\|\lambda\|^2_{l_2(I^C)}\,, \tag{18}$$

$$\forall\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in l_2\left(I^C\right).$$

Consider the family $\{\delta_\alpha\}_{\alpha\in I}$, where $\delta_\alpha = \{\delta_{\alpha\beta}\}_{\beta\in I}$ and $\delta_{\alpha\beta}$ is the Kronecker symbol, i.e.

$$\delta_{\alpha\beta} = \begin{cases} 1, & \beta = \alpha, \\ 0, & \beta \neq \alpha. \end{cases}$$

It is absolutely clear that $\delta_\alpha \in l_2\left(I^C\right)$, $\forall\alpha\in I$. The family $\{\delta_\alpha\}_{\alpha\in I}$ is called a canonical family. It is not difficult to see that for $\forall\lambda = \{\lambda_\alpha\}_{\alpha\in I} \in l_2\left(I^C\right)$ there is a representation

$$\lambda = \sum_{\alpha\in I}\lambda_\alpha\delta_\alpha. \tag{19}$$

Moreover, for $\forall\lambda \in l_2\left(I^C\right)$ the representation of the form (19) is unique, i.e. the family $\{\delta_\alpha\}_{\alpha\in I}$ forms a basis for $l_2\left(I^C\right)$. Clearly, the series (19) is unconditionally convergent. Let $\vartheta_\alpha(\cdot) : l_2\left(I^C\right) \to C-\ \vartheta_\alpha(\lambda) = \lambda_\alpha,\ \alpha\in I$, be a linear functional. From

$$|\vartheta_\alpha(\lambda)| = |\lambda_\alpha| \le \left(\sum_{\beta\in I}|\lambda_\beta|^2\right)^{1/2} = \|\lambda\|_{l_2(I^C)}\,,$$

it follows that $\vartheta_\alpha(\cdot)$ is a continuous functional. We have $\vartheta_\alpha(\delta_\beta) = \delta_{\alpha\beta},\ \forall\alpha,\beta\in I$.

Denote by $F : l_2\left(I^C\right) \to H$ the operator defined by

$$F\lambda = \sum_{\alpha\in I}\lambda_\alpha x_\alpha,\ \ \lambda = \{\lambda_\alpha\}_{\alpha\in I} \in l_2\left(I^C\right).$$

The operator F is defined correctly. From (18) it follows that it is bounded and, moreover, $\exists F^{-1}$ and F^{-1} is bounded in R_F (R_F is the range of F). If $R_F = H$, then it is clear that F is an isomorphism in H. It is not difficult to see that

$$F\delta_\alpha = x_\alpha,\ \ \forall\alpha\in I. \tag{20}$$

Conversely, if there exists an isomorphism $F \in L(H)$ such that (20) holds, then it is not difficult to see that the relation (18) is true and the family $\{x_\alpha\}_{\alpha\in I}$ is complete in H. Assume that (20) holds, where $F \in L(H)$ is some isomorphism. Let $y_\alpha = \left(F^{-1}\right)^* \delta_\alpha,\ \forall\alpha \in I$. We have

$$(x_\alpha; y_\beta)_H = (F\delta_\alpha; y_\beta)_H = (\delta_\alpha; F^* y_\beta)_{l_2(I^C)} =$$

$$= (\delta_\alpha; \delta_\beta)_{l_2(I^C)} = \delta_{\alpha\beta},\ \forall\alpha, \beta \in I.$$

Consequently, the families $\{x_\alpha; y_\alpha\}_{\alpha\in I}$ are biorthonormal in H. Take $\forall x \in H$ and let $\lambda = F^{-1}x$. We have

$$\lambda = \sum_{\alpha\in I} \lambda_\alpha \delta_\alpha \Rightarrow\ \ F\lambda = \sum_{\alpha\in I} \lambda_\alpha F\delta_\alpha \Rightarrow\ \ x = \sum_{\alpha\in I} \lambda_\alpha x_\alpha.$$

Moreover

$$\lambda_\alpha = (\lambda; \delta_\alpha)_{l_2(I^C)} = \left(F^{-1}x; \delta_\alpha\right)_{l_2(I^C)} =$$

$$= \left(x; \left(F^{-}\right)^* \delta_\alpha\right)_H = (x; y_\alpha)_H, \quad \forall\alpha \in I.$$

It is absolutely clear that the operator $\left(F^{-1}\right)^*$ is also an isomorphism in H. Then it follows from

$$y_\alpha = \left(F^{-1}\right)^* \delta_\alpha, \forall\alpha \in I,$$

that the relation

$$C\,\|\lambda\|^2_{l_2(I^C)} \le \left\|\sum_{\alpha\in I} \lambda_\alpha y_\alpha\right\|^2 \le D\,\|\lambda\|^2_{l_2(I^C)},\ \ \forall\lambda \in l_2\left(I^C\right),$$

is true for the family $\{y_\alpha\}_{\alpha\in I}$, where $C; D > 0$ are absolute constants.
So the following theorem is true.

Theorem 13. *The following properties are equivalent for a family $\{x_\alpha\}_{\alpha\in I} \subset H$ in a non-separable H-space H:*

i) $\{x_\alpha\}_{\alpha\in I}$ forms a Riesz basis for H;

ii) there exists an isomorphism $F \in L\left(l_2\left(I^C\right); H\right)\ :\ F\delta_\alpha = x_\alpha,\ \forall\alpha \in I$;

iii) there exists a family $\{y_\alpha\}_{\alpha\in I}$ biorthogonal to $\{x_\alpha\}_{\alpha\in I}$ which forms a Riesz basis for H;

iv) the families $\{x_\alpha; y_\alpha\}_{\alpha\in I}$ are biorthonormal and there exists an automorphism $T \in L\left(l_2\left(I^C\right); H\right)$, $T\delta_\alpha = y_\alpha,\ \forall\alpha \in I$.

v) there exists a scalar product $(\cdot;\cdot)'_H$, topologically equivalent to the scalar product $(\cdot;\cdot)_H$, with respect to which the family $\{x_\alpha\}_{\alpha\in I}$ forms an orthonormal basis for $H : (x_\alpha; x_\beta)'_H = \delta_{\alpha\beta},\ \ \forall\alpha, \beta \in I$.

Recall that the topological equivalence of scalar products means that there exist absolute constants $C_1; C_2 > 0$ such that

$$C_1 \|x\|_H \le \|x\|_H^{'} \le C_2 \|x\|_H \,, \ \ \forall x \in H, \tag{21}$$

where $\|x\|_H^{'} = \sqrt{(x;x)_H^{'}}$.

In fact, the equivalence of properties i)-v) is already established. Let i) be true and $\{y_\alpha\}_{\alpha\in I}$ be a biorthonormal family corresponding to $\{x_\alpha\}_{\alpha\in I}$. Let $y_\alpha(x) = (x; y_\alpha)_H \,, \ \forall \alpha \in I$. Introduce the following scalar product:

$$(x;y)_H^{'} = \left(\{y_\alpha(x)\}_{\alpha\in I}; \{y_\alpha(y)\}_{\alpha\in I}\right)_{l_2(I^C)} =$$

$$= \sum_{\alpha\in I} y_\alpha(x)\,\overline{y_\alpha(y)}, \quad \forall x; y \in H.$$

We have

$$(x;x)_H^{'} = \sum_{\alpha\in I} |y_\alpha(x)|^2 \,.$$

Clearly

$$(x_\alpha; x_\beta)_H^{'} = \delta_{\alpha\beta}, \ \forall \alpha, \beta \in I.$$

Consequently

$$\left(\left\|\sum_{\alpha\in I} y_\alpha(x)\,x_\alpha - x\right\|_H^{'}\right)^2 = \left(\sum_{\alpha\in I} y_\alpha(x)\,x_\alpha - x; \sum_{\alpha\in I} y_\alpha(x)\,x_\alpha - x\right)_H^{'} =$$

$$= \left(\|x\|_H^{'}\right)^2 - \sum_{\alpha\in I} |y_\alpha(x)|^2 = 0.$$

Hence, the family $\{x_\alpha\}_{\alpha\in I}$ forms an orthonormal basis for H-space $\left(H; (\cdot;\cdot)_H^{'}\right)$.

Conversely, let the family $\{x_\alpha\}_{\alpha\in I}$ form an orthonormal basis for $\left(H; (\cdot;\cdot)_H^{'}\right)$, where scalar products $(\cdot;\cdot)_H^{'}$ and $(\cdot;\cdot)_H$ are topologically equivalent. For convenience, we denote the space $\left(H; (\cdot;\cdot)_H^{'}\right)$ by H_1 with scalar product $(\cdot;\cdot)_{H_1} = (\cdot;\cdot)_H^{'}$. Denote by $J : H_1 \to H$ the operator which maps the element $x \in H_1$ to the element $x \in H$, considered in the space H. From (21) it follows that $J \in L(H_1; H)$ is an isomorphism from H_1 to H. Hence, the family $\{Jx_\alpha = x_\alpha\}_{\alpha\in I}$ forms a basis for H. On the other hand, for $\forall \lambda \in l_2\left(I^C\right)$ we have $\left(\lambda = \{\lambda_\alpha\}_{\alpha\in I}\right)$

$$\|\lambda\|_{l_2(I^C)}^2 = \sum_{\alpha\in I} |\lambda_\alpha|^2 = \left\|\sum_{\alpha\in I} \lambda_\alpha x_\alpha\right\|_{H_1}^2 = \left\|\sum_{\alpha\in I} \lambda_\alpha J^{-1} x_\alpha\right\|_{H_1}^2 =$$

$$= \left\| J^{-1}\left(\sum_{\alpha\in I}\lambda_\alpha x_\alpha\right)\right\|_{H_1}^2 \leq \left\|J^{-1}\right\|^2 \left\|\sum_{\alpha\in I}\lambda_\alpha x_\alpha\right\|_H^2 .$$

Similarly we have

$$\left\|\sum_{\alpha\in I}\lambda_\alpha x_\alpha\right\|_H^2 = \left\|\sum_{\alpha\in I}\lambda_\alpha J x_\alpha\right\|_H^2 = \left\|J\left(\sum_{\alpha\in I}\lambda_\alpha x_\alpha\right)\right\|_H^2 \leq$$

$$\leq \|J\|^2 \left\|\sum_{\alpha\in I}\lambda_\alpha x_\alpha\right\|_{H_1}^2 \leq \|J\|^2 \|\lambda\|_{l_2(I^C)}^2 .$$

Thus

$$\left\|J^{-1}\right\|^{-2} \|\lambda\|_{l_2(I^C)}^2 \leq \left\|\sum_{\alpha\in I}\lambda_\alpha x_\alpha\right\|_H^2 \leq \|J\|^2 \|\lambda\|_{l_2(I^C)}^2 , \forall\lambda\in l_2\left(I^C\right).$$

Theorem is fully proved.

Accept the following definition.

Definition 7. *A family $\{x_\alpha\}_{\alpha\in I}\subset H$ is called ω-linearly independent in a non-separable H-space H if $\sum_{\alpha\in\omega}\lambda_\alpha x_\alpha = 0$ implies $\lambda_\alpha = 0$, $\forall\alpha\in\omega$, for $\forall\omega\subset I$: $card\omega \leq \theta_0$.*

Now back to the frames. Let the family $\{x_\alpha\}_{\alpha\in I}\subset H$ form an exact frame for H and S be the corresponding frame operator. Then, as already stated above, $\left(x_\alpha; S^{-1}x_\beta\right) = \delta_{\alpha\beta}$, $\forall\alpha;\beta\in I$. Consequently, the family $\{x_\alpha\}_{\alpha\in I}$ forms a basis for H, and, by the definition of frame, this basis is a Riesz basis. So we get the validity of the following theorem.

Theorem 14. *Let the family $\{x_\alpha\}_{\alpha\in I}$ form a frame for a non-separable H-space H. Then the following properties are equivalent:*

i) $\{x_\alpha\}_{\alpha\in I}$ forms a Riesz basis for H;

ii) $\{x_\alpha\}_{\alpha\in I}$ is an exact frame in H;

iii) $\{x_\alpha\}_{\alpha\in I}$ is minimal in H;

iv) $\{x_\alpha\}_{\alpha\in I}$ has a biorthonormal family;

v) the family $\{x_\alpha\}_{\alpha\in I}$ is ω-linearly independent in H;

vi) $\lambda\in l_2\left(I^C\right) : \sum_{\alpha\in I}\lambda_\alpha x_\alpha = 0$ implies $\lambda = 0$;

vii) $\{x_\alpha\}_{\alpha\in I}$ forms a basis for H.

In fact, implications $i) \Rightarrow ii) \Rightarrow iii) \Rightarrow iv)$; $i) \Rightarrow vi) \Rightarrow i) \Rightarrow vii) \Rightarrow i)$ are obvious. It is clear that $vii)$ implies $v)$. Let $v)$ be true. As $\{x_\alpha\}_{\alpha\in I}$ forms a frame in H, it is clear that the arbitrary element can be decomposed with respect to this family. From $v)$ it follows that this decomposition is unique, and, consequently, the family $\{x_\alpha\}_{\alpha\in I}$ forms a basis for H. Hence, $v) \Rightarrow vii)$ is true. Theorem is proved.

Acknowledgement

This research was supported by the Azerbaijan National Academy of Sciences under the program "Approximation by neural networks and some problems of frames".

References

[1] R.J. Duffin, A.C. Schaeffer, *A class of nonharmonic Fourier series*, Trans. Amer. Math. Soc., **72**, 1952, 341-366.

[2] Ch. Chui, *Wavelets: a tutorial in theory and applications*, Academic Press, Boston, 1992.

[3] Y. Meyer, *Wavelets and operators*, Herman, Paris, 1990.

[4] I. Daubechies, *Ten lectures on wavelets*, SIAM, Philadelphia, 1992.

[5] S. Mallat, *A wavelet tour of signal processing*, Academic Press, San Diego, 1999.

[6] R. Young, *An introduction to nonharmonic Fourier series*, Academic Press, New York, 1980.

[7] Ch. Heil, *A Basis Theory Primer*, Springer, 2011.

[8] O. Christensen, *An Introduction to Frames and Riesz Bases*, Birkhauser Boston, 2002.

[9] O. Christensen, *Frames and bases. An introductory course*, Birkhauser, Boston, 2008.

[10] O. Christensen, D.T. Stoeva, *p-frames in separable Banach spaces*, Advances in Computational Mathematics, **18**, 2003, 17-126.

[11] I.M. Dremin, O.V. Ivanov, V.A. Nechitailo, *Wavelets and their uses*, Usp. phisics nauk, **171(5)**, 2001, 465-501.

[12] A. Aldroubi, Q. Sun, W.Sh. Tang, *p-frames and shift invariant subspaces of L_p*, J. Fourier Anal. Appl., **7(1)**, 2001, 1-22.

[13] N.K. Bari, *Biorthogonal systems and bases in Hilbert space*, Moscow Gos. Univ. Uceneye Zapiski **148(4)**, 1951, 69-107.

[14] B.T. Bilalov, Z.G. Guseinov, *p-Bessel and p-Hilbert systems and p-bases*, Dokl. Nats. Akad. Nauk Azerb., **64(3)**, 2008, 3-8 (in Russian).

[15] B.T. Bilalov, Z.G. Guseinov, *K -Bessel and K-Hilbert systems and K-bases*, Doklady Mathematics, **80(3)**, 2009, 826-828.

[16] B.T. Bilalov, Sh.M. Hashimov, *On Decomposition In Banach Spaces*, Proceedings of the IMM of NAS of Azerbaijan, **40(2)**, 2014, 97-106.

[17] S.R. Sadigova, *On Frame Properties Of Degenerate System Of Exponents In Hardy Classes*, Caspian J. of Appl. Math., Ecol. and Econom., **1(1)**, 2013, 97-103.

[18] S.R. Sadigova, The general solution of the homogeneous Riemann problem in the weighted Smirnov classes, Proc. of the IMM of NAS of Azerbaijan, 40(2) (2014), 115-124.

[19] S.R. Sadigova, A.I. Ismailov, *On Frames of Double and Unary Systems in Lebesgue Spaces*, Pensee Journal, **76(4)**, 2014, 189-202.

[20] S.R. Sadigova, Z.A. Kasumov, *On atomic decomposition for Hardy classes with respect to degenerate exponential systems*, Proc. of the IMM of NAS of Azerbaijan, **40(1)**, 2014, 55-67.

[21] A. Rahimi, B. Daraby, Z. Darvishi, *Construction of Continuous Frames in Hilbert spaces*, Azerb. J. of Math., **7(1)**, 2017, 49-58.

[22] S. Li, H. Ogawa, *Pseudoframes for subspaces with applications*, J. Fourier Anal. Appl., **10**, 2004, 409-431.

[23] H.G. Feichtinger, K.H. Grochening, *Banach spaces related to integrable group representations and their atomic decompositions*, I., J. of Func. Analysis, **86(2)**, 1989, 307-340

[24] H.G. Feichtinger, K.H. Grochening, *Banach spaces related to integrable group representations and their atomic decompositions*, II., Mh. Math., **108**, 1989, 129-148.

[25] K.H. Grochenig, *Describing Functions: Atomic Decompositions Versus Frames*, Mh. Math., **112**, 1991, 1-41.

[26] W. Sun, *Stability of g-frames*, J. of Math. Anal. and Appl., **326(2)**, 2007, 858-868.

[27] W. Sun, *G-frames and g-Riesz bases*, J. of Math. Anal. and Appl., **322 (1)**, 2006, 437-452.

[28] B.T. Bilalov, F.A. Guliyeva, *Noetherian perturbation of frames*, Pensee Journal, **75(12)**, 2013, 425-431.

[29] B.T. Bilalov, F.A. Guliyeva, *t-Frames and their Noetherian Perturbation*, Complex Analysis and Operator Theory, **9(7)**, 2015, 1609-1631.

[30] B.T. Bilalov, Z.V. Mamedova, *On the frame properties of some degenerate trigonometric system*, Dokl. Nats. Akad. Nauk Azerb., **68(5)**, 2012, 14-18 (in Russian).

[31] B.T. Bilalov, F.A. Guliyeva, *On The Frame Properties of Degenerate System of Sines*, J. of Funct. Spaces and Appl., **2012**, 2012, 12 pages (Article ID 184186).

[32] R. Kadison, I. Singer, *Extensions of pure states*, Amer. J. Math., **81**, 1959, 547-564.

[33] P.G. Casazza, R. Vershynin, *Kadison-Singer meets Bourgain-Tzafriri*, Preprint.

[34] P.G. Casazza, O. Christensen, A. Lindner, R. Vershynin, *Frames and the Feichtinger Conjecture*, Proceedings of AMS, **133(4)**, 2005, 1025-1033.

[35] J. Bourgain, L. Tzafriri, *On a problem of Kadison and Singer*, J. Reine Angew. Math., **420**, 1991, 1-43.

[36] A. Marcus, D.A. Spielman, N. Srivastava, *Interlacing families II: Mixed characteristic polynomials and the Kadison-Singer problem*, arXiv preprint arXiv:1306.3969, 2013.

[37] N. Weaver, *The Kadison-Singer problem in discrepancy theory*, Discrete mathematics **278(1)**, 2004, 227-239.

[38] N. Weaver, *The Kadison-Singer problem in discrepancy theory*, II. arXiv:1303.2405v1, 11 Mar 2013.

[39] P.G. Casazza, F. Matthew, C.T. Janet, W. Eric, *The Kadison-Singer problem in mathematics and engineering: a detailed account*, Contemporary Mathematics, **414**, 2006, 297-356.

[40] D.V. Cruz-Uribe, A. Fiorenza, *Variable Lebesgue Spaces: Foundations and Harmonic Analysis*, Springer, 2013.

[41] V. Kokilashvili, A. Meshki, H. Rafeiro, S. Samko, *Integral operators in nonstandard function spaces*, Birkhauser-Springer, 2016.

[42] D.R. Adams, *Morrey spaces*, Birkhauser-Springer, 2015.

[43] C. Bardaro, J. Musielak, G. Vinti, *Nonlinear integral operators and applications*, Walter de Gruyter-Berlin-New York, 2013.

[44] M.I. Ismayilov, Y.I. Nasibov, *One Generalization of Banach Frame*, Azerb. J. of Math., **6(2)**, 2016, 143-159.

[45] O. Christensen, *A Short Introduction To Frames, Gabor Systems, And Wavelet Systems*, Azerb. J. of Math., **4(1)**, 2014, 25-39.

[46] O. Christensen, M.I. Zakowicz, *Paley-Wiener type perturbations of frames and the deviation from perfect reconstruction*, Azerb. J. of Math., **7(1)**, 2017, 59-69.

[47] Z.A. Kasumov, Ch.M. Hashimov, *On the equivalent bases of cosines in generalized Lebesgue spaces*, Proc. of the IMM, Nat. Acad. of Scien. of Azerb., **41(2)**, 2015, 70-76.

Bilal Bilalov
Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9, B.Vahabzade str., AZ1141, Baku, Azerbaijan
E-mail: b_bilalov@mail.ru

Migdad Ismailov
Baku State University, 23, Z.Khalilov str., AZ1148, Baku, Azerbaijan
Institute of Mathematics and Mechanics of NAS of Azerbaijan, 9, B.Vahabzade str., AZ1141, Baku, Azerbaijan
E-mail: miqdadismailov1@rambler.ru

Zahira Mamedova
British School in Baku
E-mail: zahira_eng13@hotmail.com

Received 10 September 2017
Accepted 24 November 2017